A CRITERION FOR IRRATIONALITY

R. F. CHURCHHOUSE

1. Introduction. The question of the irrationality of functions defined by power series, for rational values of the variable, has attracted much attention for over a hundred years. Legendre, in generalizing Lambert's proof of the irrationality of tan x for rational x, proved an important theorem on the irrationality of continued fractions with integer elements. Here we use Legendre's theorem (Lemma 3) to prove that at least one of a certain pair of power series is irrational whenever the variable is rational and satisfies a further condition.

We prove the following:

THEOREM 1. *Let $\psi(n)$ be any positive, integral-valued strictly increasing function of n. Let*

$$F(x) = \sum_{p=0}^{\infty} a_p x^p, \quad G(x) = \sum_{p=0}^{\infty} b_p x^p,$$

where a_p is the number of partitions of p of the form

$$p = \psi(m_1) + \psi(m_2) + \ldots + \psi(m_k); \quad m_k > m_{k-1} > \ldots > m_1 = 1;$$
$$m_{i+1} - m_i \geqslant 2;$$

$b_0 = 1$, and for $p \geqslant 1$, b_p is the number of partitions of p of the form

$$p = \psi(m_1) + \psi(m_2) + \ldots + \psi(m_k); \quad m_k > m_{k-1} > \ldots > m_1 \geqslant 2;$$
$$m_{i+1} - m_i \geqslant 2.$$

Let

$$\gamma = \liminf_{n\to\infty} \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r).$$

Then if r, s are positive integers such that $(r, s) = 1$ and $r < s^\gamma$, the number

$$H(r/s) = F(r/s)/G(r/s)$$

is irrational.

2. Subsidiary results.

LEMMA 1. *$F(x)$ and $G(x)$ converge if $|x| < 1$.*

Proof. Since $\psi(n)$ is positive, integral-valued, and strictly increasing, $\psi(1) \geqslant 1, \psi(2) \geqslant 2, \ldots, \psi(n) \geqslant n$. Hence $a_n, b_n \leqslant p(n)$ where $p(n)$ denotes the number of partitions of n into the sum of positive integers. From Euler's product

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

Received January 31, 1952.

for $|x| < 1$, it follows that $F(x)$, $G(x)$ converge if $|x| < 1$. Henceforth we suppose that $0 < x < 1$.

LEMMA 2. *The function* $H(x) = F(x)/G(x)$ *has the continued fraction expansion*

$$H(x) = \frac{x^{\psi(1)}}{1+}\ \frac{x^{\psi(2)}}{1+}\ \frac{x^{\psi(3)}}{1+}\cdots.$$

Proof. Let $p_n(x)/q_n(x)$ denote the nth convergent to the continued fraction. Then, for $n \geqslant 3$:

$$\begin{aligned} p_n(x) &= x^{\psi(n)}p_{n-2}(x) + p_{n-1}(x), \\ q_n(x) &= x^{\psi(n)}q_{n-2}(x) + q_{n-1}(x), \end{aligned} \tag{1}$$

subject to:

$$\begin{aligned} p_1(x) &= p_2(x) = x^{\psi(1)}, \\ q_1(x) &= 1; \quad q_2(x) = 1 + x^{\psi(2)}. \end{aligned} \tag{2}$$

From (1) and (2) we see that $p_n(x)$ and $q_n(x)$ have no common factor. If $a_{r,n}$ denotes the coefficient of x^r in $p_n(x)$ it follows from (1) that

$$a_{r,n} = a_{r-\psi(n),n-2} + a_{r,n-1}\,. \tag{3}$$

Since the $a_{p,q}$ are clearly non-negative, (3) implies that

$$a_{r,n} \geqslant a_{r,n-1}\,. \tag{4}$$

On the other hand, if n_1 is that positive integer defined uniquely by the inequality

$$\psi(n_1) \leqslant r < \psi(n_1 + 1)\,, \tag{5}$$

on using (3), with $n_1 + 1$ in place of n, and noting that $a_{s,n} = 0$ if $s < 0$, we deduce that

$$a_{r,n_1+1} = a_{r,n_1}\,,$$

and in fact that

$$\begin{aligned} a_{r,n} &\leqslant a_{r,n+1}\,, && n < n_1, \\ a_{r,n} &= a_{r,n+1}\,, && n \geqslant n_1. \end{aligned} \tag{6}$$

Write

$$p_n(x) = \sum_{r=0}^{N} a_{r,n}\, x^r, \qquad N = N(n).$$

Repeated application of the recurrence formula (1) gives

$$p_n(x) = \theta_1(x)\, p_1(x) + \theta_2(x)\, p_2(x)\,,$$

where $\theta_1(x)$, $\theta_2(x)$ are polynomials in x. Since, from (2), $p_1(x) = p_2(x) = x^{\psi(1)}$ it follows that $x^{\psi(1)}$ is a factor of $p_n(x)$. This and the recurrence formula show that $a_{r,n}$ is equal to the number of decompositions of r in the form

$$r = \sum_{i=1}^{p} \psi(m_i),$$

where the m_i are positive integers satisfying

$$1 = m_1 < m_2 < \ldots < m_p \leqslant n, \quad m_{i+1} - m_i \geqslant 2.$$

Conversely, any decomposition of this type will contribute just to the coefficient of x^r in $p_n(x)$ and so $a_{r,n}$ is equal to the number of such decompositions. Then for $n \geqslant n_1$ defined by (5),

$$a_{r,n_1} = a_{r,n_1+1} = \ldots = a_{r,n} = \ldots = a_r,$$

where a_r is the coefficient of x^r in $F(x)$. Hence

$$p_n(x) = \sum_{r=0}^{N} a_{r,n}\, x^r = \sum_{r=0}^{M} a_r\, x^r + \sum_{r=M+1}^{N} a_{r,n}\, x^r,$$

where $M < N$ and $M \to \infty$ as $N \to \infty$; and so

$$|F(x) - p_n(x)| = \left| \sum_{r=M+1}^{\infty} c_r\, x^r \right|,$$

where $a_r \geqslant c_r \geqslant 0$ since $0 \leqslant a_{r,n} \leqslant a_r$. Consequently

$$|F(x) - p_n(x)| \leqslant \left| \sum_{r=M+1}^{\infty} a_r\, x^r \right| \to 0$$

as $M \to \infty$, and so $p_n(x) \to F(x)$. Similarly

$$\lim_{n\to\infty} q_n(x) = G(x).$$

We now enunciate Legendre's theorem [**5**]:

LEMMA 3. *If $m_1, m_2, \ldots$, and $n_1, n_2, \ldots$ are positive integers and $0 < m_i/n_i < 1$ for $i \geqslant i_0$, the continued fraction*

$$\frac{m_1}{n_1 +}\ \frac{m_2}{n_2 +}\ \frac{m_3}{n_3 +} \cdots$$

is irrational.

For a proof of this see, for example, [**3**]. This result has been improved upon, in particular by Bernstein and Szasz [**1**] who deduced the irrationality of the Jacobi theta series

$$\sum_{p=0}^{\infty} \left(\frac{r}{s}\right)^{p^2} \left(\frac{m}{n}\right)^{p}$$

when r, s, m, n are positive integers $s \geqslant 2$ and $0 < r^3 < s$. In the present paper we only make use of the result in its original form as no improvement is obtained by using any of the stronger forms.

3. Proof of Theorem 1. Let r, s be positive integers such that $(r, s) = 1$ and $r < s$. By Lemma 2,

$$H(x) = \frac{F(x)}{G(x)} = \frac{x^{\psi(1)}}{1 +}\ \frac{x^{\psi(2)}}{1 +}\ \frac{x^{\psi(3)}}{1 +} \cdots.$$

Put $x = r/s$. Then from the well-known equivalence

$$\frac{b_1}{a_1 +} \; \frac{b_2}{a_2 +} \; \frac{b_3}{a_3 +} \dots = \frac{c_1 b_1}{c_1 a_1 +} \; \frac{c_1 c_2 b_2}{c_2 a_2 +} \; \frac{c_2 c_3 b_3}{c_3 a_3 +} \dots \qquad (c_i \neq 0)$$

we deduce, on taking

$$a_n = 1, \quad b_n = \left(\frac{r}{s}\right)^{\psi(n)}, \quad c_1 = s^{\psi(1)},$$

and for $n \geqslant 1$,

$$c_n c_{n+1} = s^{\psi(n+1)},$$

that

$$H\left(\frac{r}{s}\right) = \frac{r^{\psi(1)}}{s^{\alpha_1} +} \; \frac{r^{\psi(2)}}{s^{\alpha_2} +} \; \dots \; \frac{r^{\psi(n)}}{s^{\alpha_n} +} \; \dots .$$

The successive exponents α_n satisfy

$$\alpha_1 = \psi(1), \quad \alpha_n = \psi(n) - \alpha_{n-1} \qquad (n \geqslant 2),$$

so that

$$\alpha_n = \sum_{r=0}^{n-1} (-1)^r \psi(n-r).$$

Now Lemma 3 is applicable if, for say $n \geqslant n_0$,

$$0 < r^{\psi(n)} < s^{\alpha_n}, \quad \text{that is,} \quad 0 < r < s^{\gamma_n},$$

where

$$\gamma_n = \alpha_n/\psi(n) = \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r) .$$

Let

$$\gamma = \liminf_{n\to\infty} \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r) .$$

Evidently $\gamma \leqslant 1$. Then $r < s^{\gamma_n}$ for $n > n_0$ if $r < s^{\gamma}$ and so

$$(r/s^{\gamma_n})^{\psi(n)} < 1.$$

This proves Theorem 1.

If $\psi(n) - \psi(n-1)$ is an increasing function we can prove that $\gamma \geqslant \frac{1}{2}$. For if $\phi(n) = \psi(n) - \psi(n-1)$ $(n \geqslant 2)$ and $\phi(1) = \psi(1)$, then

$$\psi(n) = \sum_{r=1}^{n} \phi(r) .$$

Now

$$\gamma = \liminf_{n\to\infty} \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n-r) ,$$

and so

$$\gamma = \liminf_{n\to\infty} \frac{\phi(n) + \phi(n-2) + \phi(n-4) + \dots}{\phi(n) + \phi(n-1) + \phi(n-2) + \dots + \phi(1)} = \liminf_{n\to\infty} \gamma_n,$$

say, the last term in the numerator being $\phi(2)$ or $\phi(1)$ according as n is even or odd. Since $\phi(n)$ is an increasing function of n,

$$\phi(n) + \phi(n-2) + \phi(n-4) + \ldots > \phi(n-1) + \phi(n-3) + \phi(n-5) + \ldots$$

and so $\gamma_n \geqslant \frac{1}{2}$ for

$$\sum_{r=1}^{n} \phi(r) = \sum_{k \geqslant 0}^{2k \leqslant n-1} \phi(n-2k) + \sum_{k \geqslant 0}^{2k \leqslant n-2} \phi(n-2k-1),$$

that is,

$$\sum_{r=1}^{n} \phi(r) \leqslant 2 \sum_{k \geqslant 0}^{2k \leqslant n-1} \phi(n-2k).$$

Hence

$$\gamma = \liminf_{n \to \infty} \gamma_n \geqslant \tfrac{1}{2}.$$

4. Applications. The most interesting application is obtained by taking $\psi(n) = n$. In this case we have

$$\frac{1}{1+H(x)} = \frac{1}{1+}\;\frac{x}{1+}\;\frac{x^2}{1+}\;\frac{x^3}{1+}\;\cdots \tag{7}$$

which is the Rogers-Ramanujan continued fraction [**6**; **7**]. Now

$$\gamma = \liminf_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} (-1)^r (n-r) = \tfrac{1}{2}. \tag{8}$$

The Rogers-Ramanujan identities give

$$1 + H(x) = \prod_{n=0}^{\infty} \frac{(1-x^{5n+2})(1-x^{5n+3})}{(1-x^{5n+1})(1-x^{5n+4})} \tag{9}$$

and from Theorem 1 we have at once:

THEOREM 2. *If r, s are positive integers satisfying $(r, s) = 1, r^2 < s$, the product*

$$\prod_{n=0}^{\infty} \frac{(s^{5n+2} - r^{5n+2})(s^{5n+3} - r^{5n+3})}{(s^{5n+1} - r^{5n+1})(s^{5n+4} - r^{5n+4})}$$

is irrational.

We now give another form of this result. Write

$$F(x) = \sum_{n=0}^{\infty} a_n x^n, \quad G(x) = \sum_{n=0}^{\infty} b_n x^n$$

where a_n is the number of partitions of n with least element 1 and minimal difference 2; $b_0 = 1$, and for $n \geqslant 1$, b_n is the number of partitions of n with least element not less than 2 and minimal difference 2. Then

$$F(x) + G(x) = \sum_{n=0}^{\infty} c_n x^n,$$

say, where $c_0 = 1$ and, for $n \geqslant 1$, c_n is the number of partitions of n with minimal difference 2. Now it is a consequence of the Rogers-Ramanujan identities [**4**] that the number of such partitions is equal to the number of partitions of n into parts of the forms $5m+1$ and $5m+4$ and so

$$(10)\qquad F(x)+G(x) = \sum_{n=0}^{\infty} c_n x^n = \prod_{n=0}^{\infty} (1-x^{5n+1})^{-1}(1-x^{5n+4})^{-1}.$$

Now, from (9)

$$(11)\qquad \prod_{n=0}^{\infty} \frac{(1-x^{5n+2})(1-x^{5n+3})}{(1-x^{5n+1})(1-x^{5n+4})} = 1+H(x) = \frac{F(x)+G(x)}{G(x)}$$

so that (10) and (11) enable us to determine $F(x)$ and $G(x)$ in terms of infinite products, namely,

$$G(x) = \prod_{n=0}^{\infty} (1-x^{5n+2})^{-1}(1-x^{5n+3})^{-1},$$

$$F(x) = \prod_{n=0}^{\infty} (1-x^{5n+1})^{-1}(1-x^{5n+4})^{-1} - \prod_{n=0}^{\infty} (1-x^{5n+2})^{-1}(1-x^{5n+3})^{-1}.$$

Next we note that (9) may be written in the form

$$(12)\quad 1+H(x) = \prod_{n=1}^{\infty} (1-x^n)(1-x^{5n})^{-1}\prod_{n=0}^{\infty} (1-x^{5n+1})^{-2}(1-x^{5n+4})^{-2},$$

and since the left side of (12) is irrational when $x = r/s$, $(r, s) = 1$, $r^2 < s$ at least one of the factors

$$\prod_{n=1}^{\infty} (1-x^n),\ \prod_{n=1}^{\infty} (1-x^{5n}),\ \prod_{n=0}^{\infty} (1-x^{5n+1})^{-2}(1-x^{5n+4})^{-2}$$

is irrational. The product

$$\prod_{n=1}^{\infty}(1-x^n)$$

is of importance in the theory of elliptic functions; it was considered by Euler [**2**] who proved that

$$\prod_{n=1}^{\infty} (1-x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}.$$

We see then that if we write $p'(n)$ to denote the number of partitions of n into parts of the forms $5m+1$ and $5m+4$ we have the following alternative to Theorem 2:

THEOREM 3. *If $x = r/s$ where r, s are positive integers satisfying $(r, s) = 1$, $r^2 < s$ then at least one of the numbers*

$$\sum_{n=1}^{\infty} p'(n)x^n,\quad \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2},\quad \sum_{n=-\infty}^{\infty} (-1)^n x^{5n(3n+1)/2}$$

is irrational.

Two other cases of interest are given by

(i) $\psi(n) = 2^{n-1}$,

(ii) $\{\psi(n)\} = \{1, 1, 2, 4, 6, 10, \ldots\}$, where $\psi(n+1) = \psi(n) + \psi(n-1)$ for $n \geqslant 4$.

https://doi.org/10.4153/CJM-1953-028-8 Published online by Cambridge University Press

In case (i) $F(x) = xG(x^2)$ and since

$$\gamma = \liminf_{n\to\infty} \sum_{r=0}^{n-1} (-1)^r 2^{-r} = \tfrac{2}{3}$$

we obtain the following:

THEOREM 4. *If $G(x)$ is the function defined in Theorem 1 when $\psi(n) = 2^{n-1}$ and r, s are positive integers satisfying $(r, s) = 1$, $r^3 < s^2$ then at least one of the numbers $G(r/s)$, $G(r^2/s^2)$ is irrational.*

In case (ii), $F(x) = x/(1 - x^2)$ and so is rational for rational x. It follows from Theorem 1 that $G(x)$ and also $F(x) + G(x)$ is irrational when $x = r/s$, $(r, s) = 1$, $r < s^\gamma$. As is easily seen,

$$\psi(n) = A(\alpha^{n-1} - \beta^{n-1}) \qquad (n \geqslant 3)$$

where $2\alpha = 1 + \sqrt{5}$, $2\beta = 1 - \sqrt{5}$, $2A^{-1} = \sqrt{5}$. Hence

$$\gamma_n = \frac{1}{\psi(n)} \sum_{r=0}^{n-1} (-1)^r \psi(n - r)$$

$$= \frac{(\alpha^{n-1} - \alpha^{n-2} + \alpha^{n-3} - \dots) - (\beta^{n-1} - \beta^{n-2} + \beta^{n-3} - \dots)}{\alpha^{n-1} - \beta^{n-1}}.$$

Divide throughout by α^{n-1}, then since $|\beta| < \alpha$, we have

$$\gamma_n \to 1 - \frac{1}{\alpha} + \frac{1}{\alpha^2} - \frac{1}{\alpha^3} + \dots = \frac{\alpha}{1 + \alpha} \qquad \text{as } n \to \infty$$

and so $\gamma = \alpha/(1 + \alpha) = -\beta$. Now

$$F(x) + G(x) = \sum_{p=0}^{\infty} (a_p + b_p)\, x^p = \sum_{p=0}^{\infty} c_p\, x^p$$

where $c_0 = 1$, and for $p \geqslant 1$, c_p is equal to the number of ways of expressing p as the sum of elements of the sequence 1, 1, 2, 4, 6, 10, 16, ... without repetitions and with no two consecutive elements occurring in the same decomposition. From Theorem 1 we have at once:

THEOREM 5. *If*

$$K(x) = \sum_{p=0}^{\infty} c_p\, x^p$$

is the power series just defined and r, s are positive integers satisfying $(r, s) = 1$, $r^2 < s^{\sqrt{5}-1}$ then the number $K(r/s)$ is irrational.

The details of the proofs of Theorems 4 and 5 are straightforward and I omit them.

I am indebted to Prof. L. J. Mordell and Prof. R. A. Rankin for their advice and criticism, and to the Department of Scientific and Industrial Research for a research grant.

References

1. F. Bernstein and O. Szász, *Über Irrationalität unendlicher Kettenbrüche*, Math. Ann., *76* (1915), 295–300.
2. L. Euler, *Introductio in analysin infinitorum* (Lausanne, 1748), 270.
3. H. S. Hall and S. R. Knight, *Higher algebra* (London, 1882), 366.
4. G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers* (Oxford, 1938), 289.
5. A. M. Legendre, *Géométrie et trigonométrie* (Paris, 1821), 290.
6. S. Ramanujan, *Collected papers* (Cambridge, 1927), 214–215.
7. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. (1), *25* (1894), 329.

Trinity Hall
Cambridge, England