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SUBALGEBRAS WHICH APPEAR IN QUANTUM IWASAWA DECOMPOSITIONS

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ABSTRACT. Let g be a semisimple Lie algebra. Quantum analogs of the enveloping algebra of the fixed Lie subalgebra are introduced for involutions corresponding to the negative of a diagram automorphism. These subalgebras of the quantized enveloping algebra specialize to their classical counterparts. They are used to form an Iwasawa type decompostition and begin a study of quantum Harish-Chandra modules.

Let g be a semisimple Lie algebra and θ an involution of g. The Lie subalgebra g^θ of g, consisting of those elements of g fixed by θ, plays an important role in representation theory. The classification of involutions and their invariant subalgebras, due to E. Cartan, led to a complete list of the real forms of g which, in turn, is used to describe symmetric spaces. There is also a well developed theory concerning the structure of $U(g)$-modules under the module action restricted to g^θ. When a $U(g)$-module can be written as a direct sum of finite-dimensional simple g^θ-modules, it is called a Harish-Chandra module for the pair g, g^θ. The study of Harish-Chandra modules provides an algebraic approach to understanding representations of the corresponding real Lie group of g.

The purpose of this paper is to study quantum analogs of pairs g, g^θ and their Harish-Chandra modules. More precisely, let U denote the quantized enveloping algebra associated to g introduced by Drinfeld and Jimbo. We consider subalgebras B_θ of U, corresponding to an involution θ of g, which behave similarly to the enveloping algebra of g^θ.

One of the difficulties in the quantum case is picking the correct subalgebra associated to an involution. It is unclear how to use the recently proposed quantum Lie algebras and thus we cannot start with a fixed quantum Lie subalgebra as in the classical case. On the other hand, the associative invariant subalgebra fixed under an involution of U corresponding to the original involution of g is much too large. Because of these difficulties, we have limited our attention to involutions which restrict to diagram automorphisms on the set of positive roots. Our criteria for picking a subalgebra B_θ of the correct size, given an involution θ of g, is whether B_θ can be used to form an Iwasawa type decomposition of U (see Theorem 2.4). The motivation for obtaining this result is that in the classical case the Iwasawa decomposition provides important information concerning coinduced modules; these are crucial in the study of Harish-Chandra modules. We hope to investigate coinduced modules from B_θ to U in a later paper.

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The subalgebras B_θ are not themselves quantized enveloping algebras (see Remark 2.3) nor do they have an obvious Hopf algebra structure. They do, however, satisfy the right coideal condition on comultiplication. We show that for each θ, B_θ is the unique subalgebra in a large family of possible analogs of $U(g^\theta)$ which satisfies both this right coideal condition and the quantum Iwasawa decomposition (Theorem 2.5). Moreover, we obtain some intriguing results on the structure of U-modules under the action restricted to B_θ which suggests there is an interesting Harish-Chandra module theory for the pair U, B_θ (see Section 4).

M. Nazarov has kindly pointed out to me that Noumi and Sugitani have also studied quantum analogs of the pairs g, g^θ when g is classical (see the research announcement [NS].) The methods used in [NS] for constructing the subalgebras are quite different from those presented here, so it is not obvious whether these subalgebras are the same as the B_θ (see Remark 2.4.)

This paper is organized as follows. The subalgebras B_θ are introduced in Section 2. Generators and relations for these algebras are given (implicitly) in Proposition 2.3. These subalgebras are then used to form an Iwasawa type tensor product decomposition of U (Theorem 2.4.) Recall that U specializes to $U(g)$ as the defining parameter q goes to 1. In Section 3, we show that the restriction of this specialization sends B_θ to the enveloping algebra of g^θ. In Section 4, we show that the sum of all the finite-dimensional B_θ-modules sitting inside a U-module which admits a semisimple Cartan subgroup action is equal to the sum of all the finite-dimensional simple U submodules. As an application, the locally finite part of U is the maximal Harish-Chandra module contained in U for the pair U and B_θ using the adjoint action.

We do make one additional assumption that g does not contain a factor of type G_2. This assumption is necessary for Lemma 2.2 (see Remark 2.1).

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1. **Preliminaries.** Let $g = n^- \oplus h \oplus n^+$ be a semisimple Lie algebra over an algebraically closed field k of characteristic zero. Let Δ (resp. Δ^+) denote the set of roots (resp. positive roots) associated to g and let $\alpha_1, \ldots, \alpha_l$ denote the set of positive simple weights. Write $A = [a_{ij}]$ for the $l \times l$ Cartan matrix and let d_i be a list of l pairwise relatively prime integers so that $[d_i a_{ij}]$ is symmetric. Let (,) denote the Cartan inner product where $(\alpha_i, \alpha_j) = d_i a_{ij}$ for $1 \le i, j \le l$. For the purposes of this paper, we assume that g does not contain a factor of type G_2 (see Remark 2.1). In particular, if $i \neq j$, then a_{ij} is equal to $0, -1$, or -2.

Given an indeterminate q over k, we define the quantized enveloping algebra $U = U_q(g)$ of g as the algebra over $k(q)$ generated by the elements $x_1, \ldots, x_l, t_1^{\pm 1}, \ldots, t_l^{\pm 1}$, $y_1, \ldots, y_l$ subject to the following relations.

(1.1) $t_1^{\pm 1}, \ldots t_l^{\pm 1}$ generate a free abelian group T of rank l.

(1.2) $t_ix_jt_i^{-1} = q^{(\alpha_i,\alpha_j)}x_j$ and $t_iy_jt_i^{-1} = q^{-(\alpha_i,\alpha_j)}y_j$ for all $1 \le i, j \le l$.
(1.3) $x_iy_j - y_jx_i = \delta_{ij}\frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}}$ for all $1 \le i, j \le l$.
(1.4) The $x_1, \ldots, x_l$ (resp. $y_1, \ldots, y_l$) satisfy the quantized Serre relations. For example,

$$x_ix_j - x_jx_i = 0 \text{ if } a_{ij} = 0;\ x_i^2x_j - (q^{2d_i} + q^{-2d_i})x_ix_jx_i + x_jx_i^2 = 0 \text{ if } a_{ij} = -1;$$
$$x_i^3x_j + (q^{4d_i} + 1 + q^{-4d_i})(-x_i^2x_jx_i + x_ix_jx_i^2) - x_jx_i^3 = 0 \text{ if } a_{ij} = -2.$$

Similar formulas hold with x_i and x_j replaced by y_i and y_j respectively. (For a more general form of the Serre relations using quantum binomial coefficients, see [DK, 1.2.4 and 1.2.5].)

The algebra U also has a Hopf algebra structure with comultiplication Δ, augmentation map ϵ, and antipode σ. More precisely, for $1 \le i \le l$ and $a \in U$, we have the following.

$$\Delta(x_i) = x_i \otimes t_i^{-1} + t_i \otimes x_i; \quad \Delta(y_i) = y_i \otimes t_i^{-1} + t_i \otimes y_i; \quad \Delta(t_i) = t_i \otimes t_i$$
$$\epsilon(x_i) = \epsilon(y_i) = 0; \quad \epsilon(t_i) = 1$$
$$\sigma(x_i) = -q^{-2d_i}x_i; \quad \sigma(y_i) = -q^{2d_i}y_i; \quad \sigma(t_i) = t_i^{-1}.$$

In general, we will use standard Hopf algebra notation and express the sum $\Delta(a)$ as $a_{(1)} \otimes a_{(2)}$ for a typical element $a \in U$ (see for example [JL1, Section 2].) Using the Hopf algebra structure, one can define an adjoint action by $(\operatorname{ad} b)a = b_{(1)}a\sigma(b_{(2)})$ for a and b in U. On the generators, the adjoint action takes the following form.

$$(\operatorname{ad} x_i)a = x_iat_i - q^{-2d_i}t_iax_i; \quad (\operatorname{ad} y_i)a = y_iat_i - q^{2d_i}t_iay_i; \quad (\operatorname{ad} t_i)a = t_iat_i^{-1}$$

Let U^+ (resp. U^-; U^o), denote the $k(q)$ subalgebra of U generated by $x_1, \ldots x_l$ (resp. $y_1, \ldots y_l$; $t_1^{\pm 1}, \ldots, t_l^{\pm 1}$.) As a vector space, U is isomorphic to the tensor product $U^- \otimes U^o \otimes U^+$ over $k(q)$ ([R]).

An element $v \in U$ is called a weight vector of weight $\lambda \in h^*$ if $(\operatorname{ad} t_i)v = q^{(\lambda,\alpha_i)}v$ for each $1 \le i \le l$. Set $Q = \sum_{1\le i\le l} \mathbf{Z}\alpha_i$ and $Q^+ = \sum_{1\le i\le l} \mathbf{N}\alpha_i$. It is well known that the algebra U^+ (resp. U^-) is a direct sum of its finite-dimensional weight spaces where the weights that appear are exactly the elements (resp. the negative of the elements) in Q^+. Recall the standard ordering on weights in h^*: $\beta \ge \gamma$ if $\beta - \gamma \in Q^+$. For $\beta \in h^*$, denote the β weight space of an ad T submodule M of U by M_β.

2. **Involutions and Subalgebras.** Given an involution θ of g, let g^θ denote the fixed Lie subalgebra of g. In this section we look at certain involutions and (associative) subalgebras B_θ of the quantized enveloping algebras which correspond to the enveloping algebra $U(g^\theta)$. We give generators and relations for these subalgebras and use them to form an Iwasawa type decomposition of U. We further show that the B_θ are the unique algebras among a large class of potential quantum analogs of $U(g^\theta)$ to satisfy both the Iwasawa decomposition and the right coideal property on comultiplication. First, we describe the involutions that we will be working with throughout this paper.

Let θ be an automorphism of the root system of g which preserves the positive simple roots Δ. It is standard to call θ a diagram automorphism since θ uniquely determines an automorphism of the Dynkin diagram associated to g (see [Hu, Section 12.2]). Assume further that $\theta^2 = \mathrm{Id}$ where Id denotes the identity. Clearly θ induces a permutation of the indices $1, \ldots, l$ for the positive simple roots. Since the correspondence is faithful, we frequently identify θ with this permutation. Note that if g is simple, the only possibilities for θ are ([Hu, Section 12.2]):

Case 1: $\theta = \mathrm{Id}$.

Case 2: g is of type A_n and $\theta(\alpha_i) = \alpha_{n+1-i}$ for $1 \leq i \leq n$.

Case 3: g is of type E_6: $\theta(\alpha_1) = \alpha_6; \theta(\alpha_3) = \alpha_5; \theta(\alpha_2) = \alpha_2$ and $\theta(\alpha_4) = \alpha_4$.

Case 4: g is of type D_l: $\theta(\alpha_i) = \alpha_i$ for $1 \leq i \leq l-2$ and $\theta(\alpha_{l-1}) = \alpha_l$. Also when $l = 4$ we have two additional possibilities: $\theta'(\alpha_1) = \alpha_3$ and $\theta'(\alpha_i) = \alpha_i$ for $i = 2, 4$; $\theta''(\alpha_1) = \alpha_4$ and $\theta''(\alpha_i) = \alpha_i$ for $i = 2, 3$.

A diagram involution θ induces an involution, which we also call θ, on g defined by $\theta(e_i) = f_{\theta(i)}$, $\theta(f_i) = e_{\theta(i)}$, and $\theta(h_i) = -h_{\theta(i)}$ where $e_i, f_i, h_i, 1 \leq i \leq l$ are standard generators for g. We are now ready to define the quantum subalgebras corresponding to $U(g^\theta)$ which are the focus of this paper.

DEFINITION 2.1. Let θ be a diagram automorphism of Δ such that $\theta^2 = \mathrm{Id}$. For each i between 1 and l, set

$$B_i = x_i t_{\theta(i)}^{-1} + y_{\theta(i)} t_i^{-1}.$$

Let B_θ be the subalgebra of U generated by $\{B_i \mid 1 \leq i \leq l\}$ and $\{t_i t_{\theta(i)}^{-1} \mid 1 \leq i \leq l\}$.

Throughout this paper we will be proving results about B_θ that correspond to facts about the classical enveloping algebra of g^θ. Unlike the classical case, however, there is no obvious $k(q)$ algebra automorphism of U which specializes to θ and fixes the elements of B_θ. One can, though, lift θ to a k-algebra automorphism $\tilde{\theta}$ such that $\tilde{\theta}(x_i) = y_{\theta(i)}$, $\tilde{\theta}(t_i) = t_{\theta(i)}$, and $\tilde{\theta}(q) = q^{-1}$. Note that $\tilde{\theta}(B_i) = B_i$ for each i. Although $\tilde{\theta}(t_i t_{\theta(i)}^{-1}) \neq t_i t_{\theta(i)}^{-1}$ for $i \neq \theta(i)$, we do have $\tilde{\theta}(t_i t_{\theta(i)}^{-1} - t_i^{-1} t_{\theta(i)})/(q - q^{-1}) = (t_i t_{\theta(i)}^{-1} - t_i^{-1} t_{\theta(i)})/(q - q^{-1})$. Hence $\tilde{\theta}$ fixes the elements of the k subalgebra of U generated by B_i and $(t_i t_{\theta(i)}^{-1} - t_i^{-1} t_{\theta(i)})/(q - q^{-1})$, $1 \leq i \leq l$. It follows from Theorem 3.1 below that this k subalgebra of B_θ specializes to $U(g^\theta)$. Thus the fact that this large k subalgebra of B_θ consists of elements fixed by the involution $\tilde{\theta}$ can be considered a quantum analog of the fact that g^θ is the set of elements in g fixed by θ.

It would be nice to use the newly proposed quantum Lie algebras to define B_θ (see for example [S] and [DG]). One of the problems however is understanding the precise connection between the associative algebra generated by the quantum Lie algebra and the quantized enveloping algebra U. When $g = sl\,2$, the quantum Lie algebra generates the locally finite part of U (see [S]); such a result is unknown in general.

Although B_θ is not defined directly using an involution which specializes to θ, and so the choice of B_θ might seem arbitrary, it is exactly this subalgebra which is small enough to be used in a Iwasawa type decomposition of U (Theorem 2.4) and specializes to the enveloping algebra of the corresponding fixed Lie subalgebra of g (Theorem 3.1). The next lemma, which is quite computational, describes crucial relations satisfied by the B_i.

LEMMA 2.2. *Set* $A = B_i$ *and* $B = B_j$.
(2.1) If $a_{ij} = 0$, *then* $AB - BA = \delta_{i,\theta(j)}(t_i^2 t_{\theta(i)}^{-2} - t_{\theta(i)}^2 t_i^{-2}) / (q^{2d_i} - q^{-2d_i})$.
(2.2) If $a_{ij} = -1$, *then*

$$A^2B - (q^{2d_i} + q^{-2d_i})ABA + BA^2 = \delta_{i,\theta(i)}B - \delta_{i,\theta(j)}(q^{2d_i} + q^{-2d_i})A(q^{3d_i}t_i^2 t_{\theta(i)}^{-2} + q^{-3d_i}t_{\theta(i)}^2 t_i^{-2}).$$

(2.3) If $a_{ij} = -2$, *then*

$$A^3B + (q^{4d_i} + 1 + q^{-4d_i})(-A^2BA + ABA^2) - BA^3 = (q^{2d_i} + q^{-2d_i})^2(AB - BA).$$

PROOF. Statement (2.1) follows quickly from the definition of the B_i. Assume that $a_{ij} = -1$. Consider the terms on the left hand side of the identity in (2.2) of weight $2\alpha_i + \alpha_j$.

$$x_i t_{\theta(i)}^{-1} x_i t_{\theta(i)}^{-1} x_j t_{\theta(j)}^{-1} - (q^{2d_i} + q^{-2d_i}) x_i t_{\theta(i)}^{-1} x_j t_{\theta(j)}^{-1} x_i t_{\theta(i)}^{-1} + x_j t_{\theta(j)}^{-1} x_i t_{\theta(i)}^{-1} x_i t_{\theta(i)}^{-1}$$

This expression simplifies to

$$\begin{aligned} &q^{(-\alpha_{\theta(i)},\alpha_i+2\alpha_j)} x_i x_i x_j t_{\theta(i)}^{-2} t_{\theta(j)}^{-1} - (q^{2d_i} + q^{-2d_i}) q^{(-\alpha_{\theta(i)},\alpha_i+\alpha_j)+(-\alpha_{\theta(j)},\alpha_i)} x_i x_j x_i t_{\theta(i)}^{-2} t_{\theta(j)}^{-1} \\ &\qquad + q^{(-\alpha_{\theta(j)},2\alpha_i)+(-\alpha_{\theta(i)},\alpha_i)} x_j x_i x_i t_{\theta(i)}^{-2} t_{\theta(j)}^{-1}. \end{aligned} \tag{2.4}$$

Note that $(\alpha_{\theta(i)}, \alpha_j) = (\alpha_i, \alpha_{\theta(j)})$ since θ is an automorphism of Δ. Hence (2.4) equals

$$q^{-2(\alpha_{\theta(i)},\alpha_j)-(\alpha_{\theta(i)},\alpha_i)} \big(x_i x_i x_j - (q^{2d_i} + q^{-2d_i}) x_i x_j x_i + x_j x_i x_i \big) t_{\theta(i)}^{-2} t_{\theta(j)}^{-1}$$

which is just the quantum Serre relation (identity (1.4)) and so equals zero. This agrees with the sum of the terms of weight $2\alpha_i + \alpha_j$ on the right hand side of (2.2). A similar argument using the quantum Serre relations shows that the sum of the terms on both the left hand and the right hand side of identity (2.2) of weight $-\theta(2\alpha_i + \alpha_j)$ equals zero.

Now consider the terms of weight $2\alpha_i - \alpha_{\theta(j)}$ on the left hand side of the identity in (2.2).

$$x_i t_{\theta(i)}^{-1} x_i t_{\theta(i)}^{-1} y_{\theta(j)} t_j^{-1} - (q^{2d_i} + q^{-2d_i}) x_i t_{\theta(i)}^{-1} y_{\theta(j)} t_j^{-1} x_i t_{\theta(i)}^{-1} + y_{\theta(j)} t_j^{-1} x_i t_{\theta(i)}^{-1} x_i t_{\theta(i)}^{-1}.$$

This expression simplifies to

$$\begin{aligned} &q^{(-\alpha_{\theta(i)},\alpha_i-2\alpha_{\theta(j)})} x_i x_i y_{\theta(j)} t_{\theta(i)}^{-2} t_{\theta(j)}^{-1} \\ &\qquad - (q^{2d_i} + q^{-2d_i}) q^{(-\alpha_{\theta(i)},\alpha_i-\alpha_{\theta(j)})+(-\alpha_j,\alpha_i)} x_i y_{\theta(j)} x_i t_{\theta(i)}^{-2} t_{\theta(j)}^{-1} \\ &\qquad + q^{(-\alpha_j,2\alpha_i)+(-\alpha_{\theta(i)},\alpha_i)} y_{\theta(j)} x_i x_i t_{\theta(i)}^{-2} t_{\theta(j)}^{-1}. \end{aligned} \tag{2.5}$$

Using the fact that $(\alpha_i, \alpha_j) = d_i a_{ij} = -d_i$, expression (2.5) reduces to

$$\begin{aligned} &\big[q^{-(\alpha_{\theta(i)},\alpha_i)-2d_i} - (q^{2d_i} + q^{-2d_i}) q^{-(\alpha_{\theta(i)},\alpha_i)} + q^{-(\alpha_{\theta(i)},\alpha_i)+2d_i} \big] x_i^2 y_{\theta(j)} t_{\theta(i)}^{-2} t_j^{-1} \\ &\quad + \delta_{i,\theta(j)} \left[q^{-d_i} x_i \left(\frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right) + \left(q^{-d_i} - (q^{2d_i} + q^{-2d_i}) q^{d_i} \right) \left(\frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right) x_i \right] t_{\theta(i)}^{-3} \end{aligned}$$

which equals $-q^{3d_i}(q^{2d_i}+q^{-2d_i})x_it^{-3}_{\theta(i)}t^2_i$ when $i=\theta(j)$ and zero otherwise. In either case, expression (2.5) simplifies to the same value as the sum of the terms of weight $2\alpha_i-\alpha_{\theta(j)}$ on the right hand side of identity (2.2). The computation for showing that the sum of terms of weight $-\theta(2\alpha_i-\alpha_{\theta(j)})$ on both the right and left hand side of (2.2) equals $-q^{-3d_i}(q^{2d_i}+q^{-2d_i})y_{\theta(i)}t^3_{\theta(i)}t^{-2}_i$ when $i=\theta(j)$ and zero otherwise is the same.

Now consider the terms on the left hand side of (2.2) of weight $\alpha_i-\alpha_{\theta(i)}+\alpha_j$:

$$\begin{aligned}x_it^{-1}_{\theta(i)}y_{\theta(i)}t^{-1}_ix_jt^{-1}_{\theta(j)} &+ y_{\theta(i)}t^{-1}_ix_it^{-1}_{\theta(i)}x_jt^{-1}_{\theta(j)}\\ &- (q^{2d_i}+q^{-2d_i})[x_it^{-1}_{\theta(i)}x_jt^{-1}_{\theta(j)}y_{\theta(i)}t^{-1}_i+y_{\theta(i)}t^{-1}_ix_jt^{-1}_{\theta(j)}x_it^{-1}_{\theta(i)}]\\ &+ x_jt^{-1}_{\theta(j)}x_it^{-1}_{\theta(i)}y_{\theta(i)}t^{-1}_i+x_jt^{-1}_{\theta(j)}y_{\theta(i)}t^{-1}_ix_it^{-1}_{\theta(i)}.\end{aligned}$$

This expression simplifies to

(2.6)

$$\begin{aligned}&\Big[q^{(-\alpha_{\theta(i)},-\alpha_{\theta(i)}+\alpha_j)-(\alpha_i,\alpha_j)}x_iy_{\theta(i)}x_j+q^{(-\alpha_i,\alpha_i+\alpha_j)+(-\alpha_{\theta(i)},\alpha_j)}y_{\theta(i)}x_ix_j\\ &\quad-(q^{2d_i}+q^{-2d_i})[q^{(-\alpha_{\theta(i)},-\alpha_{\theta(i)}+\alpha_j)+(\alpha_{\theta(j)},\alpha_{\theta(i)})}x_ix_jy_{\theta(i)}+q^{(-\alpha_i,\alpha_j+\alpha_i)-(\alpha_{\theta(j)},\alpha_i)}y_{\theta(i)}x_jx_i]\\ &\quad+q^{(-\alpha_{\theta(j)},\alpha_i-\alpha_{\theta(i)})+(\alpha_{\theta(i)},\alpha_{\theta(i)})}x_jx_iy_{\theta(i)}+q^{(-\alpha_{\theta(j)},\alpha_i-\alpha_{\theta(i)})+(-\alpha_i,\alpha_i)}x_jy_{\theta(i)}x_i\Big]t^{-1}_{\theta(j)}t^{-1}_it^{-1}_{\theta(i)}.\end{aligned}$$

We can rewrite (2.6) so that each term is in the form $a^-a^+a^o$ where $a^\pm\in U^\pm$ and $a^o\in U^o$. In particular, the coefficent of $y_{\theta(i)}x_ix_jt^{-1}_it^{-1}_{\theta(i)}t^{-1}_{\theta(j)}$ in (2.6) is

$$q^{-(\alpha_{\theta(i)},\alpha_j)}\Big[q^{2d_i+d_i}+q^{-2d_i+d_i}-(q^{2d_i}+q^{-2d_i})(q^{2d_i-d_i}+q^{d_i-2d_i})+q^{2d_i-d_i}+q^{-d_i-2d_i}\Big]$$

which simplifies to zero. Similarly, the coefficient of $y_{\theta(i)}x_jx_it^{-1}_it^{-1}_{\theta(i)}t^{-1}_{\theta(j)}$ in (2.6) equals zero. Hence, the sum of the terms of weight $\alpha_i-\alpha_{\theta(i)}+\alpha_j$ (and not contained in either x_iU^o or x_jU^o) on both the right and left hand side of hand side of (2.2) equals zero. A similar argument shows that the coefficients of terms of weight $-\alpha_{\theta(i)}-\alpha_{\theta(j)}+\alpha_i$ (and not contained in either $y_{\theta(j)}U^o$ or $y_{\theta(i)}U^o$) on both the right and left hand side of (2.2) equals zero.

To further simplify (2.6), one needs to consider what happens when $i=\theta(j)$ and $i=\theta(i)$. (Note that these are mutually exclusive cases since $a_{ij}=-1$ implies that $i\neq j$.) We first assume that $\theta(i)=i$. Using the previous paragraph, (2.6) simplifies to

$$(q^{d_i}-q^{-d_i})^{-1}\Big[\Big(q^{4d_i}-(q^{2d_i}+q^{-2d_i})q^{2d_i}\Big)(t^2_i-t^{-2}_i)x_j+q^{2d_i}x_j(t^2_i-t^{-2}_i)\Big]t^{-2}_it^{-1}_j$$

which after some cancellation equals $x_jt^{-1}_j$.

Now assume that $i=\theta(j)$. Expression (2.6) simplifies to

$$(q^{-3d_i}+q^{d_i})x_i\left(\frac{t^2_{\theta(i)}-t^{-2}_{\theta(i)}}{q^{2d_i}-q^{-2d_i}}\right)t^{-2}_it^{-1}_{\theta(i)}-(q^{2d_i}+q^{-2d_i})q^{3d_i}\left(\frac{t^2_{\theta(i)}-t^{-2}_{\theta(i)}}{q^{2d_i}-q^{-2d_i}}\right)x_it^{-2}_it^{-1}_{\theta(i)}$$

which reduces to $-q^{-3d_i}(q^{2d_i}+q^{-2d_i})x_it^{-1}_{\theta(i)}t^{-2}_it^2_{\theta(i)}$. Hence (2.6) is equal to $x_jt^{-1}_j$ if $i=\theta(i)$; $-q^{-3d_i}(q^{2d_i}+q^{-2d_i})x_it^{-1}_{\theta(i)}t^{-2}_it^2_{\theta(i)}$ if $\theta(i)=j$; and zero otherwise. A similar computation shows that the sum of the terms of weight $-\alpha_{\theta(i)}+\alpha_i-\alpha_{\theta(j)}$ equals $y_jt_j^{-1}$ when $\theta(i)=i$, $-q^{3d_i}(q^{-2d_i}+q^{+2d_i})y_{\theta(i)}t_i^{-1}t_{\theta(i)}^{-2}t_i^2$ if $\theta(i)=j$, and zero otherwise. Statement (2.2) now follows.

Note that when g is simple and not simply laced, we must have $\theta=\mathrm{Id}$. Hence if $a_{ij}=-2$, it follows that $\theta(i)=i$ and $\theta(j)=j$. The identity in statement (2.3) can be proved using a routine though lengthy computation similar to the calculation for (2.2). We only sketch the proof here and omit the details. Using the quantum Serre relations, one can show that the terms of weight $3\alpha_i+\alpha_j$ (resp. $-3\alpha_i-\alpha_j$) on the left hand side of (2.3) add to zero. A straightforward computation shows that the terms of weight $3\alpha_i-\alpha_j$ (resp. $-3\alpha_i+\alpha_j$) on the left hand side of (2.3) cancel. What remains is checking that the terms of weight $\alpha_j-\alpha_i$, $-\alpha_j+\alpha_i$, $\alpha_j+\alpha_i$, and $-\alpha_j-\alpha_i$ agree on the left and right hand sides of (2.3). ∎

REMARK 2.1. It seems likely and preliminary computations suggest that Lemma 2.2 extends to include a fourth identity when g contains a subfactor of type G_2, and B_i, B_j are chosen such that $a_{ij}=-3$. However, the calculations are lengthy and tedious, so the case where g has a subfactor of type G_2 has been left out of this paper.

Our next goal is to show that the relations in Lemma 2.2 are sufficient to define B_θ. Let $\tilde{\mathcal{B}}^\theta$ be the free algebra generated by the indeterminates $\tilde{B}_i$ for $1\le i\le l$ and let $\mathcal{K}$ be the free abelian multiplicative group generated by the indeterminates $\{K_i \mid 1\le i<\theta(i)\le l\}$. Consider the automorphism ϕ from $\mathcal{K}$ to Aut $\tilde{\mathcal{B}}^\theta$ defined by $\phi(K_i)(\tilde{B}_j)=q^{(\alpha_i,\alpha_j)-(\alpha_{\theta(i)},\alpha_j)}\tilde{B}_j$. Let $\mathcal{B}=\tilde{\mathcal{B}}^\theta_\phi[\mathcal{K}]$ denote the skew group ring generated by $\tilde{\mathcal{B}}^\theta$ and $\mathcal{K}$ using ϕ to glue the two together. (Here the automorphism θ is understood from context.) Note that the group generated by the K_i is isomorphic to the group generated by the $t_it_{\theta(i)}^{-1}$ under the map which sends K_i to $t_it_{\theta(i)}^{-1}$.

Let I be the ideal of $\mathcal{B}$ generated by the elements

$$\tilde{B}_i\tilde{B}_j-\tilde{B}_j\tilde{B}_i-\delta_{i,\theta(j)}(K_i^2-K_i^{-2})/(q^{2d_i}-q^{-2d_i})$$

for all i,j such that $a_{ij}=0$;

$$\begin{gathered}\tilde{B}_i^2\tilde{B}_j-(q^{2d_i}+q^{-2d_i})\tilde{B}_i\tilde{B}_j\tilde{B}_i+\tilde{B}_j\tilde{B}_i^2-\delta_{i,\theta(i)}\tilde{B}_j\\ +\delta_{i,\theta(j)}(q^{2d_i}+q^{-2d_i})\tilde{B}_i(q^{3d_i}K_i^2+q^{-3d_i}K_i^{-2})\end{gathered}$$

for all i,j such that $a_{ij}=-1$; and

$$\begin{gathered}\tilde{B}_i^3\tilde{B}_j+(q^{4d_i}+1+q^{-4d_i})(-\tilde{B}_i^2\tilde{B}_j\tilde{B}_i+\tilde{B}_i\tilde{B}_j\tilde{B}_i^2)-\tilde{B}_i\tilde{B}_j^3\\ -(q^{2d_i}+q^{-2d_i})^2(\tilde{B}_i\tilde{B}_j-\tilde{B}_j\tilde{B}_i)\end{gathered}$$

for all i,j such that $a_{ij}=-2$.

We now define some notation which will be used in Proposition 2.3 below as well as in Theorem 2.4 and Theorem 3.1. Let $\mathcal{M}$ denote the free monoid generated by the letters $w_1,\ldots,w_l$. Given $w=w_{i_1}\ldots w_{i_m}$, set

$$\tilde{B}_w=\tilde{B}_{i_1}\ldots\tilde{B}_{i_m} \text{ and } B_w=B_{i_1}\ldots B_{i_m}.$$

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Let $\mathbf{S}$ be the subset of $\mathbf{Z}^l$ consisting of all l-tuples with zeros in the j-th place whenever $j \geq \theta(j)$. For $S \in \mathbf{S}$, set $K^S = \prod_{1\leq i<\theta(i)\leq l} K_i^{s_i}$. Note that $\{K^S \mid S \in \mathbf{S}\}$ is a basis for the group algebra $k(q)[\mathcal{K}]$. Each element in $\mathcal{B}$ can be written in the form $\sum_{w,S} a_{w,S}\tilde{B}_w K^S$ where $w \in \mathcal{M}$, $S \in \mathbf{S}$, and $a_{w,S} \in k(q)$. We can filter $\mathcal{B}$ by highest degree where $\deg \tilde{B}_w = m = \deg w$ for $w = w_{i_1} \dots w_{i_m}$ and $\deg K^S = 0$ for all $S \in \mathbf{S}$.

We will be using a subalgebra of U which is a slight variation of U^-. Set $F_i = y_i t_{\theta(i)}^{-1}$ for $1 \leq i \leq l$ and let U_θ^- denote the algebra generated by the F_i, $1 \leq i \leq l$. For $w = w_{i_1} \dots w_{i_m} \in \mathcal{M}$ set $F_w = F_{i_1} \dots F_{i_2}$ and $y_w = y_{i_1} \dots y_{i_m}$. One can check that F_w is just a power of q times $y_w t_{i_1}^{-1} \cdots t_{i_m}^{-1}$ where both the power of q and $t_{i_1}^{-1} \cdots t_{i_m}^{-1}$ only depend on the weight of y_w. Hence the same quantum Serre relations hold for U_θ^- and, moreover, $U_\theta^- \cong U^-$. Let $\tilde{U}_\theta^-$ be the free algebra generated by the indeterminates $\{\tilde{F}_i \mid 1 \leq i \leq l\}$ and set $\tilde{F}_w = \tilde{F}_{i_1} \dots \tilde{F}_{i_m}$. By [JL1, Section 4], U_θ^- is isomorphic to the $\tilde{U}_\theta^-$ modulo the ideal generated by the Serre relations in the $\tilde{F}_i$, $1 \leq i \leq l$.

Define a degree function on U by setting $\operatorname{Deg} F_i = 1$ for $1 \leq i \leq l$ and $\operatorname{Deg} a = 0$ for all $a \in U^0 U^+$. Note that every element of U can be written as a (finite) sum of homogeneous elements using this degree function.

PROPOSITION 2.3. *The algebra $\mathcal{B}/I$ is isomorphic to the subalgebra B_θ of U.*

PROOF. Let ψ be the map from $\mathcal{B}$ to B_θ defined by sending $\tilde{B}_i$ to B_i and K_i to $t_i t_{\theta(i)}^{-1}$ for all $1 \leq i \leq l$. Since $t_i t_{\theta(i)}^{-1} B_j t_i^{-1} t_{\theta(i)} = q^{(\alpha_i,\alpha_j)-(\alpha_{\theta(i)},\alpha_j)} B_j$ for all $1 \leq i, j \leq l$, the map ψ is a well-defined algebra homomorphism. By Lemma 2.2, $\psi(I) = 0$. We show that the kernel of ψ is exactly I. Suppose that $X = \sum_{w,S} a_{w,S}\tilde{B}_w K^S$ is an element of minimal degree in the kernel of ψ but not contained in I where the $a_{w,S}$ are elements of $k(q)$, $w \in \mathcal{M}$, and $S \in \mathbf{S}$. Set $n = \deg X$. Given $w = w_{i_1} \dots w_{i_m}$ in $\mathcal{M}$, set $\theta w = w_{\theta(i_1)} \dots w_{\theta(i_m)}$. By the definition of the B_i, it follows that

$$\psi(X) = \sum_{w,S;\deg w=n} a_{w,S} F_{\theta w} \prod (t_i t_{\theta(i)}^{-1})^{s_i} + \text{ lower degree terms in } U. \tag{2.7}$$

Now the sum of all the terms in the expansion of $\psi(X)$ of degree n must be zero (where here we are using the degree function Deg on U). From the triangular decomposition and defining relations of U (see Section 1) we have that $U_\theta^- U^o \cong U_\theta^- \otimes U^o$ is a free U_θ^--module with basis $\{\prod_{i<\theta(i)} (t_i t_{\theta(i)}^{-1})^{s_i} \mid S \in \mathbf{S}\}$. Hence $\sum_{w;\deg w=n} a_{w,S} F_{\theta w} = 0$ for each $S \in \mathbf{S}$. Therefore $\sum_{w;\deg w=n} a_{w,S}\tilde{F}_{\theta w}$ must be an element of the ideal in $\tilde{U}_\theta^-$ generated by the Serre relations using the $\tilde{F}_i$. A comparison of the Serre relations (1.4) with the elements in the ideal I of $\mathcal{B}$ shows that

$$\sum_{w;\deg w=n} a_{w,S}\tilde{B}_{\theta w}$$

is equal to $x + I$ for some $x \in \mathcal{B}$ of lower degree than the degree of X. This contradicts the choice of a minimal degree element X. Hence the kernel of ψ equals I. ∎

In the rest of the paper we will identify the group $\mathcal{K}$ with the subgroup of B_θ generated by $t_i t_{\theta(i)}^{-1}$, $1 \leq i \leq l$. In particular, we set $K_i = t_i t_{\theta(i)}^{-1}$ for each i.

REMARK 2.2. The proof of Proposition 2.3 shows that ψ induces a map of degree filtrations from $\mathcal{B}$ to U. Restrict the filtrations to B_θ and $U_\theta^- k(q)[\mathcal{K}]$ and form the corresponding associated graded algebras. It is straightforward to check that $\mathrm{gr}\, U_\theta^{-1} k(q)[\mathcal{K}] \simeq U_\theta^{-1} k(q)[\mathcal{K}]$ and so by the proof of the above proposition it follows that $\mathrm{gr}\, B_\theta \simeq U_\theta^- k(q)[\mathcal{K}]$. We can exploit this isomorphism to construct a good basis for B_θ which will be used in Theorem 2.4 and Theorem 3.1. Now $B_w = F_w +$ terms of lower degree in U. Let $\mathcal{V}$ be a basis for $U_\theta^- k(q)[\mathcal{K}]$ over $k(q)$ consisting of elements of the form $F_w K^S$ where $w \in \mathcal{M}$ and $S \in \mathbf{S}$. Note that these elements are homogeneous with respect to the degree function on U. It follows that there exists a basis $\mathcal{W}$ of B_θ consisting of elements of the form $B_w K^S$ where $w \in \mathcal{M}$ and $S \in \mathbf{S}$ with the following property. The map χ sending F_w to $B_{\theta w}$ and fixing elements of $\mathcal{K}$ induces a bijection from $\mathcal{V}$ to $\mathcal{W}$ such that

$$\chi(b) = b + \text{ terms of lower degree in } U$$

for each $b \in \mathcal{V}$. We may further assume that the subset $\bar{\mathcal{V}} = \mathcal{V} \cap U_\theta^-$ of $\mathcal{V}$ is a basis for U_θ^- and that $\mathcal{V} = \{bK^S \mid b \in \bar{\mathcal{V}} \text{ and } S \in \mathbf{S}\}$. Similarly, set $\bar{\mathcal{W}} = \{\chi(b) \mid b \in \bar{\mathcal{V}}\}$. Then $\mathcal{W} = \{bK^S \mid b \in \bar{\mathcal{W}} \text{ and } S \in \mathbf{S}\}$.

The next theorem is a quantum analog of the Iwasawa decomposition. Remark 2.2 is crucial in the proof. We also need some additional notation. Let $\mathcal{A}$ be the group generated by $\{t_i t_{\theta(i)} \mid 1 \le i \le l\}$. Let T' be the subgroup of T generated by $\mathcal{K}$ and $\mathcal{A}$. Set $U_\theta^o = k(q)[T']$ and $U_\theta^+ = k(q)[x_1 t_{\theta(1)}^{-1}, \ldots, x_l t_{\theta(l)}^{-1}]$. One checks that $U_\theta = U_\theta^- U_\theta^o U_\theta^+$ is a subalgebra of U. Furthermore, U is a finitely generated U_θ-module with generators from T. Note that the group T' is isomorphic to the product $\mathcal{K} \times \mathcal{A}$ and so $U_\theta^o \cong k(q)[\mathcal{K}] \otimes k(q)[\mathcal{A}]$.

THEOREM 2.4. *The linear map from $B_\theta \otimes \mathcal{A} \otimes U_\theta^+$ to U which sends elements of the form $b \otimes a \otimes u$ to the product bau defines a vector space isomorphism $B_\theta \otimes \mathcal{A} \otimes U_\theta^+ \cong U_\theta$ where the tensor product is taken over $k(q)$.*

PROOF. We keep the notation of Remark 2.2. It follows from the triangular decomposition of U (Section 1) and the definitions of $U_\theta^\pm$ and U_θ^o that there is an isomorphism of vector spaces

$$U_\theta \cong U_\theta^- \otimes U_\theta^o \otimes U_\theta^+ \tag{2.8}$$

over $k(q)$. Identity (2.8) implies that $\mathcal{A}U^+$ is isomorphic to $\mathcal{A} \otimes U_\theta^+$ as vector spaces. Let M be a basis for the vector space $\mathcal{A}U_\theta^+$. To prove the theorem, it is sufficient to show that the set $\{bm \mid b \in \mathcal{W}, m \in M\}$ is a basis for U_θ. This follows from (2.8), Remark 2.2 and induction on degree. ∎

Unfortunately, B_θ is not a Hopf subalgebra of $U_q(g)$. Though $\Delta(t_i t_\theta^{-1}) = t_i t_\theta^{-1} \otimes t_i t_\theta^{-1}$ is an element of $B_\theta \otimes B_\theta$, the same does not hold true for $\Delta(B_i)$. However, the image of B_i under Δ is still rather nice. In particular,

$$\Delta(B_i) = (x_i t_{\theta(i)}^{-1} + y_{\theta(i)} t_i^{-1}) \otimes t_i^{-1} t_{\theta(i)}^{-1} + t_i t_{\theta(i)}^{-1} \otimes x_i t_{\theta(i)}^{-1} + t_i^{-1} t_{\theta(i)} \otimes y_{\theta(i)} t_i^{-1}.$$

Hence each generator b (equal to either B_i or $t_i t_\theta^{-1}$) of B_θ satisfies

$$\Delta(b) \in b \otimes T + (B_\theta \cap T) \otimes U. \tag{2.9}$$

Condition (2.9) implies the right coideal property on comultiplication:

$$\Delta(b) \in B_\theta \otimes U \tag{2.10}$$

for all $b \in B_\theta$. We shall see below that (2.10) combined with the Iwasawa decomposition of Theorem 2.4 are enough to distinguish B_θ from other subalgebras as potential candidates for the quantum analog of $U(g^\theta)$.

Given arbitrary elements r_i and s_i of T, let B be the subalgebra of U generated by C_i and $(t_i t_{\theta(i)}^{-1})^{\pm 1}$ for $1 \leq i \leq l$ where $C_i = x_i r_i + y_{\theta(i)} s_i$. Assume further that B satisfies $\Delta(B) \subset B \otimes U$. Recall that $\mathcal{K}$ is the group generated by $t_i t_{\theta(i)}^{-1}$ for $1 \leq i \leq l$.

For each i,

$$\Delta(C_i) = x_i r_i \otimes r_i t_i^{-1} + y_{\theta(i)} s_i \otimes s_i t_{\theta(i)}^{-1} + r_i t_i \otimes x_i r_i + s_i t_{\theta(i)} \otimes y_{\theta(i)} s_i. \tag{2.11}$$

We can rewrite (2.11) $\Delta(C_i)$ as

$$\Delta(C_i) = x_i r_i \otimes (r_i t_i^{-1} - s_i t_{\theta(i)}^{-1}) + B_i \otimes s_i t_{\theta(i)}^{-1} + r_i t_i \otimes x_i r_i + s_i t_{\theta(i)} \otimes y_{\theta(i)} s_i.$$

Hence $\Delta(C_i) \in B \otimes U$ if and only if $x_i r_i \in B$ or $r_i t_i^{-1} = s_i t_{\theta(i)}^{-1}$ and both $r_i t_i, s_i t_{\theta(i)} \in B \cap T$. When $C_i = B_i$ for each $1 \leq i \leq l$, then $B = B_\theta$ and we know that $x_i r_i \notin B_\theta$ and $B_\theta \cap T = \mathcal{K}$ by Theorem 2.4. So if we assume in addition (using the notation of Theorem 2.4) that B is used to form a quantum Iwasawa type decomposition, in particular $U_\theta \cong B \otimes U_\theta^o \otimes U_\theta^+$, then B also satisfies $x_i r_i \notin B$ and $B \cap T = \mathcal{K}$. Thus under this assumption, $\Delta(C_i)$ satisfies the stronger coproduct condition (2.9).

The following theorem shows that B is actually equal to B_θ.

THEOREM 2.5. *Let B be the subalgebra of U generated by C_i and $(t_i t_{\theta(i)}^{-1})^{\pm 1}$ for $1 \leq i \leq l$ where $C_i = x_i r_i + y_{\theta(i)} s_i$ and r_i and s_i are some elements of T. If B satisfies the right coideal property (2.10) and can be used to form a quantum Iwasawa decomposition, then $B = B_\theta$.*

PROOF. Since the image of C_i under Δ satisfies (2.9), we must have $s_i t_{\theta(i)}^{-1} = r_i t_i^{-1}$. Hence there exists $a_i \in T$ such that $t_{\theta(i)}^{-1} a_i = r_i$ and $t_i^{-1} a_i = s_i$. But we also have $r_i t_i = a_i t_{\theta(i)}^{-1} t_i \in \mathcal{K}$ and since $t_{\theta(i)}^{-1} t_i \in \mathcal{K}$, it follows that $a_i \in \mathcal{K}$. Therefore $C_i a_i^{-1} = B_i$ is an element of B and B is generated by B_i and $(t_i t_{\theta(i)}^{-1})^{\pm 1}$ for $1 \leq i \leq l$. Thus $B = B_\theta$. ∎

REMARK 2.3. In the next section we show that the subalgebras B_θ specialize to the enveloping algebra of g^θ at $q = 1$. However, when g^θ is semisimple, B_θ is not isomorphic to the Drinfeld-Jimbo quantization $U_q(g^\theta)$. To see this, recall [JL1] that the set of invertible elements of U is exactly T. Hence the set of invertible elements of B_θ is $B_\theta \cap T$. By Theorem 2.4, this group is just the group $\mathcal{K}$ generated by the $t_i t_{\theta(i)}^{-1}$ for $1 \leq i \leq l$. This is a maximal subgroup of B_θ which, one checks using the specializations in Corollary 3.2, has rank strictly smaller than the rank of the Cartan subgroup of $U_q(g_\theta)$.

REMARK 2.4. When $g = sl(n)$ and $\theta = \mathrm{Id}$, it is easy to show that B_θ is isomorphic to the quantization of so(n) introduced by [GK]. By [NS, pp. 4, 6] it follows that in this case B_θ is also isomorphic to the subalgebra constructed by Noumi and Sugitani. For other pairs of classical Lie algebra g and involution θ it is not clear whether the subalgebras constructed in [NS] have a connection to the B_θ. However, the B_θ satisfy the right coideal condition (4.10), an important property of the subalgebras considered in [NS].

3. **Specialization.** In this section, we determine the specialization at $q = 1$ of the subalgebras B_θ defined in Section 1. More precisely, set A equal to the localization of $k[q, q^{-1}]$ at the ideal generated by $q-1$. Let $\hat{U}$ be the free A-module generated as an algebra by $x_i, y_i, t_i^{\pm 1}, (t_i - 1)/(q-1)$ for $1 \leq i \leq l$. Note that $U^o \cap \hat{U}$ is generated by the elements t_i and $(t_i - 1)/(q-1)$ as i ranges from 1 to l. The tensor product $\hat{U} \otimes_A k$ is isomorphic to $U(g)[t_1^{\pm}, \ldots, t_l^{\pm}]$ where the t_i are now central elements such that $t_i^4 = 1$. Further modding out by the ideal generated by $\{t_i - 1 \mid 1 \leq i \leq l\}$ recovers the ordinary enveloping algebra $U(g)$. (See for example [JL2, 6.11] or [KD, Prop 1.5].) This process of tensoring and then modding out is called specialization at $q = 1$.

Let θ be a diagram automorphism as in Section 2. Recall that θ induces a Lie algebra involution on g which sends e_i to $f_{\theta(i)}$ and h_i to $-h_{\theta(i)}$. Here e_i, f_i, h_i, $1 \leq i \leq l$ are standard generators of g such that x_i (resp. y_i) specializes to e_i (resp. f_i.) We also refer to this induced automorphism as θ with the meaning clear from context. The Lie subalgebra fixed by θ is denoted by g^θ.

THEOREM 3.1. *The subalgebra $B_\theta \cap \hat{U}$ of $\hat{U}$ specializes to the subalgebra $U(g^\theta)$ of $U(g)$.*

PROOF. Let $\bar{B}_\theta$ denote the image of $B_\theta \cap \hat{U}$ in $U(g)$ under the specialization of $\hat{U}$ at $q = 1$. Note that B_i specializes to $e_i + f_{\theta(i)}$ for each $1 \leq i \leq l$. Furthermore, for each i,

$$\frac{t_i^2 t_{\theta(i)}^{-2} - t_i^{-2} t_{\theta(i)}^2}{q-1} = \frac{t_{\theta(i)}^{-2}(t_i^2 - t_i^{-2}) - t_i^{-2}(t_{\theta(i)}^2 - t_{\theta(i)}^{-2})}{q-1}$$

is an element of $\hat{U} \cap B_\theta$ which specializes to $h_i + h_{\theta(i)}$ up to a (nonzero) scalar. Now $[e_i + f_{\theta(i)}, e_j + f_{\theta(j)}] = [e_i, e_j] + [f_{\theta(i)}, f_{\theta(j)}] + \delta_{i\theta(j)}(h_i - h_{\theta(i)})$. Hence $e_\alpha + f_{\theta(\alpha)}$ is also an element of $\bar{B}_\theta$ for all positive roots α. Therefore $U(g^\theta)$ is a subalgebra of $\bar{B}_\theta$.

Let $\mathbf{S}^+$ denote the intersection of $\mathbf{S}$ with the set of l-tuples with nonnegative entries. For $S \in \mathbf{S}^+$, set

$$\left(\frac{K-1}{q-1}\right)^S = \prod_{1 \leq i < \theta(i) \leq l} \left(\frac{K_i - 1}{q-1}\right)^{s_i}$$

where recall that K_i is identified with $t_i t_{\theta(i)}^{-1}$. Recall the notation of Remark 2.2 of Section 2. Given $X \in B_\theta$ one can find $M \in \mathbf{S}$ such that XK^M is a linear combination over $k(q)$ of elements of the form $B_w \left(\frac{K-1}{q-1}\right)^S$ with $B_w \in \bar{W}$ and $S \in \mathbf{S}^+$. Hence X can be written in the form

$$X = \left(\sum_{B_w \in \bar{W}, S \in \mathbf{S}^+} a_{w,S} B_w \left(\frac{K-1}{q-1}\right)^S\right) K^{-M}$$

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where $a_{w,S} \in k(q)$. Assume that $X \in B_\theta \cap \hat{U}$ and let n be the maximum degree such there exists a nonzero $a_{w,S}$ with $\operatorname{Deg} B_w = n$. It follows that the highest degree term of X,

$$\sum_{\{w,S:\operatorname{Deg} B_w = n\}} a_{w,S} F_{\theta w},$$

is an element of $\hat{U}$ for each choice of $S \in \mathbf{S}^+$. By Remark 2.2 of Section 2, the set $\bar{V} = V \cap U_\theta^- = \{\underline{\chi}^{-1}(B_w) \mid B_w \in \bar{W}\}$ is a basis for U_θ^-. Therefore $a_{w,S} \in A$ for each $S \in \mathbf{S}^+$ and $B_w \in \bar{W}$ with $\operatorname{Deg} B_w = n$. Hence

$$X - \Big(\sum_{B_w \in \bar{W}, S \in \mathbf{S}^+; \operatorname{Deg} B_w = n} a_{w,S} B_w \left(\frac{K-1}{q-1}\right)^S\Big) K^{-M}$$

is an element of $B_\theta \cap \hat{U}$ of lower degree. By (reverse) induction on Deg X, we have that $X \in B_\theta \cap \hat{U}$ if and only if $a_{w,S} \in A$ for all B_w and S. In particular $B_\theta \cap \hat{U}$ is spanned over A by elements of the form $B_w \left(\frac{K-1}{q-1}\right)^S K^M$ where $B_w \in \bar{W}$ and $S \in \mathbf{S}^+$. It is straightforward to check that the specialization of $B_w \left(\frac{K-1}{q-1}\right)^S K^M$ is an element of $U(g^\theta)$. It follows that the specialization of B_θ is a subalgebra of $U(g^\theta)$ which completes the proof of the theorem. ∎

REMARK 3.1. Consider the A subalgebra $\hat{B}_\theta$ of B_θ generated by the B_i, $\big((t_i t_{\theta(i)}^{-1}) - 1\big)/(q-1)$, and $(t_i t_{\theta(i)}^{-1})^{\pm 1}$ for $1 \leq i \leq l$. We can specialize B_θ directly to the algebra $(\hat{B}_\theta \otimes_A k)/\langle t_i t_{\theta(i)}^{-1} - 1\rangle$. It follows from the proof of Theorem 3.1 that the direct specialization of B_θ is isomorphic to the specialization of $B_\theta \cap \hat{U}$ considered as a subalgebra of $\hat{U}$.

We can now use the description of such algebras in the classical case (due to E. Cartan) to determine the specializations of the quantum subalgebras B_θ. For a simple Lie algebra g and an involution θ induced from a diagram automorphism as above, it is straightforward to show that the dimension of g^θ is equal to $|\Delta^+| + \#\{i \mid i < \theta(i)\}$.

COROLLARY 3.2. *Let g be a simple Lie algebra over the complex numbers $k = \mathbf{C}$ and let θ be a diagram automorphism. The image $\bar{B}_\theta \cong U(g^\theta)$ of $B_\theta \cap \hat{U}$ under the specialization of $\hat{U}$ at $q = 1$ can be described as follows.*

(3.1) If $g = sl(n)$ and $\theta = \mathrm{Id}$, then $\bar{B}_\theta \cong U\big(so(n)\big)$.

(3.2) If $g = sl(n)$ and θ is defined as in Case 2 of Section 1, then $\bar{B}_\theta \cong R$ where R is the subalgebra generated by h_p and e_i, f_i for $1 \leq i \leq n-1$ and $i \neq p$ with $p = n/2$ (resp. $(n-1)/2$) if n is even (resp. odd).

(3.3) If $g = so(2n+1)$ $(n \geq 2)$ and $\theta = \mathrm{Id}$ then $\bar{B}_\theta \cong U\big(so(n) \oplus so(n+1)\big)$.

(3.4) If $g = sp(n)$, $n = 2m \geq 2$, and $\theta = \mathrm{Id}$ then $\bar{B}_\theta \cong U\big(gl(m)\big)$.

(3.5) If $g = so(2n)$ and $\theta = \mathrm{Id}$ then $\bar{B}_\theta \cong U\big(so(n) \oplus so(n)\big)$.

(3.6) If $g = so(2n)$ and $\theta \neq \mathrm{Id}$ then $\bar{B}_\theta \cong U\big(so(n-1) \oplus so(n+1)\big)$.

(3.7) If g is of type E_6 and $\theta = \mathrm{Id}$, then $\bar{B}_\theta \cong U\big(sp(4)\big)$.

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(3.8) *If g is of type E_6 and $\theta \neq \mathrm{Id}$, then* $\overline{B}_\theta \cong U\big(sl(6) \oplus sl(2)\big)$.
(3.9) *If g is of type E_7 and $\theta = \mathrm{Id}$, then* $\overline{B}_\theta \cong U\big(sl(8)\big)$.
(3.10) *If g is of type E_8 and $\theta = \mathrm{Id}$, then* $\overline{B}_\theta \cong U\big(so(16)\big)$.
(3.11) *If g is of type F_4 and $\theta = \mathrm{Id}$, then* $\overline{B}_\theta \cong U\big(sp(3) \oplus sl(2)\big)$.

PROOF. When $\theta = \mathrm{Id}$, the algebra B_θ specializes to the Lie subalgebra g^θ generated by $e_j + f_j$, $1 \le j \le l$. By [He, Definition on p. 426] and [He, p. 183, Theorem 7.2], g^θ corresponds to the complexification of the fixed subalgebra in the Cartan decomposition of the normal (or split) real form of g. Thus the cases when $\theta = \mathrm{Id}$ follows from [He, pp. 451–455, 517, 518] by taking the complexification of the appropriate fixed subalgebras. For $\theta \neq \mathrm{Id}$, we can compute the dimensions of the g^θ and this is enough to determine the desired fixed Lie algebras. In particular, for (3.2), $\dim g^\theta = (n^2 - 2)/2$ when n is even and equals $(n^2 - 1)/2$ when n is odd; for (3.6), $\dim g^\theta = n^2 - n + 1$; and for (3.8), $\dim g^\theta = 38$. The result now follows by comparing dimensions of the complexified fixed subalgebras in [He, pp. 451–455, 518]. ∎

4. **Quantum Harish-Chandra Modules.** In this section we begin the the study of quantum Harish-Chandra modules for the pair U, B_θ. These modules are defined exactly as in the classical case (see [D, Section 9.1.4]). In particular, we have the following.

DEFINITION 4.1. Let V be a subalgebra of U. A Harish-Chandra module for V is a U-module which can be written as a direct sum of finite-dimensional simple V-modules.

In the classical case, Harish-Chandra modules arise naturally inside a typical g module as the sum of all the finite-dimensional simple g^θ submodules. The proof uses the Hopf structure, or more precisely, the diagonal map of $U(g^\theta)$. Though B_θ satisfies the coideal property on comultiplication (4.10), it is not enough to make the same argument work for the pair U, B_θ. In this section, using different methods, we prove a quantum analog for U-modules on which the subgroup T acts semisimply.

One of the basic U-modules is U itself considered as a module using the adjoint action. Recall [JL1] that U contains a large locally finite subalgebra which we denote $F(U)$. Note that $F(U)$ is the sum of all the finite-dimensional simple $(\mathrm{ad}\, U)$-modules contained in U. As a consequence of our results on general U-modules in this section, it follows that the $(\mathrm{ad}\, U)$-module $F(U)$ is the maximal Harish-Chandra module for the pair U, B_θ inside U.

For the purposes of this section, we assume that $k = \mathbf{C}$. The Chevalley anti-automorphism κ on U is a $k(q)$ anti-automorphism of U which sends x_i to y_i, y_i to x_i, and fixes elements of T. Let $U_{\mathbf{R}(q)}$ denote the $\mathbf{R}(q)$ subalgebra of U generated over $\mathbf{R}(q)$ by x_i, y_i, and $t_i^{\pm 1}$ for $1 \le i \le l$. The anti-automorphism κ can be restricted to the real quantized enveloping algebra $U_{\mathbf{R}(q)}$ and then extended to the anti-automorphism κ^* of U using complex conjugation“ $^-$". In particular, $\kappa^*(av) = \bar{a}\kappa(v)$ where $a \in \mathbf{C}$ and $v \in U_{\mathbf{R}(q)}$.

Our first goal is to show that any finite-dimensional U-module is a Harish-Chandra module. We use a twisted version of the quantum Shapovalov form which behaves nicely in terms of the B_θ action. To define this form, first twist the antiautomorphism κ^* using elements of h^* as follows. Set $h^*_{\mathbf{Z}} = \{\lambda \in h^* \mid (\lambda, \alpha_i) \in \mathbf{Z}$ for each $1 \le i \le l\}$. For $\lambda \in h^*_{\mathbf{Z}}$ and a vector v of weight β in $U_{\mathbf{R}(q)}$, set

$$\kappa_\lambda(v) = q^{(\lambda,\beta)}\kappa(v). \tag{4.1}$$

Define $\kappa^*_\lambda(aw)$ as $\bar{a}\kappa_\lambda(w)$ where $a \in \mathbf{C}$ and $w \in U_{\mathbf{R}(q)}$.

Let φ denote the Harish-Chandra map of U onto U^o using the direct sum decomposition $U = (UU^+_+ + U^-_+U) \oplus U^o$. (Here U^+_+, U^-_+ denote the augmentation ideals of U^+, U^- respectively.) Set $G^- = k(q)[y_it_i \mid 1 \le i \le l]$ and $G^+ = k(q)[x_it_i \mid 1 \le i \le l]$. Note that $\kappa^*_\lambda(G^-) = G^+$ and $\kappa^*_\lambda(G^+) = G^-$.

Recall that finite-dimensional simple U-modules can be parametrized by pairs γ, $(\xi_1, \dots \xi_l)$ where γ is a dominant integral weight and $(\xi_1, \dots \xi_l)$ is a sequence of fourth roots of unity (see [L] and [R]). In particular, given such a pair γ and $(\xi_1, \dots \xi_l)$, set Λ equal to the algebra homomorphism from U^o to $\mathbf{C}(q)$ defined by $\Lambda(t_i) = \xi_i q^{(\gamma,\alpha_i)}$. The corresponding finite-dimensional simple U module $L(\Lambda)$ is generated by a highest weight vector v_Λ such that $x_iv_\Lambda = 0$ and $t_iv_\Lambda = \xi_i q^{(\gamma,\alpha_i)}v_\Lambda$ for $1 \le i \le l$. It is easy to check that $G^-v_\Lambda = L(\Lambda)$. Define a conjugate linear form $S = S^\Lambda_\lambda$ on $L(\Lambda)$ using κ^*_λ as follows. Set $S(v_\Lambda, v_\Lambda) = 1$ and $S(fv_\Lambda, gv_\Lambda) = \Lambda\Big(\varphi\big(\kappa^*_\lambda(f)g\big)\Big)$ where f and g are elements of $G^- \cap U_{\mathbf{R}(q)}$. It is straightforward to check that $S(fw, v) = S\big(w, \kappa^*_\lambda(f)v\big)$ for any w, v in $L(\Lambda)$ and $f \in U$. When $\xi_i = 1$ for each i, we simply write $L(\gamma)$ for the corresponding finite-dimensional simple module with highest weight vector v_γ and S^γ_λ for the corresponding bilinear form.

The next nondegeneracy result uses specialization to the classical case.

LEMMA 4.2. *Let $S = S^\Lambda_\lambda$ be the conjugate linear form on $L(\Lambda)$ as described above. Then $S(w, w) \neq 0$ for any nonzero w in $L(\Lambda)$.*

PROOF. Let γ be a dominant integral weight and let $\xi_1, \dots, \xi_l$ be a sequence of fourth roots of unity. Set Λ equal to the homomorphism from U^o to $\mathbf{C}(q)$ such that $\Lambda(t_i) = \xi_i q^{(\gamma,\alpha_i)}$ for each $1 \le i \le l$.

Note that $\varphi\big(\kappa^*_\lambda(a)b\big) \in \mathbf{C}(q)[t_1^4, \dots, t_l^4]$ for any choice of $a, b \in G^-$. Since $\Lambda(t_i^4) = q^{4(\gamma,\alpha_i)}$ for each i, it follows that $S^\Lambda_\lambda(fv_\Lambda, bv_\Lambda) = S^\gamma_\lambda(fv_\gamma, bv_\gamma)$ for each $f, b \in G^-$. Since $G^-v_\Lambda = L(\Lambda)$, it is sufficient to prove the lemma for simple modules of the form $L(\gamma)$ where γ is a dominant integral weight.

Recall the definition of $\hat{U}$ (Section 3) and set $\widehat{L(\gamma)} = \hat{U}v_\gamma$. Note that $\widehat{L(\gamma)}$ specializes to the simple finite-dimensional $U(g)$-module $L_1(\gamma)$ with highest weight generating vector v_1 which is the image of v_γ under specialization (see for example [JL1, Lemma 5.10]).

Let κ_1 denote the conjugate linear antiautomorphism of $U(g)$ defined by $\kappa_1(e_i) = f_i$, $\kappa_1(f_i) = e_i$, $\kappa_1(h_i) = h_i$ for each $1 \le i \le l$. Write φ_1 for the classical Harish-Chandra map (see, for example, [D, Section 7.4.3]). Define a conjugate linear form S_1 on $L_1(\gamma)$ by $S_1(v_1, v_1) = 1$ and $S_1(fv_1, bv_1) = \gamma\Big(\varphi_1\big(\kappa_1(f)b\big)\Big)$ for $f, b \in U(g)$. Note that $S_1(f_iw, m) = S(w, e_im)$ and $S_1(e_iw, m) = S_1(w, f_im)$ for all $w, m \in L_1(\gamma)$ and $1 \le i \le l$. Hence this form agrees with the one defined in [K, Section 11.5]. By [K, Theorem 11.7], S_1 is positive definite on $L_1(\gamma)$.

For any $b \in \hat{U}$, $\kappa_\lambda(b)$ specializes to $\kappa_1(b_1)$ where b_1 denotes the specialization of b. Furthermore for $w, m \in \widehat{L(\gamma)}$, we have $S_\lambda(m, w)$ specializes to $S_1(m_1, w_1)$ where m_1, w_1 denotes the images of m and w under specialization. Now consider a nonzero element $w \in L(\gamma)$. Multiplying by a suitable integer power of $q-1$ we may assume that $w \in \widehat{L(\gamma)}$ and $w \notin (q-1)\widehat{L(\gamma)}$. In particular the image w_1 of w under specialization is nonzero. Hence $S_\lambda(w, w)$ is nonzero since it specializes to $S_1(w_1, w_1)$ which is positive. It follows that $S(w, w) \neq 0$ for all nonzero $w \in L(\gamma)$. ∎

Now choose λ such that $(\lambda, \alpha_i) = -(\alpha_{\theta(i)}, \alpha_i)$ for each $1 \leq i \leq l$. Note that

$$\begin{aligned}\kappa_\lambda^*(B_i) &= q^{-(\alpha_{\theta(i)}, \alpha_i)} t_{\theta(i)}^{-1} y_i + q^{(\alpha_i, \alpha_{\theta(i)})} t_i^{-1} x_{\theta(i)} \\ &= y_i t_{\theta(i)}^{-1} + x_{\theta(i)} t_i^{-1} = B_i\end{aligned}$$

and hence $\kappa_\lambda^*(b) = b$ for all $b \in B_\theta \cap U_{\mathbf{R}(q)}$. It follows that

$$\kappa_\lambda^*(B_\theta) = B_\theta. \tag{4.2}$$

LEMMA 4.3. *When $k = \mathbf{C}$, each finite-dimensional simple U-module is a Harish-Chandra module for the pair U, B_θ.*

PROOF. Let $L = L(\Lambda)$ be a finite-dimensional simple U-module and write S for the corresponding conjugate linear form S_λ^Λ on L. Let M be a nonzero B_θ submodule of L. If $M = L$ then we are done. Otherwise set $M^\perp = \{m \in L \mid S(m, M) = 0\}$. Recall that S satisfies $S(afv_\gamma, bv_\gamma) = S\big(fv_\gamma, \kappa_\lambda^*(a)bv_\gamma\big)$ for all $a \in U$. Using (4.2) it follows that $S(B_\theta r, s) = S\big(r, \kappa_\lambda^*(B_\theta)s\big) = S(r, B_\theta s)$ for all $r, s \in L$. Hence $M^\perp$ is a B_θ module. By the previous lemma, S restricted to M is nondegenerate so $M \cap M^\perp = 0$ and $M \oplus M^\perp$ is isomorphic to L. Now apply induction to M and $M^\perp$. ∎

Consider U-modules on which T acts semisimply. The next lemma allows one to embed finite-dimensional B_θ submodules of U-modules which admit a semisimple T action inside a direct sum of simple finite-dimensional U-modules.

Let $k(q)^*$ denote the multiplicative group generated by the nonzero elements in $k(q)$. We expand the notion of weight given in Section 1. Let M be a U-module. For $\Lambda \in \mathrm{Hom}\big(T, k(q)^*\big)$, set $M_\Lambda = \{v \in M \mid \ t_i v = \Lambda(t_i)v$ for all t_i, $1 \leq i \leq l.\}$ Given $\beta \in h^*$, let q^β denote the element of $\mathrm{Hom}\big(T, k(q)^*\big)$ which sends t_i to $q^{(\beta, \alpha_i)}$. (Note that the notation here differs from the notation defined in Section 1 concerning weight spaces.) We can extend the standard partial order on weights in h^* to $\mathrm{Hom}\big(T, k(q)^*\big)$ as follows: $\Lambda \geq \Lambda'$ if $\Lambda = q^\beta \Lambda'$ for some $\beta \in Q^+$. Now let M be a U-module which admits a semisimple T action. We can write $M = \oplus M_\Lambda$ where Λ runs over nonzero elements of $\mathrm{Hom}\big(T, k(q)^*\big)$. So each element in M can be written as a sum of T weight vectors $\sum_{\Lambda \in \mathrm{Hom}(T, k(q)^*)} v_\Lambda$ where $v_\Lambda \in M_\Lambda$. Define the support of v by $\mathrm{supp}(v) = \{\Lambda \mid \ v_\Lambda \neq 0\}$ and set $\max\big(\mathrm{supp}(v)\big)$

equal to the set of maximal weights in the support of v with respect to the partial order on $\mathrm{Hom}\big(T, k(q)^*\big)$. Write $\overline{v}$ for the element $\sum_{\Lambda\in\max\left(\mathrm{supp}(v)\right)} v_\Lambda$.

LEMMA 4.4. *Let M be a U-module such that the action of T on M is semisimple. Let W be a finite-dimensional B_θ submodule of M. Then W generates a semisimple U submodule of M.*

PROOF. Let $v \in W$. Choose $1 \leq i \leq l$. Note that $B_i^m v = (x_i t_{\theta(i)}^{-1})^m \overline{v}+$ terms of non-maximal weight. By assumption, $\{B_i^m v \mid m \geq 0\}$ spans a finite-dimensional vector space. Now the supports of $(x_i t_{\theta(i)}^{-1})^m \overline{v}$ are distinct for different choices of m, hence for large m, $(x_i t_{\theta(i)}^{-1})^m \overline{v} = 0$. It follows that for large m, $x_i^m v_\Lambda = (x_i t_{\theta(i)}^{-1})^m v_\Lambda = 0$ for all Λ in the support of $\overline{v}$.

Set $v_1 = v - \overline{v}$. Given $\Lambda \in \mathrm{Hom}\big(T, k(q)^*\big)$, $(B_i^m \overline{v})_\Lambda$ is a linear combination of terms of the form

$$x_i^{m_1} y_{\theta(i)}^{n_1} \cdots x_i^{m_r} y_{\theta(i)}^{n_r} v_{\Lambda'} \tag{4.3}$$

where $M = \sum m_i$, $N = \sum n_i$, $q^{M\alpha_i - N\alpha_{\theta(i)}}\Lambda' = \Lambda$, and $\Lambda' \in \mathrm{supp}(\overline{v})$. Using the defining relation (1.3) of Section 1, we have $x_i^{m_1} y_{\theta(i)}^{n_1} \cdots x_i^{m_r} y_{\theta(i)}^{n_r} \in U x_i^{M-N}$ for $M \geq N$. Hence (4.3) is zero whenever $M - N$ is large. It follows that $\mathrm{supp}\big((x_i t_{\theta(i)}^{-1})^m \overline{v}_1\big)$ has zero intersection with $\mathrm{supp}\, B_i^m \overline{v}$ for m large enough. As before, the supports of $(x_i t_{\theta(i)}^{-1})^m \overline{v}_1$ are disjoint for different choices of m and so $(x_i t_{\theta(i)}^{-1})^m \overline{v}_1 = 0$ for large m. Therefore for m large enough, $x_i^m v_\Lambda = (x_i t_{\theta(i)}^{-1})^m v_\Lambda = 0$ for all Λ in the support of $\overline{v}_1$. Repeat this argument using the fact that $\mathrm{supp}(v)$ is a finite set to show that $x_i^m v_\Lambda = (x_i t_{\theta(i)}^{-1})^m v_\Lambda = 0$ for very large m and for all $\Lambda \in \mathrm{supp}(v)$.

We have shown that for each $1 \leq i \leq l$, x_i acts nilpotently on Tv, the T-module generated by v, and hence on the T-module TW generated by W. A similar argument holds for y_i.

Recall that $F(U)$ denotes the (maximal) locally finite subalgebra of U (see [JL1]). Using the Hopf algebra structure on U, it is straightforward to check that $aF(U)TW = \big((\mathrm{ad}\, a_{(1)})F(U)\big)a_{(2)}TW$ for $a \in U$. In particular using standard Hopf algebra notation, properties of Hopf algebras (see for example [JL1, Section 2], and the definition of ad (see Section 1),

$$\begin{aligned}\big((\mathrm{ad}\, a_{(1)})F(U)\big)a_{(2)}TW &= a_{(1)}F(U)\sigma(a_{(2)})a_{(3)}TW \\ &= a_{(1)}F(U)\epsilon(a_{(2)})TW \\ &= a_{(1)}\epsilon(a_{(2)})F(U)TW = aF(U)TW.\end{aligned}$$

Given the image of x_i^m and y_i^m under Δ (see [JL1, Section 3.7]) and the fact that x_i and y_i act locally nilpotent on $F(U)$ using the adjoint action, it follows that x_i and y_i act locally

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nilpotent on $F(U)TW$. Clearly, T acts semisimply on $F(U)TW$. Now $F(U) * (TW) = \big(F(U)T\big) * W$ and $F(U)T = U$ by [JL1]. Hence $F(U)TW$ is just the U-module generated by W and the theorem follows from applying [JL1, Theorem 5.9]. ■

Putting Lemma 4.3 and Lemma 4.4 together yields:

THEOREM 4.5. *Assume $k = \mathbf{C}$. Let M be a U-module with a semisimple T action. Then the sum of all the finite-dimensional simple B_θ submodules is a (maximal) Harish-Chandra module for B_θ and equals the sum of all the finite-dimensional simple U submodules of M.*

PROOF. By Lemma 4.4, each simple finite-dimensional B_θ submodule is contained in a direct sum of finite-dimensional simple U submodules of M. By Lemma 4.3, each of these simple U-modules can be written as a direct sum of finite-dimensional simple B_θ-modules. Hence the sum of all the simple B_θ submodules of M is equal to the sum of all the simple finite-dimensional U submodules of M, which clearly is a U-module. It follows immediately that this sum is the maximal Harish-Chandra module for the pair U, B_θ contained inside M. ■

The locally finite part $F(U)$ of U can be realized as a direct sum of all the finite-dimensional $\operatorname{ad} U$ simple submodules of U. Hence Theorem 4.5 implies the following.

COROLLARY 4.6. *Assume $k = \mathbf{C}$. The $(\operatorname{ad} U)$-module $F(U)$ is the maximal Harish-Chandra module for the pair U, B_θ inside of U.*

REMARK 4.1. The results of this section can be extended to $F(\check{U})$ the locally finite part of the simply connected quantized enveloping algebra. For definition of the simply connected quantized enveloping algebra, see [JL2, Section 3.1]. The reader is also referred to comments following Remark 8.3 in [JL3].

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