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J. ARCHIBALD, Esq., M.A., President, in the Chair.

On the Cartesian Coordinates of Classes of Tortuous Curves.

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The following notation is here used for the quantities occurring in the discussion of tortuous curves.

Length $= s$; curvature $= \frac{1}{R}$; torsion $= \frac{1}{T}$; direction cosines of tangent, principal normal and binormal:—a, β, γ; l, m, n; λ, μ, ν.

Frenet's formulæ are therefore,

$$\frac{da}{ds} = \frac{l}{R};\ \frac{dl}{ds} = -\frac{a}{R} - \frac{\lambda}{T};\ \frac{d\lambda}{ds} = \frac{l}{T}$$

with two corresponding sets.

To find the cartesians of a tortuous curve from the values of R and T as functions of s, Hoppe in *Crelle's Journal* (1862) reduced these equations to the discussion of a differential equation of the second order. Lie reduced them to a Riccati. The detailed process is given in Scheffers' *Einführung in die Theorie der Curven*, but only two cases are worked out: (i) the helix, (ii) the general helix on any cylinder that is the curves $\frac{R}{T} =$ a constant. These examples are trivial and need no elaborate theory. I shall return to the Riccati equation at the end of this paper.

Integral forms have been given for the cartesians of several classes of curves. Thus when R is a constant

$$x = R\int a d\sigma,\quad y = R\int \beta d\sigma,\quad z = R\int \gamma d\sigma$$

where $d\sigma^2 = da^2 + d\beta^2 + d\gamma^2$, and $a^2 + \beta^2 + \gamma^2 = 1$.

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For T = a constant, Darboux gives the beautiful forms

$$x = \mathrm{T}\int\frac{ldk - kdl}{h^2 + k^2 + l^2}, \quad y = \mathrm{T}\int\frac{hdl - ldh}{h^2 + k^2 + l^2}, \quad z = \mathrm{T}\int\frac{kdh - hdk}{h^2 + k^2 + l},$$

where h, k, and l are arbitrary functions of a single variable.

For Bertrand's curves $\frac{a}{\mathrm{R}} + \frac{b}{\mathrm{T}} = c$ we have

$$x = \frac{a}{c}\int u d\sigma + \frac{b}{c}\int(wv' - vw')d\sigma,$$

$$y = \frac{a}{c}\int v d\sigma + \frac{b}{c}\int(uw' - wu')d\sigma,$$

$$z = \frac{a}{c}\int w d\sigma + \frac{b}{c}\int(vu' - uv')d\sigma,$$

where u, v and w are functions of σ such that $u^2 + v^2 + w^2 = 1$ and $u'^2 + v'^2 + w'^2 = 1$.

Finally Scheffers gives integral expressions of a very involved form for the curves $\frac{1}{\mathrm{R}^2} + \frac{1}{\mathrm{T}^2} = \text{constant}$.

It is seen that with the exception of the curves of constant torsion none of these can claim to be very explicit, and I can find no actual examples worked out except for T = constant. [See Darboux *Théorie des Surfaces*, Vol. IV., Appendix.] The first part of the present paper gives integral expressions of an explicit nature for these and other classes of curves by one uniform simple method which shows the reason of the occurrence of such integrals.

Let $x = \int\cos\theta ds, \quad y = \int\sin\theta\cos\phi ds, \quad z = \int\sin\theta\sin\phi ds.$

Then $\frac{1}{\mathrm{R}} = \sqrt{\left\{1 + \left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}\frac{d\theta}{ds}$, the positive sign of the root being taken so that, as R is to be considered positive, θ and s increase together. If θ is a constant we shall have from

$$\frac{1}{\mathrm{R}} = \sqrt{\left\{\left(\frac{d\theta}{ds}\right)^2 + \left(\sin\theta\frac{d\phi}{ds}\right)^2\right\}}, \quad \frac{1}{\mathrm{R}} = \sin\theta\frac{d\phi}{ds}.$$

$$\frac{da}{ds} - \frac{l}{\mathrm{R}} = -\sin\theta\frac{d\theta}{ds}.$$

$$\therefore \quad l = -\frac{\sin\theta}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}.$$

$$\therefore \quad \lambda = \frac{\sin^2\theta\frac{d\phi}{d\theta}}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}.$$

$$\frac{dl}{ds} = -\frac{a}{R} - \frac{\lambda}{T}.$$

From this, $\frac{1}{T} = -\left\{\cos\theta\frac{d\phi}{d\theta} + \frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}\frac{d\theta}{ds}.$

When R is a constant and θ is not a constant, the cartesians are

$$x = R\int\cos\theta\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta;$$

$$y = R\int\sin\theta\cos\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta;$$

$$z = R\int\sin\theta\sin\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta.$$

In these ϕ is an arbitrary function of θ.

I give some examples integrable in terms of ordinary functions.

1st Case. θ a constant. The spherical indicatrix is a circle and the curve is a helix.

$$x = \frac{R\phi}{2}\sin 2\theta, \quad y = R\sin^2\theta\sin\phi, \quad z = -R\sin^2\theta\cos\phi.$$

Let us transform the integrals by making $\tan\frac{\theta}{2} = v$ and $\tan\frac{\phi}{2} = u$ where u and v are arbitrary functions of a variable t.

$$a, \beta, \gamma \text{ are } \frac{1-v^2}{1+v^2}, \quad \frac{2v(1-u^2)}{(1+u^2)(1+v^2)}, \quad \frac{4uv}{(1+u^2)(1+v^2)}.$$

Then
$$\frac{x}{R} = 2\int\frac{1-v^2}{(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt,$$

$$\frac{y}{R} = 4\int\frac{(1-u^2)v}{(1+u^2)(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt,$$

$$\frac{z}{R} = 8\int\frac{uv}{(1+u^2)(1+v^2)^2}\sqrt{\left\{v'^2+\frac{4u'^2v^2}{(1+u^2)^2}\right\}}dt.$$

2nd Case. $v = 1 + u^2$ and $u^2 = t$. The spherical indicatrix is

$$\tan\frac{\theta}{2} = \sec^2\frac{\phi}{2}.$$

$$\frac{x}{R} = -2\int\frac{t^2+2t}{(t^2+2t+2)^2}\sqrt{\left(1+\frac{1}{t}\right)}dt,$$

$$\frac{y}{R} = 4\int\frac{1-t}{(t^2+2t+2t)^2}\sqrt{\left(1+\frac{1}{t}\right)}dt,$$

$$\frac{z}{R} = 8\int\frac{\sqrt{(1+t)}dt}{(t^2+2t+2)^2}.$$

3rd Case. $v = \dfrac{1+u^2}{u}$, that is $\tan\dfrac{\theta}{2} = 2\operatorname{cosec}\phi$.

$$\frac{x}{R} = -2\int\frac{(u^4+u^2+1)(1+u^2)udu}{(u^4+3u^2+1)^2},$$

$$\frac{y}{R} = 4\int\frac{(1-u^4)u^2du}{(u^4+3u^2+1)^2},$$

$$\frac{z}{R} = 8\int\frac{(1+u^2)u^3du}{(u^4+3u^2+1)^2}.$$

4th Case. $v = \dfrac{1+u^2}{u^2}$, that is $\tan\dfrac{\theta}{2} = \operatorname{cosec}^2\dfrac{\phi}{2}$.

$$\frac{x}{R} = -4\int\frac{u(2u^2+1)\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du,$$

$$\frac{y}{R} = 8\int\frac{(1-u^2)u^3\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du,$$

$$\frac{z}{R} = 16\int\frac{u^4\sqrt{(1+u^2)}}{(2u^4+2u^2+1)^2}du.$$

There is no need to give the results which involve logarithmic and inverse circular functions.

The locus of the centre of curvature (ξ, η, ζ) of a curve of constant curvature is a similar curve and, if T and T′ be the radii of torsion, $TT' = R^2$.

As an example, take the 2nd case.

$$ds = \frac{2R\sqrt{\left(1+\frac{1}{t}\right)}}{t^2+2t+2}dt; \quad a = -1 + \frac{2}{t^2+2t+2}; \quad \frac{da}{ds} = -\frac{2\sqrt{t(1+t)}}{R(t^2+2t+2)} = \frac{l}{R}.$$

Similarly $m=\frac{t(t^2-2t-4)}{2(t^2+2t+2)\sqrt{t(1+t)}}$, $n=\frac{2-2t-3t^2}{(t^2+2t+2)\sqrt{(1+t)}}$.

$$\xi=x+l\mathrm{R},\quad \eta=y+m\mathrm{R},\quad \zeta=z+n\mathrm{R}.$$

When T is a constant the cartesians are

$$x=-\mathrm{T}\int\cos^2\theta d\phi-\mathrm{T}\int\cos\theta d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right),$$

$$y=-\mathrm{T}\int\sin\theta\cos\theta\cos\phi d\phi-\mathrm{T}\int\sin\theta\cos\phi d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right),$$

$$z=-\mathrm{T}\int\sin\theta\cos\theta\sin\phi d\phi-\mathrm{T}\int\sin\theta\sin\phi d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

One integrable class occurs evidently when $\phi=n\theta+c$, n a positive integer. We give the results for $\phi=\theta$.

$$\frac{x}{\mathrm{T}}=\frac{\theta}{2}-\tfrac{1}{4}\sin2\theta-\frac{1}{\sqrt{2}}\cos^{-1}\frac{3\cos2\theta-1}{3-\cos2\theta},$$

$$\frac{y}{\mathrm{T}}=\tfrac{1}{3}\cos^3\theta-\cos\theta+\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+\cos\theta}{\sqrt{2}-\cos\theta},$$

$$\frac{z}{\mathrm{T}}=-\tfrac{1}{3}\sin^3\theta-\sin\theta+\tan^{-1}\sin\theta.$$

It may be easily proved that $\mu=\frac{\sin\phi+\cos\theta\cos\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$

and that $\nu=\frac{\cos\phi-\sin\phi\cos\theta\left(\sin\theta\frac{d\phi}{d\theta}\right)}{\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$.

By making

$$\frac{\sin^2\theta\frac{d\phi}{d\theta}}{h\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}=\frac{\sin\psi+\cos\theta\cos\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{k\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}$$

$$=\frac{\cos\phi-\cos\theta\sin\phi\left(\sin\theta\frac{d\phi}{d\theta}\right)}{l\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}}=\frac{1}{\sqrt{(h^2+k^2+l^2)}}$$

we get Darboux's formulae.

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For Bertrand's curves we have

$$ds=\frac{a}{c}\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta+\frac{b}{c}\cos\theta d\phi+\frac{b}{c}d\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

$$x=\frac{a}{c}\int\cos\theta\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta+\frac{b}{c}\int\left\{\cos^2\theta\frac{d\phi}{d\theta}+\cos\theta\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta,$$

$$y=\frac{a}{c}\int\sin\theta\cos\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta+\frac{b}{c}\int\sin\theta\cos\phi\left\{\cos\theta\frac{d\phi}{d\theta}+\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta,$$

$$z=\frac{a}{c}\int\sin\theta\sin\phi\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}d\theta+\frac{b}{c}\int\sin\theta\sin\phi\left\{\cos\theta\frac{d\phi}{d\theta}+\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}d\theta.$$

For shortness let

$$\mathrm{P}=\sqrt{\left\{1+\left(\sin\theta\frac{d\phi}{d\theta}\right)^2\right\}}\quad\text{and}\quad\mathrm{Q}=-\cos\theta\frac{d\phi}{d\theta}-\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right).$$

For the curves whose radius of screw (Frost) is constant, that is the curves $\frac{1}{\mathrm{R}^2}+\frac{1}{\mathrm{T}^2}=\text{constant}=\frac{1}{c^2}$,

$$ds=c\sqrt{(\mathrm{P}^2+\mathrm{Q}^2)}d\theta$$

$$=c\sqrt{\left[1+\left(\frac{d\phi}{d\theta}\right)^2+2\cos\theta\frac{d\phi}{d\theta}\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)+\left\{\frac{d}{d\theta}\tan^{-1}\left(\sin\theta\frac{d\phi}{d\theta}\right)\right\}^2\right]}d\theta.$$

The curves whose principal normals are the binormals of a second curve,

$$\frac{1}{\mathrm{R}^2}+\frac{1}{\mathrm{T}^2}=\frac{1}{a\mathrm{R}}$$

give
$$ds=a\left(\mathrm{P}+\frac{\mathrm{Q}^2}{\mathrm{P}}\right)d\theta.$$

The curves $\frac{a}{\mathrm{RT}}+\frac{b}{\mathrm{T}^2}+\frac{c}{\mathrm{R}^2}+\frac{d}{\mathrm{R}}=0$, the axis of whose osculating helix with the same torsion describes, with reference to the tangent, principal normal and binormal as axes, a Plücker's conoid [Demoulin, *Paris Soc. Math. Bull.* 21 (1893)] give

$$ds=-\frac{1}{d}\left\{a\mathrm{Q}+\frac{b\mathrm{Q}^2}{\mathrm{P}}+c\mathrm{P}\right\}d\theta.$$

The curves $\frac{a}{\mathrm{RT}}+\frac{b}{\mathrm{T}^2}+\frac{c}{\mathrm{R}^2}+\frac{d}{\mathrm{R}}+\frac{e}{\mathrm{T}}=0$ in which a straight line fixed relatively to the tangent, principal normal and binormal generates a developable surface [E. Cesàro, *Natürliche Geometrie*] give

$$ds=-\{a\mathrm{PQ}+b\mathrm{Q}^2+c\mathrm{P}^2\}d\theta/(d\mathrm{P}+e\mathrm{Q}).$$

Hence expressions as integrals, although very cumbrous, can be given for the cartesians.

Let us now consider the curves $\frac{R}{T}=f(s)$ given by Enneper (*Mathematische Annalen*, 1882) and Pirondini (*Crelle's Journal* 1892) as geodetics on developable surfaces.

If $\frac{d\theta}{\sin\theta}=du$, that is $\tan\frac{\theta}{2}=e^u$,

then
$$\frac{1}{R}=\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}\frac{1}{\cosh u}\frac{du}{ds},$$

and
$$-\frac{1}{T}=\left\{-\tanh u\frac{d\phi}{du}+\frac{d}{du}\tan^{-1}\left(\frac{d\phi}{du}\right)\right\}\frac{du}{ds}.$$

$$\therefore\quad -\frac{R}{T}=-\frac{\sinh u\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}+\cosh u\frac{d}{du}\left[\frac{\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}\right].$$

$$\therefore\quad \frac{\frac{d\phi}{du}}{\sqrt{\left\{1+\left(\frac{d\phi}{du}\right)^2\right\}}}=\left\{\text{const.}-\int\frac{R\,du}{T\cosh^2 u}\right\}\cosh u.$$

Hence if $\cosh u$ be made an arbitrary function of s, $\frac{d\phi}{du}$ and therefore ϕ are known in terms of s. The whole is now reduced to a question of quadratures.

Let $\sin\theta\frac{d\phi}{d\theta}=\tan\zeta$ where ζ is an arbitrary function of θ.

Then $\frac{1}{R}=\sec\zeta\frac{d\theta}{ds}$ and $-\frac{1}{T}=\left\{\cot\theta\tan\zeta+\frac{d\zeta}{d\theta}\right\}\frac{d\theta}{ds}$.

$$\therefore\quad \phi=\int\frac{\tan\zeta}{\sin\theta}d\theta,$$

$$\int\frac{ds}{R}=\int\frac{d\theta}{\cos\zeta},$$

$$-\int\frac{ds}{T}=\zeta+\int\frac{\tan\zeta}{\tan\theta}d\theta.$$

The elimination of ζ or θ gives an involved differential equation of the second order for θ or ζ. We may, however, get some solutions

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when R or T is given as a function of s by giving a value to ζ. The corresponding value of T or R can then be got. Thus if $\zeta=\theta$, $\phi=\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ and $\int\frac{ds}{R}=\log c\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$. Hence, if $R=s$ so that $s=c\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ and $\phi=\log\frac{s}{c}$, then $T=-\frac{c^2+s^2}{4c}$.

Again if $\phi=\theta$ and $R=c\sqrt{\left(\frac{c^2-s^2}{c^2+s^2}\right)}$ we have $\sin\theta=\frac{s}{c}$ and $T=-\frac{c(c^2+s^2)}{2c^2+s^2}$.

The coordinates are then

$$x=c\int\cos^2\theta d\theta \quad = \quad \frac{c}{2}(\theta+\sin\theta\cos\theta),$$

$$y=c\int\sin\theta\cos^2\theta d\theta = -\frac{c}{3}\cos^3\theta,$$

$$z=c\int\sin^2\theta\cos\theta d\theta = \frac{c}{3}\sin^3\theta.$$

We wish here to give a note on the Riccati equation given by Lie, although the work is not directly connected with the preceding method. Before giving his substitution we shall slightly change the form of Frenet's formulae by writing $R=\frac{ds}{d\theta}$ or alternatively $T=\frac{ds}{d\omega}$ where $d\theta$ and $d\omega$ are the angles of contingence and torsion.

Thus $$\frac{da}{d\theta}=l, \quad \frac{dl}{d\theta}=-a-\frac{R}{T}\lambda, \quad \frac{d\lambda}{d\theta}=\frac{R}{T}l, \text{ etc.,}$$

or $$\frac{da}{d\omega}=\frac{T}{R}l, \quad \frac{dl}{d\omega}=-\lambda-\frac{T}{R}a, \quad \frac{d\lambda}{d\omega}= \quad l, \text{ etc.}$$

Then, as pointed out by Hoppe, from any expressions involving a, λ, R, T and θ we get corresponding ones with ω by interchanging a and λ, R and T, θ and ω,

Since $$a^2+l^2+\lambda^2=1$$

we may put $$\frac{a+il}{1-\lambda}=\xi.$$

Then $$\frac{d\xi}{d\omega}=\frac{i}{2}(1-\xi^2)-i\frac{T}{R}\xi.$$

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If $\phi + i\psi$ is a particular solution, ϕ and ψ being real functions of ω,

$$\frac{d\psi}{d\omega} = \tfrac{1}{2}(1 - \phi^2 + \psi^2) - \frac{T}{R}\phi,$$

and
$$\frac{d\phi}{d\omega} = \phi\psi + \frac{T}{R}\psi.$$

By elimination of $\dfrac{T}{R}$

$$\phi\frac{d\phi}{d\omega} + \psi\frac{d\psi}{d\omega} = \tfrac{1}{2}\psi(1 + \phi^2 + \psi^2).$$

$\therefore\ 1 + \phi^2 + \psi^2 = ce^{\int\psi d\omega}$, c a positive constant.

If ϕ or ψ be taken arbitrarily, a particular solution is determined and the general solution can be found corresponding to the value $\dfrac{1}{\psi}\dfrac{d\phi}{d\omega} - \phi$ of $\dfrac{T}{R}$. We refer to Scheffers' *Einführung in die Theorie der Curven* for the deduction of coordinates from the general integral. To take a simple case, let $\phi = \cosh\theta$, $\psi = -\sin\theta$ and $c = 2$. By changing the variable from ω to θ, the equation is

$$\frac{d\xi}{d\theta} = \frac{iR}{2T}(1 - \xi^2) - i\xi.$$

From $\quad 1 + \phi^2 + \psi^2 = 2e^{\int\psi d\omega}$

we have
$$\frac{d\omega}{d\theta} = -\frac{2}{\cosh\theta}.$$

$\therefore$
$$\frac{T}{R} = -\tfrac{1}{2}\cosh\theta.$$

The equation now takes the form

$$\frac{d\xi}{d\theta} + \frac{i(1 - \xi^2)}{\cosh\theta} + i\xi = 0$$

of which a particular solution is $\cosh\theta - i\sinh\theta$.

The general solution is therefore

$$\frac{d\{\cosh\theta - i\sin\theta\} + e^{i\theta}}{d - e^{i\theta}\{\cosh\theta + i\sinh\theta\}} \text{ where } d \text{ is a constant.}$$

The coordinates are

$$x = -\int \tanh\theta \sin\theta \mathrm{R} d\theta,$$

$$y = -\int \tanh\theta \cos\theta \mathrm{R} d\theta,$$

$$z = -\int \mathrm{sech}\theta \mathrm{R} d\theta,$$

where R is an arbitrary function of θ.

When a case is worked out with ω as the variable, T will occur in the final integrals. Hence if R or T can be expressed as a function of $\frac{\mathrm{T}}{\mathrm{R}}$, as in the classes of curves given earlier, we shall have from any given solution of the equation an example of each of such classes.