ON (v, k, λ)-CONFIGURATIONS WITH $v = 4p^e$

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1. Introduction. A (v, k, λ)*-configuration*, also called a *symmetric balanced incomplete block design*, is an arrangement of v distinct objects called *points* or *varieties* into v subsets called *lines* or *blocks* such that each line contains exactly k points and each pair of distinct lines contains exactly λ points in common. To avoid certain trivial configurations, one assumes that $0 < \lambda < k < v-1$.

In this paper we show that a (v, k, λ)-configuration with v of the form $4p^e$ where p is a prime must have the parameters

$$(v, k, \lambda) = (4m^2, 2m^2 - m, m^2 - m), \tag{1}$$

where $m = \pm p^{\frac{1}{2}e}$. Hence, in particular, e must be even. This result was proved for $p = 2$ by Mann [**2**, pp. 72–73] or [**3**, p. 213]. The method of proof can be extended to determine all possible parameter values when v is of the form $2^d p^e$ and d is a small positive integer. We list the possible parameter values for $v = 2p^e$, $8p^e$ and $16p^e$.

The ambiguity in the sign of m in (1) arises from the fact that replacing m by $-m$ in (1) yields the parameters of the complementary (v, k, λ)-configuration, that is, the (v, k, λ)-configuration obtained by replacing each line by its complement.

The parameters (1) for arbitrary integral m are of some interest since the incidence matrix of such a (v, k, λ)-configuration yields a Hadamard matrix on replacing each 0 by -1 (see, e.g., Mann [**2**, p. 71]).

For further general information on (v, k, λ)-configurations and Hadamard matrices, see, e.g., Hall [**1**, Chapters 10 and 14].

2. Main result. We shall use the following two well-known facts concerning the parameters of a (v, k, λ)-configuration (see, e.g., Hall [**1**, Chapter 10]). First, the parameters are related by

$$k(k-1) = \lambda(v-1). \tag{2}$$

As is customary, set

$$n = k - \lambda. \tag{3}$$

Then (2) can be written as

$$n = k^2 - \lambda v. \tag{4}$$

Second, if v is even, then n is a square.

We now show that if the parameters of a (v, k, λ)-configuration satisfy $v = 4n$, then the parameters are of the form (1) for some integer m, a result first noted by Menon [**4**, pp. 739–740]. If $v = 4n$, then (3) and (4) yield

$$n = k^2 - \lambda v = (n+\lambda)^2 - 4\lambda n = (n-\lambda)^2.$$

Let $m = n - \lambda$. Then $n = m^2$, $\lambda = n - m = m^2 - m$, $k = n + \lambda = 2m^2 - m$ and $v = 4n = 4m^2$, as desired. Note that $m > 0$ if and only if $k < \frac{1}{2}v$.

THEOREM. *Suppose there exists a (v, k, λ)-configuration with v of the form $4p^e$, where p is a prime and e is a positive integer. Then e is even and the parameters are of the form*

$$(v, k, \lambda) = (4m^2, 2m^2-m, m^2-m),$$

where $m = \pm p^{\frac{1}{2}e}$.

Proof. Replace the (v, k, λ)-configuration by its complementary configuration if necessary so that

$$k < \tfrac{1}{2}v. \tag{5}$$

Since $v = 4p^e$ is even, $n = k-\lambda$ must be a square, say

$$n = p^{2f}n_1^2,$$

where n_1 is not divisible by p. First suppose that $2f \geqq e$. Then (3) and (5) yield

$$p^{2f}n_1^2 = n < k < \tfrac{1}{2}v = 2p^e. \tag{6}$$

Thus $1 \leqq p^{2f-e}n_1^2 < 2$, so that $2f = e$ and $n_1 = 1$. Therefore $v = 4p^{2f} = 4n$ and, as noted above, the parameters (1) result with $m = p^{\frac{1}{2}e}$. The complementary parameters with $k > \frac{1}{2}v$ correspond to $m = -p^{\frac{1}{2}e}$.

Now assume that

$$2f < e. \tag{7}$$

We complete the proof by showing that (7) is impossible. Substituting for n and v in (4) yields

$$p^{2f}n_1^2 = k^2-4p^e\lambda. \tag{8}$$

Hence

$$k = p^f k_1$$

for some integer k_1. Let

$$\lambda_1 = k_1 - p^f n_1^2. \tag{9}$$

Then

$$\lambda = k-n = p^f k_1 - p^{2f}n_1^2 = p^f\lambda_1.$$

Substituting the above expressions for k and λ in (8) yields

$$4p^{e-f}\lambda_1 = k_1^2 - n_1^2 = (k_1+n_1)(k_1-n_1). \tag{10}$$

Since $(k_1+n_1)-(k_1-n_1) = 2n_1$ and $p \nmid n_1$, (10) implies that

$$p^{e-f} \mid k_1+n_1 \quad \text{or} \quad p^{e-f} \mid k_1-n_1, \tag{11}$$

even if $p = 2$. Now

$$k_1 = k/p^f < \tfrac{1}{2}v/p^f = 2p^{e-f},$$

$$n_1 = \sqrt{n}/p^f < \sqrt{\tfrac{1}{2}v}/p^f = \sqrt{2}p^{\frac{1}{2}e-f} \leqq p^{e-f}. \tag{12}$$

Thus

$$0 < k_1+n_1 < 3p^{e-f}, \tag{13}$$

$$0 < k_1-n_1 < 2p^{e-f}. \tag{14}$$

Then (10), (11), (13) and (14) imply that either

$$p^{e-f} = k_1 \pm n_1, \quad 4\lambda_1 = k_1 \mp n_1 \tag{15}$$

or

$$2p^{e-f} = k_1 + n_1, \quad 2\lambda_1 = k_1 - n_1. \tag{16}$$

Suppose that (15) holds. Then eliminating k_1 yields

$$p^{e-f} \mp n_1 = k_1 = 4\lambda_1 \pm n_1,$$
$$p^{e-f} = 4\lambda_1 \pm 2n_1.$$

Therefore $2 \mid p$, and so $p = 2$. But $p \nmid n_1$, so that $e-f = 1$. Then (7) implies that $e = 1$, so that $v = 4p^e = 8$. There are no nontrivial parameter values satisfying (2) with $v = 8$. Suppose that (16) holds. First eliminate λ_1 using (9) and then eliminate the quantity $k_1 + n_1$.

$$k_1 - n_1 = 2\lambda_1 = 2(k_1 - p^f n_1^2),$$
$$2p^f n_1^2 = k_1 + n_1 = 2p^{e-f},$$
$$n_1^2 = p^{e-2f}.$$

But $p \nmid n_1$, so that $e = 2f$ in opposition to (7). This completes the proof.

3. Further results. As noted previously, Mann [2] or [3] has shown that a (v, k, λ)-configuration with v a power of 2 must have parameters of the form

$$(v, k, \lambda) = (2^{2f+2}, 2^{2f+1} \pm 2^f, 2^{2f} \pm 2^f). \tag{17}$$

G. F. Stahly has pointed out (written communication via a third party) that Mann's proof actually shows (on replacing most occurrences of 2 by p) that a (v, k, λ)-configuration with v of the form $2p^e$, where p is a prime, must have $p = 2$ and hence parameters of the form (17).

The proof of this paper can be extended (straightforward but tedious) to determine all possible parameter values of a (v, k, λ)-configuration when v is of the form $8p^e$ or $16p^e$, where p is a prime. The results are: If v is of the form $8p^e$, then the parameters are of the form (17), or up to complementation the (v, k, λ) parameters are (40, 13, 4) or (56, 11, 2). If v is of the form $16p^e$, then the parameters are of the form (1) with $m = \pm 2p^{\frac{1}{2}e}$ so that e must be even in this case, or up to complementation the parameters are one of the following.

v	k	λ	n
112	37	12	25
176	50	14	36
208	46	10	36
400	57	8	49
496	55	6	49
944	369	144	225
976	351	126	225
3888	507	66	441

In proving the theorem we used the inequality $n < \frac{1}{2}v$ in (6) and (12); in these further calculations it is better to use instead the inequality $n \leqq \frac{1}{4}(v+1)$ at the corresponding steps. This inequality is a consequence of the relation

$$\lambda(2n-1) \leqq n^2-n+\lambda^2 = \lambda(v-2n),$$

in which the inequality is equivalent to $n-\lambda \leqq (n-\lambda)^2$ and the equality follows from (2) and (3).

Note added in proof. A proof of G. F. Stahly's result that a (v, k, λ)-configuration with v of the form $2p^e$, where p is a prime, must have $p = 2$ can be found in J. F. Dillon, *Elementary Hadamard difference sets*, Ph.D. Thesis (University of Maryland, 1974).

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