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Inverse subsemigroups of Rees matrix semigroups

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According to the Rees Theorem, every completely 0-simple semigroup can be represented by a Rees matrix semigroup over a group with zero. A characterization of all subsemigroups of the latter is given in terms of the structure group, structure sets, and two mappings. Next all congruences on such subsemigroups are described, along with conditions for comparability. Finally, an algorithm for computing the number of nonisomorphic inverse subsemigroups is constructed.

1. Introduction and summary

In [10] Tamura and Chrislock posed the problem of determining the structure of all subsemigroups of a completely 0-simple semigroup S. They resolved it for 0-simple subsemigroups of S when the structure group of S is finite. The purpose of this paper is to describe the structure of all inverse subsemigroups of S. This is achieved by means of the Rees Theorem, which asserts that a semigroup is completely 0-simple if and only if it is isomorphic to a regular Rees matrix semigroup $M^0(I, G, M; P)$ (see [9], or Theorem 3.5 of [1]). Note that we are writing nonzero elements of S as triples (i, a, μ), with multiplication defined by

$$(i, a, \mu)(j, b, \nu) = \begin{cases} (i, ap_{\mu j}b, \nu) & \text{if } p_{\mu j} \neq 0 , \\ 0 & \text{otherwise.} \end{cases}$$

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Our main results are contained in Theorems 3.2 and 3.9. The former indicates that inverse subsemigroups without zero correspond to subgroups, while the latter describes those with zero. To carry out this description we introduce the concept of an inverse block subsemigroup and show that, in a sense, these semigroups form a basis for the structure of an arbitrary inverse subsemigroup. The term "block" has been adopted here because of its similarity to the decomposition of a matrix into block submatrices. In fact our decomposition is reminiscent of that described by Hall in Theorem 6 of [3].

Section 2 lays the groundwork for our main results. The structure of H-classes of an inverse subsemigroup T of S is given in terms of certain elements and subsets of G. Proposition 2.9 provides a useful characterization for specific H-classes. Section 3 contains the basic results of this paper. We give two structure theorems according to whether T has a zero, and describe methods for constructing T in both cases.

The remaining two sections deal with some properties of the subsemigroups in the title. In Section 4, making use of results due to Ljapin [5] and Preston [6], we characterize their congruences and give conditions for comparability. The last section develops an algorithm for computing the number of nonisomorphic inverse subsemigroups. It depends on the number-theoretic concept of a partition, and we apply it to rectangular 0-bands.

All undefined concepts and symbols can be found in [1, 2]. However, for Rees matrix semigroups we will use the notation of Petrich (see §V.3 of [7]), which varies somewhat from that used in [1, 2].

2. H-classes

Let $S = M^0(I, G, M; P)$ be an arbitrary Rees matrix semigroup. Since S is regular the set $V(i, a, \mu)$ of all inverses of (i, a, μ) is nonempty for each $(i, a, \mu) \in S$. The proofs of the next two results are evident.

LEMMA 2.1.

$$V(i, a, \mu) = \left\{(j, x, \lambda) \in S \mid j \in I, p_{\mu j} \neq 0, \lambda \in M, \right.$$
$$\left. p_{\lambda i} \neq 0, x = p_{\mu j}^{-1} a^{-1} p_{\lambda i}^{-1}\right\} .$$

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COROLLARY 2.2. *An element* $(i, a, \mu) \in S$ *has a unique inverse if and only if* $p_{\lambda i} \neq 0$ *for exactly one* $\lambda \in M$ *and* $p_{\mu j} \neq 0$ *for exactly one* $j \in I$. *In such a case,*

$$(i, a, \mu)^{-1} = \left(j, p_{\mu j}^{-1}a^{-1}p_{\lambda i}^{-1}, \lambda\right) .$$

Let T be an inverse subsemigroup of S. For $i \in I$, $\mu \in M$, define

$$H_{i\mu} = \{(i, a, \mu) \in S \mid a \in G\} ,$$

$$T_{i\mu} = \{(i, a, \mu) \in T \mid a \in G\} .$$

Of course the $H_{i\mu}$ are the H-classes of S and $T_{i\mu} = T \cap H_{i\mu}$. It is well known that the H-relation on a regular subsemigroup of any S is the restriction of the H-relation on S (see, for example, Lemma 4.4.1 of [3]). As a special case we have the following fundamental result.

LEMMA 2.3. *The* $T_{i\mu}$ *are the* H-*classes of an inverse subsemigroup* T *of* S.

For T an inverse subsemigroup of S put

$$I' = \{i \in I \mid T_{i\lambda} \neq \emptyset \text{ for some } \lambda \in M\} ,$$

$$M' = \{\mu \in M \mid T_{j\mu} \neq \emptyset \text{ for some } j \in I\} .$$

For each $i \in I'$ there exist $a \in G$ and $\mu \in M'$ such that $(i, a, \mu) \in T$. Since (i, a, μ) has a unique inverse in T it follows from Corollary 2.2 that $p_{\lambda i} \neq 0$ for precisely one $\lambda \in M$. This remark enables us to define the function below.

LEMMA 2.4. *For each* $i \in I'$, *define*

(1) $$\psi i = \lambda \text{ if and only if } p_{\lambda i} \neq 0 .$$

Then ψ *is a one-to-one function of* I' *onto* M'.

Proof. If $i \in I'$ and $\psi i = \lambda$ then $p_{\lambda i} \neq 0$. We will show that $\lambda \in M'$. Since $(i, a, \mu) \in T$ for some $a \in G$, $\mu \in M$, Corollary 2.2 and the fact that $p_{\lambda i} \neq 0$ yield that $(i, a, \mu)^{-1} = (\;,\; , \lambda) \in T$. By

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definition this means that $\lambda \in M'$.

Next let $k, l \in I'$ with $\psi k = \psi l = \mu$. Then $p_{\mu k} \neq 0$ and $p_{\mu l} \neq 0$. Further, $\mu \in M'$ implies that $(i, a, \mu) \in T$ for some $i \in I'$, $a \in G$. Since $i \in I$ and P is regular it follows that $p_{\lambda i} \neq 0$ for some $\lambda \in M$. According to Lemma 2.1 this means that both $\left(k, p_{\mu k}^{-1}a^{-1}p_{\lambda i}^{-1}, \lambda\right)$ and $\left(l, p_{\mu l}^{-1}a^{-1}p_{\lambda i}^{-1}, \lambda\right)$ are inverses of (i, a, μ) in T , which of course implies that $k = l$.

Finally, if $\mu \in M'$ then $(i, a, \mu) \in T$ for some $i \in I$, $a \in G$. It follows from Corollary 2.2 that $(i, a, \mu)^{-1} = (j, b, \lambda)$ where $p_{\mu j} \neq 0$, $p_{\lambda i} \neq 0$, and $b = p_{\mu j}^{-1}a^{-1}p_{\lambda i}^{-1}$. But $(j, b, \lambda) \in T$, which implies that $j \in I'$. Since also $p_{\mu j} \neq 0$ it follows that $\psi j = \mu$. Thus ψ is onto and consequently is a bijection of I' onto M' .

As a result of this lemma we can identify M' with I' . Thus we will write p_{ij} for $p_{(\psi i)j}$ and T_{ij} for $T_{i(\psi j)}$. Set $p_i = p_{ii}$. Then (1) implies that $p_i \neq 0$ for all $i \in I'$ and $p_{ij} = 0$ for $i \neq j \in I'$. Moreover Corollary 2.2 becomes

$$(2) \qquad (i, a, j)^{-1} = \left(j, p_j^{-1}a^{-1}p_i^{-1}, i\right) \quad (i, j \in I') .$$

Note that this element is the unique inverse of (i, a, j) in T ; of course (i, a, j) may have other inverses in S .

The next result is an analogue of Corollary 2.52 a) of [1].

PROPOSITION 2.5. *Let T be an inverse subsemigroup of $M^0(I, G, M; P)$ and $i, j, k, l \in I'$. Then the following assertions hold:*

(i) T_{ii} *is a maximal subgroup of* T *,*

(ii) $T_{ij}^2 = 0$ *if* $i \neq j$ *;*

(iii) the product $T_{ij}T_{kl}$ *is equal to* T_{il} *if* $j = k$ *and to* 0 *if* $j \neq k$ *.*

Proof. Since $i \in I'$ there exist $j \in I'$ and $a \in G$ such that

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$(i, a, j) \in T$. Using (2) we obtain

$$(i, a, j)(i, a, j)^{-1} = \left(i, p_i^{-1}, i\right) \in T .$$

Thus T_{ii} is an H-class containing an idempotent, so *(i)* follows from Exercise 1, §2.3, [*1*]. Both *(ii)* and *(iii)* hold since the analogous properties hold for the H_{ij} .

By making use of Lemma 2.3, it can be seen that our next result is a particular case of Green's Theorem. Its proof is straightforward.

PROPOSITION 2.6. *Let* T *be an inverse subsemigroup of* $M^0(I, G, M; P)$ *and* $(i, a, j), (i, b, k), (l, c, k) \in T$. *Then the mapping*

$$(i, x, j) \to \left(l, cb^{-1}p_i^{-1}, i\right)(i, x, j)\left(j, p_j^{-1}a^{-1}b, k\right)$$

is a bijection of T_{ij} *onto* T_{lk} .

Define

$$C_{ij} = \{a \in G \mid (i, a, j) \in T\} \quad (i, j \in I') .$$

We can then write

$$T_{ij} = \left(i, C_{ij}, j\right) ,$$

where C_{ij} is a subset of G which, in general, is not a subgroup. However we can characterize the C_{ii} in a useful way.

LEMMA 2.7. *If* $x, y, z \in C_{ii}$ *then* $xy^{-1}z \in C_{ii}$.

Proof. This follows from Proposition 2.5 and the fact that

$$(i, x, i)(i, y, i)^{-1}(i, z, i) = \left(i, xy^{-1}z, i\right) \in T .$$

LEMMA 2.8. *A nonempty subset* C *of a group* G *is a right coset of* G *if and only if* $xy^{-1}z \in C$ *for all* $x, y, z \in C$.

This last result is a restatement of Exercises 4.1.22-23 of [*6*]. These two lemmas are fundamental in the next proof.

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PROPOSITION 2.9. *For each* $i \in I'$, C_{ii} *is a right coset of* G *and* $C_{ii}p_i$ *is a subgroup of* G.

Proof. Let $H = C_{ii}p_i$. Then $e \in H$ since $p_i^{-1} \in C_{ii}$. If $x = ap_i$, $y = bp_i \in H$ for $a, b \in C_{ii}$ then it follows from Lemma 2.7 that $abp_i^{-1} \in C_{ii}$. Thus

$$xy^{-1} = ab^{-1} = \left(ab^{-1}p_i^{-1}\right)p_i \in H ,$$

which implies that H is a subgroup of G.

PROPOSITION 2.10. *For every pair of elements* $i, j \in I'$,

$$C_{ij}^{-1} = p_jC_{ji}p_i .$$

Proof. If $a \in C_{ij}^{-1}$ then $a^{-1} \in C_{ij}$, so $\left(i, a^{-1}, j\right) \in T_{ij}$. Using (2) and the fact that T is an inverse semigroup, we obtain $\left(i, a^{-1}, j\right)^{-1} = \left(j, p_j^{-1}ap_i^{-1}, i\right) \in T_{ji}$. This means that $p_j^{-1}ap_i^{-1} \in C_{ji}$ so that $a \in p_jC_{ji}p_i$. Thus $C_{ij}^{-1} \subseteq p_jC_{ji}p_i$. The opposite inclusion follows by retracing the above steps, so the given equality holds.

3. Structure theorems

Recall that a primitive inverse semigroup is an inverse semigroup in which every nonzero idempotent is primitive. Every Brandt semigroup is obviously a primitive inverse semigroup. To express the converse relationship we need the next definition. It was first introduced by Ljapin [5], who used the term "mutually annihilating sum". Another term frequently used is "orthogonal sum" [4].

DEFINITION. A semigroup S with zero is a 0-*direct union* of semigroups S_α $(\alpha \in A)$ if $S = \bigcup_{\alpha\in A} S_\alpha$, $S_\alpha \cap S_\beta = 0$ and $S_\alpha S_\beta = 0$ for all $\alpha, \beta \in A$, $\alpha \neq \beta$.

The next result is due to Venkatesan [11], *cf.* also Corollaire 5.17 of [4].

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LEMMA 3.1. *A semigroup is a primitive inverse semigroup if and only if it is a* 0-*direct union of Brandt semigroups.*

For the remainder of this section let $S = M^0(I, G, M; P)$. Suppose that T is an inverse subsemigroup of S without zero. If $T_{ij} \neq \emptyset$ then $(i, a, j) \in T$ for some $a \in G$. But $(i, a, j)^2 = 0$ if $i \neq j$, so that $i = j$. Now if $T_{ii} \neq \emptyset$ and $T_{jj} \neq \emptyset$ then Proposition 2.5 implies that $T_{ii}T_{jj} = 0$ if $i \neq j$, whence we conclude again that $i = j$. Thus $T = T_{ii}$ for some $i \in I$, which according to Proposition 2.5 means that T is a subgroup of S . We have proved

THEOREM 3.2. *A subsemigroup T of a completely* 0-*simple semigroup S is an inverse semigroup without zero if and only if T is a nonzero subgroup.*

COROLLARY 3.3. *A subsemigroup of a completely simple semigroup is an inverse subsemigroup if and only if it is a subgroup.*

We can now construct all inverse subsemigroups T of S without zero by combining Proposition 2.9 and Theorem 3.2. One merely takes a subgroup H of G and elements $i \in I$, $\mu \in M$ where $p_{\mu i} \neq 0$, then forms

$$T = \left\{\left(i, hp_{\mu i}^{-1}, \mu\right) \in S \mid h \in H\right\} .$$

For the remainder of this section we will let T be an inverse subsemigroup of S which contains 0 but is not a subgroup of S . We shall call such a subsemigroup *nontrivial*. Since all of the idempotents of S are primitive, naturally those of T are also, so T is a primitive inverse semigroup. With Lemma 3.1 in mind, we introduce the next basic concept.

DEFINITION. Let J and L be nonempty subsets of I and M , respectively, with $|J| = |L|$. We say that T is an *inverse block subsemigroup* of S corresponding to J and L if

$$T_{jl} \neq \emptyset \quad \text{for all} \quad j \in J,\ l \in L ,$$

$$T_{jl} = \emptyset \quad \text{if either} \quad j \notin J \quad \text{or} \quad l \notin L .$$

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To ensure that the function ψ defined below exists, we will assume that $|I| \leq |M|$. Similar arguments will apply in the case $|M| \leq |I|$. Let J be a nonempty subset of I, H a subgroup of G,

$$u : J \to G \text{ any function,} \qquad u : j \to u_j ,$$

$$\psi : J \to M \text{ a one-to-one function,} \quad \psi j = \hat{j}$$

satisfying

$$p_{\mu j} \neq 0 \text{ if and only if } \mu = \hat{j} .$$

Take $p_j = p_{\hat{j}j}$ and define

$$[J, H, u, \psi] = \left\{(i, a, \hat{j}) \in S \mid i, j \in J, a \in p_i^{-1}u_i^{-1}Hu_j\right\} \cup \{0\} .$$

PROPOSITION 3.4. $[J, H, u, \psi]$ *is a nontrivial inverse block subsemigroup of* S *corresponding to* J *and* $\hat{J}$.

Proof. Let $T = [J, H, u, \psi]$. For each $j \in J$, $\left(j, p_j^{-1}, \hat{j}\right) \in T$ since $e \in H$, so that T is nonzero. If $(i, x, \hat{j})$, $(j, y, \hat{k}) \in T$ then

$$x = p_i^{-1}u_i^{-1}au_j , \quad y = p_j^{-1}u_j^{-1}bu_k \tag{3}$$

for some $a, b \in H$. Thus $(i, x, \hat{j})(j, y, \hat{k}) = \left(i, p_i^{-1}u_i^{-1}abu_k, \hat{k}\right) \in T$. Of course $(i, x, \hat{j})(l, y, \hat{k}) = 0$ if $j \neq l$, so that T is a subsemigroup of S.

For $(i, x, \hat{j}) \in T$ with x as in (3), direct computation yields that $\left(j, p_j^{-1}u_j^{-1}a^{-1}u_i, \hat{i}\right)$ is an inverse of $(i, x, \hat{j})$. However, if $(i, x, \hat{j})(k, y, \hat{l})(i, x, \hat{j}) = (i, x, \hat{j})$ then $j = k$, $l = i$, and $y = p_j^{-1}x^{-1}p_i^{-1} = p_j^{-1}u_j^{-1}a^{-1}u_i$. Thus each element of T has a unique inverse.

If $T_{kl} \neq \emptyset$ for $k \in I$, $l \in M$ then $(k, x, l) \in T$ for some $x \in G$. Thus by definition $k \in J$ and $l = \hat{j}$ for some $j \in J$. On the other hand, if $k \in J$ and $l \in \hat{J}$, $l = \hat{j}$, then $\left(k, p_k^{-1}u_k^{-1}u_j, \hat{j}\right) \in T$. All the parts have now been put together, so the proposition holds.

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THEOREM 3.5. *A subset* T *of* $S = M^0(I, G, M; P)$ *is a nontrivial inverse block subsemigroup if and only if it is of the form* $T = [J, H, u, \psi]$.

Proof. In view of the proposition above, it suffices to show that a nontrivial inverse block subsemigroup T corresponding to $J \subseteq I$ and $L \subseteq M$ is of the desired form. First, the function ψ defined in Lemma 2.4 supplies us with a bijection of I' onto M' satisfying the necessary condition. Since T corresponds to J and L we obtain immediately that $J = I'$ and $L = M' = \hat{J}$.

Fix an element $l \in J$. By setting $C = C_{l\hat{l}}$ and $p = p_{\hat{l}l}$ we can conclude from Proposition 2.9 that $T_{l\hat{l}} = (l, C, \hat{l})$, where

$$H = Cp \tag{4}$$

is a subgroup of G . For each $i \in J$, $T_{l\hat{i}} \neq \emptyset$ since T corresponds to J and $\hat{J}$. Let $(l, u_i, \hat{i})$ be an arbitrary but fixed element of $T_{l\hat{i}}$. Then define $u : J \to G$ by $u : i \to u_i$ for $i \neq l$ and $u(l) = e$, the identity of G . Take $p_i = p_{\hat{i}i}$. Since $u_i \in C_{li}$ it follows from Proposition 2.10 that $pu_ip_i \in C_{i\hat{l}}^{-1}$, which implies that $\left(i, p_i^{-1}u_i^{-1}p^{-1}, \hat{l}\right) \in T_{i\hat{l}} \subseteq T$. Hence for any $j \in J$,

$$\left(i, p_i^{-1}u_i^{-1}p^{-1}, \hat{l}\right)(l, u_j, \hat{j}) = \left(i, p_i^{-1}u_i^{-1}u_j, \hat{j}\right) \in T .$$

Clearly, $(l, p^{-1}, \hat{l})R(l, u_j, \hat{j})$ and $(l, u_j, \hat{j})L\left(i, p_i^{-1}u_i^{-1}u_j, \hat{j}\right)$. Thus we conclude from Lemma 2.6 that the mapping

$$(l, x, \hat{l}) \to \left(i, p_i^{-1}u_i^{-1}p^{-1}, \hat{l}\right)(l, x, \hat{l})(l, u_j, \hat{j}) \tag{5}$$

is a bijection of $T_{l\hat{l}}$ onto $T_{i\hat{j}}$. Expanding the right-hand side of (5) yields

$$C_{i\hat{j}} = \left\{p_i^{-1}u_i^{-1}xpu_j \in G \mid x \in C\right\} . \tag{6}$$

Now (4) and (6) imply that

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$$T_{i\hat{j}} = \left\{(i,\ a,\ \hat{j}) \in S \mid a \in p_i^{-1}u_i^{-1}Hu_j\right\} .$$

However, Lemma 2.3 indicates that the $T_{i\hat{j}}$ are the H-classes of T, and every semigroup is the union of its H-classes. Since the i and j were chosen arbitrarily in J it follows that

$$T = \left\{(i,\ a,\ \hat{j}) \in S \mid i,\ j \in J,\ a \in p_i^{-1}u_i^{-1}Hu_j\right\} .$$

Consequently $T = [J,\ H,\ u,\ \psi]$ which completes the proof of the theorem.

Let $B(I,\ G)$ denote the Brandt semigroup $M^0(I,\ G,\ I;\ \Delta)$. Recall that with our notation multiplication in $B(I,\ G)$ is defined by

$$(i,\ a,\ j)(k,\ b,\ l) = \begin{cases}(i,\ ab,\ l) & \text{if } j = k ,\\ 0 & \text{if } j \neq k .\end{cases}$$

PROPOSITION 3.6. $[J,\ H,\ u,\ \psi] \cong B(J,\ H)$.

Proof. Let $T = [J,\ H,\ u,\ \psi]$ and $B = B(J,\ H)$. Define $\chi : T \to B$ by $(i,\ a,\ j) \to \left(i,\ u_ip_iau_j^{-1},\ j\right)$ and $0 \to 0$. If $(i,\ a,\ j) \in T$ then $i,\ j \in J$ and $a \in p_i^{-1}u_i^{-1}Hu_j$, which implies that $u_ip_iau_j^{-1} \in H$. Thus χ maps T into B. The remaining details showing that χ is an isomorphism of T onto B are straightforward.

COROLLARY 3.7. *A subsemigroup of a completely* 0*-simple semigroup is a nontrivial inverse block subsemigroup if and only if it is a Brandt subsemigroup.*

COROLLARY 3.8. $[J,\ H,\ u,\ \psi] \cong [L,\ K,\ v,\ \eta]$ *if and only if* $|J| = |L|$ *and* $H \cong K$.

THEOREM 3.9. *Let* $S = M^0(I,\ G,\ M;\ P)$, *let* $\{J_\alpha\}_{\alpha \in A}$ *form a partition of some* $J \subseteq I$, *and let* H_α *be a subgroup of* G *for each* $\alpha \in A$. *Take*

$u : J \to G$ *any function,*

$\psi : J \to M$ *a one-to-one function satisfying* $p_{\mu j} \neq 0$ *if and only if* $\mu = \psi j$,

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and form

$$B_\alpha = [J_\alpha, H_\alpha, u_\alpha, \psi_\alpha] ,$$

where u_α *and* ψ_α *are the restrictions of* u *and* ψ *, respectively, to* J *. Then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is a nontrivial inverse subsemigroup of S *.*

Conversely every nontrivial inverse subsemigroup of S *is of this form.*

Proof. Let $T = \bigcup_{\alpha \in A} B_\alpha$ be defined as in the statement of the first part of the theorem. It follows from Proposition 3.4 that all of the B_α are subsemigroups of S, so that the product of elements in the same B_α again falls into B_α. Since the J_α are mutually disjoint the product of elements in different B_α is zero. Thus T is a subsemigroup of S. But the B_α, being Brandt semigroups, are inverse semigroups, and hence T is an inverse subsemigroup of S which is obviously nontrivial.

Conversely, if T is a nontrivial inverse subsemigroup of S then T is a 0-direct union of Brandt subsemigroups B_α of S. According to Corollary 3.7 each B_α is an inverse block subsemigroup, so by Theorem 3.5 each B_α is of the desired form. Since T is a 0-direct union of the B_α it follows immediately that the J_α are mutually disjoint, which completes the proof of the theorem.

Theorem 3.9 yields the complete structure of inverse subsemigroups T of a Rees matrix semigroup $M^0(I, G, M; P)$, where $|I| \le |M|$. The structure of T is determined by five independent parameters:

(1) a subset J of I;

(2) a partition of J into subsets $\{J_\alpha\}_{\alpha \in A}$;

(3) a function of J into G;

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(4) a one-to-one function of J into M, and

(5) subgroups H_α of G for each set J_α.

The problem of characterizing inverse subsemigroups of a Rees matrix semigroup is thus completely solved as far as the theory of semigroups is concerned.

4. Congruences

In view of Theorem 3.9, to describe congruences on inverse subsemigroups of a Rees matrix semigroup it suffices to consider congruences on a 0-direct union of Brandt semigroups. We rely heavily on the notions and results due to Ljapin [5].

DEFINITION. A nonempty subset A of a semigroup S is called a *normal complex* of S if $xAy \cap A \neq \emptyset$ implies $xAy \subseteq A$ for all $x, y \in S^1$; A is an *anti-ideal* of S if

$$AS \cap A = SA \cap A = SAS \cap A = \emptyset .$$

For any semigroup S let ω_S denote the universal congruence on S, $C(S)$ the set of all congruences on S, and $C'(S)$ those different from ω_S.

PROPOSITION 4.1. *A nonempty subset A of a semigroup S is a normal complex of S if and only if A is a σ-class for some $\sigma \in C(S)$.*

PROPOSITION 4.2. *Let S be a 0-direct union of semigroups S_α, $\alpha \in Y$. A subset A of S is a normal complex if and only if A is either an ideal of S, an anti-ideal of S, or a normal complex of some S_α.*

Let T be a 0-direct union of 0-simple semigroups S_α, $\alpha \in Y$, and for each $X \subseteq Y$ let $J(X)$ denote the 0-direct union of S_α for $\alpha \in X$. Clearly $J(X)$ is an ideal of T. We will prove the converse of this statement.

LEMMA 4.3. *J is an ideal of T if and only if $J = J(X)$ for some nonempty subset X of Y.*

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Proof. It is easy to see that, in general, ideals of a 0-direct union of S_α are themselves 0-direct unions of ideals of the S_α (*cf.* [5]). Thus if J is an ideal of T, the 0-simplicity of the S_α implies that $J = J(X)$, where X consists of all α for which $J \cap S_\alpha \neq \emptyset$.

DEFINITION. A semigroup is called *antisimple* if it has no proper anti-ideals.

LEMMA 4.4. *If a semigroup S has the property that each element has a left or right identity, then S is antisimple.*

Proof. If $A \subseteq S$, $a \in A$, and $ax = a$, then $ax \in AS \cap A$.

COROLLARY 4.5. *Every 0-direct union whose components are either completely 0-simple or inverse semigroups is antisimple.*

PROPOSITION 4.6 (Preston [8]). *Let $B = B(I, G)$. For each normal subgroup N of G define σ_N by*

$$(i, a, j)\sigma_N(k, b, l) \text{ if and only if } i = k,\ j = l,\ a \equiv b \ (\mathrm{mod}\ N)\ .$$

Then $\sigma \in C'(B)$ if and only if $\sigma = \sigma_N$ for some normal subgroup N of G. Moreover, $\sigma_K \subseteq \sigma_N$ if and only if $K \subseteq N$.

In what follows let T be a 0-direct union of Brandt semigroups $B_\alpha = B(I_\alpha, G_\alpha)$, $\alpha \in Y$, some indexing set. Let $X \subseteq Y$ and for each $\alpha \in Y\backslash X$ let N_α be a normal subgroup of G_α and σ_α represent the congruence σ_{N_α} defined as above.

Define on T the binary relation

$$x\rho y \leftrightarrow \begin{cases} x, y \in J(X)\ , \text{ or} \\ x\sigma_{N_\alpha} y \quad \text{for some} \quad \alpha \in Y\backslash X\ . \end{cases}$$

LEMMA 4.7. $\rho \in C(T)$.

Proof. It is obvious that ρ is an equivalence relation on T. Let $x\rho y$, $x \in B_\alpha$, and $z \in T$. If $\alpha \in X$ then $B_\alpha \subseteq J(X)$ and hence by

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definition $x, y \in J(X)$. Thus $xz, yz \in J(X)$ since $J(X)$ is an ideal of T, and so $(xz)\rho(yz)$. If $\alpha \notin X$ then it follows that $x, y \in B_\alpha$ and $x\sigma_{N_\alpha} y$. If $z \notin B$ then $xz = 0 = yz$ while if $z \in B_\alpha$ then $(xz)\sigma_{N_\alpha}(yz)$ since $\sigma_{N_\alpha} \in C(B_\alpha)$. Thus in all cases $(xz)\rho(yz)$ which implies that ρ is right compatible. Similarly ρ is left compatible, and hence $\rho \in C(T)$.

For brevity we will write $\rho = [X, N_\alpha]$. Our aim is to show that every congruence on T is of this form.

LEMMA 4.8. *If* $\sigma \in C(T)$ *then each* σ*-class is either a normal complex of some* B_α *or equals* $J(X)$ *for some* $X \subseteq Y$.

LEMMA 4.9. *For* S *any semigroup and* $\sigma \in C(S)$ *there is at most one* σ*-class which is an ideal of* S.

THEOREM 4.10. *If* T *is a* 0*-direct union of Brandt semigroups* $B_\alpha = B(I_\alpha, G_\alpha)$, $\alpha \in Y$, *then a relation* σ *on* T *is a congruence if and only if* $\sigma = [X, N_\alpha]$ *for some* $X \subseteq Y$ *and some family* $\{N_\alpha \mid \alpha \in Y\backslash X\}$ *of normal subgroups of* G.

Proof. In view of Lemma 4.7 it suffices to show that every $\sigma \in C(T)$ is of the desired form. It follows from Lemma 4.8 that each σ-class is either a normal complex of some B_α or an ideal of T, while Lemma 4.9 indicates that there can be at most one such ideal. If no ideal σ-class exists, set $X = \emptyset$. On the other hand, if $J(X)$ is an ideal σ-class then $\sigma_\alpha = \sigma|_{B_\alpha} \in C(B_\alpha)$ for all $\alpha \in Y$. For $\alpha \in Y\backslash X$, no element of B_α is σ-related to any element of $J(X)$ and if $a \neq 0 \in B_\alpha$, $b \in B_\beta$, $\alpha \neq \beta$ then $a \notin b(\sigma)$. Furthermore, $\sigma_\alpha \neq \omega_{B_\alpha}$ since 0 forms a σ_α-class. Thus from Proposition 4.6, $\sigma_\alpha = \sigma_{N_\alpha}$ for some normal subgroup N_α of G_α. It is now easy to verify that $\sigma = [X, N_\alpha]$.

Next we will prove a result on the ordering of congruences described by the above theorem. Note first that if $U, V \subseteq Y$ then clearly

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$J(U) \subseteq J(V)$ if and only if $U \subseteq V$. For each $\rho \in C(T)$ write $\rho_\alpha = \rho|_{B_\alpha}$. It is immediate from our proof of Theorem 4.10 that if $\rho = [X, N_\alpha]$ then $X = \{\alpha \in Y \mid \rho_\alpha = \omega_\alpha\}$.

THEOREM 4.11. *$[U, K_\alpha] \subseteq [X, N_\alpha]$ if and only if $U \subseteq X$ and $K_\alpha \subseteq N_\alpha$ for all $\alpha \in Y\backslash X$.*

Proof. Let $\mu = [U, K_\alpha]$, $\rho = [X, N_\alpha]$ and assume that $\mu \subseteq \rho$. If $\alpha \in U$ then $\mu_\alpha = \omega_\alpha$, which implies that $\rho_\alpha = \omega_\alpha$. But then $\alpha \in X$, so $U \subseteq X$. If $\alpha \notin X$ then $\rho_\alpha = \sigma_{N_\alpha}$ and $\mu_\alpha = \sigma_{K_\alpha}$. That $K_\alpha \subseteq N_\alpha$ follows directly from Proposition 4.6.

Conversely, if $a\mu b$ then either $a, b \in J(U)$ or $a\sigma_{K_\alpha} b$ for some $\alpha \in Y\backslash U$. In the first case $a, b \in J(X)$, while in the second either $a\omega_\alpha b$ or $a\sigma_{N_\alpha} b$. But all three situations yield $a\rho b$, whence $\mu \subseteq \rho$.

5. Number

Throughout this section S will denote the Rees matrix semigroup $M^0(I, G, M; P)$ where I, G and M are finite, and unless otherwise stated, $|I| \leq |M|$. The next result is basic to our consideration.

LEMMA 5.1. *Let T and T' be inverse subsemigroups of S,*

$$T = \bigcup_{i=1}^{r} [J_i, H_i, u, \psi], \quad T' = \bigcup_{i=1}^{s} [J_i', H_i', u', \psi'],$$

for some nonempty subsets J_i, J_i' of I, subgroups H_i, H_i' of G, mappings $u : \bigcup_{i=1}^{r} J_i \to G$, $u' : \bigcup_{i=1}^{s} J_i' \to G$, and one-to-one mappings $\psi : \bigcup_{i=1}^{r} J_i \to M$, $\psi' : \bigcup_{i=1}^{s} J_i' \to M$. Then $T \cong T'$ if and only if $r = s$, $|J_i| = |J_i'|$, and $H_i \cong H_i'$ for $i = 1, 2, \ldots, r$ (after a suitable rearrangement of indices).

We will show that the number of nonisomorphic subsemigroups of S is

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intimately connected to the basic number-theoretic notion of a partition. Recall that a partition of a natural number n is a decomposition of n into the sum of any number of positive integral parts. We will denote the number of partitions of n by $p(n)$.

Let there be given a partition of a number n into r parts

$$(*) \qquad n = n_1 + n_2 + \dots + n_r .$$

DEFINITION 3.18. We will say that an inverse subsemigroup T of S is *induced by* $(*)$ if $T = \bigcup_{i=1}^{r} [J_i, H_i, u, \psi]$ where $|J_i| = n_i$, $i = 1, 2, \dots, r$. The subgroup H_i is said to *correspond to the part* n_i.

Clearly every inverse subsemigroup of S is induced by a partition of n for some $n \le |I|$. It can be seen immediately from Lemma 5.1 that distinct partitions induce nonisomorphic inverse subsemigroups. We will first determine the number of nonisomorphic inverse subsemigroups of S which are induced by a given partition. The total number of such subsemigroups of S is then obtained by summing the obtained numbers over all partitions of all numbers n, $1 \le n \le |I|$. We will state next the main result of this section, the proof of which will be supplied by the two lemmas following it.

THEOREM 5.2. *Let* $S = M^0(I, G, M; P)$, $m = \min\{|I|, |M|\}$, *and let* l *be the number of nonisomorphic subgroups of* G. *For each number* n, $1 \le n \le m$, *consider a partition of* n *into parts* $n_1, n_2, \dots, n_s$:

$$n = r_1 n_1 + r_2 n_2 + \dots + r_s n_s .$$

The number of nonisomorphic inverse subsemigroups of S *is given by*

$$\sum_P \prod_{k=1}^{s} \binom{l+r_k-1}{r_k}$$

where P *runs over all partitions of* n *for* $n = 1, 2, \dots, m$.

We will first consider partitions of n into r equal parts, say

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$$(**) \qquad n = \underbrace{k + k + \ldots + k}_{r} .$$

It follows from Lemma 5.1 that two inverse subsemigroups T and T' of S induced by the partition $(**)$ are isomorphic if and only if the r structure groups composing T are isomorphic in pairs to the r structure groups of T'. Thus the number of nonisomorphic inverse subsemigroups of S induced by $(**)$ is equal to the number of sets consisting of r subgroups of G without regard to order, where the subgroups are chosen from the set of l nonisomorphic subgroups of G with repetition allowed. But the number of such sets can be obtained from the classical occupancy problem, and we state this as our next lemma.

LEMMA 5.3. *The number of nonisomorphic inverse subsemigroups of* S *induced by* $(**)$ *is equal to* $\binom{l+r-1}{r}$.

This result tells us the number of nonisomorphic inverse subsemigroups of S having r blocks whose structure sets all have the same cardinality. We next consider such subsemigroups of S having r_i blocks with structure sets of cardinality n_i, for $i = 1, 2, \ldots, s$. To this end, let

$$(***) \qquad n = r_1n_1 + r_2n_2 + \ldots + r_sn_s , \quad n_1 > n_2 > \ldots > n_s .$$

LEMMA 5.4. *The number of nonisomorphic inverse subsemigroups of* S *induced by* $(***)$ *is equal to*

$$\prod_{k=1}^{s} \binom{l+r_k-1}{r_k} .$$

Proof. Two inverse subsemigroups T and T' of S induced by $(***)$ are isomorphic if and only if the r_i groups in T corresponding to n_i are isomorphic in pairs to the r_i groups in T' corresponding to n_i for each $i = 1, 2, \ldots, s$. By Lemma 5.3 there are $\binom{l-r_i-1}{r_i}$ nonisomorphic inverse subsemigroups induced by the part n_i. Clearly every arrangement of subgroups of G corresponding to n_i is independent

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of every arrangement of subgroups corresponding to n_j , $i \neq j$. Thus the total number of such subsemigroups is the product of each of these expressions. This completes the proof of the lemma.

EXAMPLE. By a *rectangular* 0-*band* we mean a Rees matrix semigroup over the trivial group. Let $R = M^0(I, e, M; P)$ and $q = \min\{|I|, |M|\}$. For each $n \leq q$ any two inverse subsemigroups of R induced by the same partition of n are isomorphic, while distinct partitions induce nonisomorphic inverse subsemigroups. Thus there is a one-to-one correspondence between such subsemigroups and partitions of numbers $n \leq q$. Consequently the number of nonisomorphic inverse subsemigroups of R is equal to

$$\sum_{n=1}^{q} p(n) .$$

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