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A remark on Ribenboim's paper 'On the extension of orders in ordered modules'

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We give another formulation of Theorem 1 of Ribenboim's paper, *Bull. Austral. Math. Soc.* 2 (1970), 81-88. Our condition on a module to be totally ordered such that this order extends a given strict order seems to be more useful for applications than condition (2) of Ribenboim. This is shown for example in Corollaries 3 and 4 of our paper. Corollary 4 is due to Ribenboim. Corollary 3 generalizes the corresponding well known result on abelian groups.

Let A be a commutative ordered ring with $1 \in P_A$, where P_A is the *positive cone* of A satisfying $P_A + P_A \subset P_A$, $P_A P_A \subset P_A$, $P_A \cap -P_A = 0$.

Let M be an ordered A-module with positive cone P_M satisfying $P_M + P_M \subset P_M$, $P_A P_M \subset P_M$ and $P_M \cap -P_M = 0$.

A (or M) is said to be *strictly ordered* iff for $a, b \in A$, $a > 0$ and $b > 0$ implies $ab > 0$ (or iff for $a \in A$, $m \in M$, $a > 0$ and $m > 0$ implies $am > 0$). Let $P_A^* = P_A \setminus \{0\}$, $P_M^* = P_M \setminus \{0\}$.

LEMMA 1. *Let M be an A-module with strict orders P_M and Q_M. If $P_M \cap -Q_M = 0$, then $S_M = P_M + Q_M$ is a strict order on M such that*

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$P_M, Q_M \subset S_M$.

Proof. $S_M + S_M \subset S_M$ and $P_A S_M \subset S_M$ are immediate from the definition of S_M . Let $x \in S_M \cap -S_M$; then $x = p + q = -p' + (-q')$ with $p, p' \in P_M$, $q, q' \in Q_M$. Hence $p + p' = -(q+q') \in P_M \cap -Q_M = 0$, from which we deduce $p = 0$, $q = 0$. Thus $S_M \cap -S_M = 0$. Let $a \in P_A$, $x = p + q \in S_M$; then $0 = ax = ap + aq$ implies $ap = -aq \in P_M \cap -Q_M = 0$. As P_M is a strict order, this shows $a = 0$; hence S_M is a strict order.

THEOREM 2. *Let A be a commutative strictly ordered ring with $1 \in P_A$. Let M be a strictly ordered A-module with order P_M . The following statements are equivalent:*

(1) *there exists a total strict order T_M on M such that $P_M \subset T_M$;*

(2) $\forall\, m \in M \setminus \{0\} : \mathrm{Ann}(m) \cap P_A = 0$.

($\mathrm{Ann}(m)$ denotes the *annihilator* of $m \in M$).

Proof. (1) ⇒ (2). Let $0 \neq m \in M$; we may assume $m > 0$. Let $a \in \mathrm{Ann}(m) \cap P_A$; then $am = 0$ implies $a = 0$, since T_M is a strict order.

(2) ⇒ (1). Let P be the set of strict orders P with $P_M \subset P$. Let $(P_i)_{i\in I}$ be a chain in P . Define $P_I = \bigcup_{i\in I} P_i$; then evidently $P_I \in \mathsf{P}$, and P is inductive. By Zorn's Lemma there exists a maximal strict order $T_M \supset P_M$. We state: T_M is a total order.

Assume on the contrary that there exists $m \in M$ such that $m \notin T_M \cup -T_M$. Without loss of generality we consider the following two cases:

(a). $\forall\, a \in P_A^* : am \notin -T_M$. Now $P_A m$ defines a strict order on M ,

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for $x \in P_A m \cap -P_A m$ implies $x = am = -bm$ with $a, b \in P_A$; thus $(a+b)m = 0$. Since $\mathrm{Ann}(m) \cap P_A = 0$, we have $a + b = 0$; hence $a = -b = 0$. Let $a \in P_A^*$, $b \in P_A$. Then $b(am) = 0$ implies $ba = 0$, since $\mathrm{Ann}(m) \cap P_A = 0$. Since A is strictly ordered, we have $b = 0$. These are the only two nontrivial conditions to be proved. Now from Lemma 1 it follows that $T'_M = T_M + P_A m$ is a strict order on M , a contradiction to the maximality of T_M .

(b). $\exists\, a \in P_A^*$ such that $am \in -T_M$. Then $\forall\, b \in P_A^* : b(-m) \notin -T_M$. Assume on the contrary that there is a $b \in P_A^*$ with $b(-m) \in -T_M$; then with the above $a \in P_A^*$ we have $b(-amb) = ab(-m) \in T_M \cap -T_M = 0$; hence $ba(-m) = 0$. $\mathrm{Ann}(-m) \cap P_A = 0$ implies $ba = 0$. Since A is strictly ordered this leads to a contradiction. Thus $-m$ instead of m leads back to case (a). This completes the proof.

COROLLARY 3. *Every torsionfree module over a strictly ordered ring can be totally ordered such that this order is strict.*

COROLLARY 4 (see Ribenboim [1]). *Let* (R, P_R) *be a strictly ordered, totally ordered ring. Let* $M = R^I$ *be the ordered* R*-module with pointwise order* P_M . *Then there exists a total order* T_M *on* M *such that* $P_M \subset T_M$ *and* (M, T_M) *is a strictly ordered module over* (R, P_R) .

Proof. Immediate, since $\mathrm{Ann}\left((r_i)\right) = \bigcap_{i \in I} \mathrm{Ann}(r_i)$, $(r_i) \in R^I$.

Reference

[1] P. Ribenboim, "On the extension of orders in ordered modules", *Bull. Austral. Math. Soc.* 2 (1970), 81-88.

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