On Finite Groups with Dismantlable Subgroup Lattices

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Abstract. In this note we study the finite groups whose subgroup lattices are dismantlable.

1   Introduction

The relation between the structure of a group and the structure of its lattice of subgroups constitutes an important domain of research in group theory. The topic has enjoyed a rapid development starting with the first half of the 20th century. Many classes of groups determined by different properties of partially ordered subsets of their subgroups (especially lattices of (normal) subgroups) have been identified. We refer to Suzuki's book [9], Schmidt's book [7], or the more recent book [11] by the author for more information about this theory.

A finite lattice L of n elements is called *dismantlable* if there is a chain $L_1 \subset L_2 \subset \cdots \subset L_n = L$ of sublattices of L such that $|L_i| = i$, for all $i = 1, 2, \ldots, n$ (see [5,6]). It is well known that every lattice with at most seven elements is dismantlable, while for every integer $n \geq 8$, there is a lattice of n elements that is not dismantlable. Two basic properties of these lattices are the following:

- The class of dismantlable lattices is closed under the formation of sublattices and homomorphic images.
- If a dismantlable lattice is not a chain, then it contains at least two incomparable doubly irreducible elements.

Several characterizations of dismantlable lattices are known. One of the most significant is given by Kelly and Rival [5] and uses some particular partially ordered sets, namely crowns. Recall that for an integer $n \geq 3$, a *crown* of order $2n$ is a poset $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ in which the only comparability relations are $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$, $y_i \geq x_{i+1}$ for all $i = 1, 2, \ldots, n-1$, and $x_1 \leq y_n$.

Theorem 1.1 *A finite lattice is dismantlable if and only if it contains no crown. In particular, a finite modular lattice is dismantlable if and only if it contains no crown of order* 6 *or, equivalently, no sublattice isomorphic to the boolean lattice* $\mathbf{2}^3$.

The connections between dismantlable lattices and planar lattices (*i.e.,* lattices having a planar diagram) are powerful (see [1,5]). In this way, every finite planar lattice

is dismantlable. The converse is not necessary true; more precisely, for every integer $n \geq 9$ there is an n-element dismantlable lattice that is not planar. However, for distributive lattices the above two concepts are equivalent.

Theorem 1.2 *A finite distributive lattice is dismantlable if and only if it is planar.*

Starting with the lattice-theoretical concept of "planar lattice" in [8], the classification of finite groups whose subgroup lattices are planar has been made.

Theorem 1.3 *A finite group G has planar subgroup lattice if and only if it satisfies one of the following properties (where p and q are primes, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$):*

(i) *G is cyclic of order p^n or $p^n q^m$;*
(ii) *G is a p-group lattice-isomorphic to $\mathbb{Z}_{p^m} \times \mathbb{Z}_p$;*
(iii) *G is dihedral of order 8 or quaternion of order 8 or 16;*
(iv) *$G = PQ$, where $P \trianglelefteq G$, $|P| = p$, Q is cyclic of order q^m, and $|Q : C_Q(P)| = q$;*
(v) *$G = PQ$, where $P \trianglelefteq G$, $|P| = p^2$, $|Q| = q$, and Q operates irreducibly on P.*

Inspired by the above classification and because dismantlable lattices and planar lattices are closely connected, the following question is very natural.

Question *Which are the finite groups G with dismantlable subgroup lattices?*

This suggests that we should consider the class $\mathcal{D}$ of groups satisfying the above property. Obviously, finite groups with at most seven subgroups are contained in $\mathcal{D}$, but at first sight it is difficult to determine all finite groups in $\mathcal{D}$. Therefore they must be investigated more carefully. Their study is the main goal of the current note.

Most of our notation is standard and will usually not be repeated here. Basic definitions and results on lattices and groups can be found in [2,3] and [4,10], respectively.

2 Main Results

We start with the following easy but important lemma.

Lemma 2.1 *Let G be a finite group in $\mathcal{D}$. Then all elements of G are of order p^n or $p^n q^m$, where p, q are primes and $n, m \in \mathbb{N}$.*

Proof Let $a \in G$. Then $L(\langle a \rangle)$ is a sublattice of $L(G)$ and, consequently, is dismantlable. Since $L(\langle a \rangle)$ is also distributive (see *e.g.,* [7, Corollary 1.2.4]), we infer that it is planar by Theorem 1.2. Moreover, Theorem 1.3 shows that $o(a) = |\langle a \rangle|$ is of type p^n or $p^n q^m$, as desired. ∎

It is clear that the groups A_n for $n \leq 4$ and S_n for $n \leq 3$ belong to $\mathcal{D}$. On the other hand, we can easily check that the subgroups $H_1 = \langle (12)(34) \rangle$, $K_1 = \langle (125), (12)(34) \rangle$, $H_2 = \langle (25)(34) \rangle$, $K_2 = \langle (143), (25)(34) \rangle$, $H_3 = \langle (13)(25) \rangle$, and $K_3 = \langle (15234), (13)(25) \rangle$ form a crown of order 6 in A_5, while the subgroups $H_1 = \langle (12) \rangle$, $K_1 = \langle (123), (12) \rangle$, $H_2 = \langle (13) \rangle$, $K_2 = \langle (134), (13) \rangle$, $H_3 = \langle (14) \rangle$, and

$K_3 = \langle (124), (14) \rangle$ form a crown of order 6 in S_4. Consequently, A_5 and S_4 are not contained in $\mathcal{D}$. These remarks lead to the following result.

Theorem 2.2 *The alternating group A_n is contained in $\mathcal{D}$ if and only if $n \leq 4$, while the symmetric group S_n is contained in $\mathcal{D}$ if and only if $n \leq 3$.*

Next we will focus on describing abelian groups G contained in $\mathcal{D}$. Since the subgroup lattice of an abelian group is modular, Theorem 1.1 implies that G belongs to $\mathcal{D}$ if and only if $L(G)$ does not contain sublattices isomorphic to the boolean lattice $\mathbf{2}^3$. We remark that for abelian p-groups this condition is equivalent to the fact that G has no section of type (p, p, p); *i.e.*, it is of rank ≤ 2.

Theorem 2.3 *A finite abelian group G belongs to $\mathcal{D}$ if and only if either it is cyclic and $|G| \in \{p^n, p^n q^m\}$, where p, q are primes and $n, m \in \mathbb{N}$, or G is a rank 2 abelian p-group.*

Proof The subgroup lattice of a cyclic group of order p^n or $p^n q^m$ is planar by Theorem 1.3 and therefore such a group is contained in $\mathcal{D}$. On the other hand, we have already seen that rank 2 abelian p-groups belong to $\mathcal{D}$.

Conversely, assume that G is contained in $\mathcal{D}$. Then Lemma 2.1 and the above remark show that either G is an abelian p-group of rank ≤ 2 or a direct product of two p-groups, say $G \cong G_1 \times G_2$ with $|G_1| = p^n$ and $|G_2| = q^m$, $n, m \in \mathbb{N}^*$. In the second case, if both G_1 and G_2 are cyclic, so is G, and we are done. Assume now that G_1 is not cyclic and take two maximal subgroups M and M' of G_1. Then it is easy to see that the subgroups $H_1 = M$, $K_1 = G_1$, $H_2 = M'$, $K_2 = M' \times G_2$, $H_3 = (M \cap M') \times G_2$, and $K_3 = M \times G_2$ form a crown of order 6 of $L(G)$, a contradiction. This completes the proof. ∎

An immediate consequence of Theorems 1.3 and 2.3 is the following corollary.

Corollary 2.4 *For every integer $n = p^m$, where p is a prime and $m \geq 4$, there is a non-planar dismantlable lattice of n elements that is isomorphic to the subgroup lattice of a certain finite abelian p-group.*

We also observe that almost the same thing can be said about nilpotent groups contained in $\mathcal{D}$.

Theorem 2.5 *Let G be a finite nilpotent group contained in $\mathcal{D}$. Then either G is cyclic and $|G| \in \{p^n, p^n q^m\}$, where p, q are primes and $n, m \in \mathbb{N}$, or G is a non-cyclic p-group that does not contain abelian sections of type (p, p, p).*

Unfortunately, we were not able to determine all non-cyclic p-groups with the above property. It is obvious that this is satisfied by rank 2 abelian p-groups, but several non-abelian p-groups also satisfy it, as the following theorem shows.

Theorem 2.6 *All finite p-groups having a cyclic maximal subgroup are contained in $\mathcal{D}$. In particular, the non-abelian p-groups $M(p^n)$, D_{2^n}, Q_{2^n} for $n \geq 3$, and QD_{2^n} for $n \geq 4$ are contained in $\mathcal{D}$.*

Proof Let $n \geq 3$ be an integer and denore by $\mathcal{G}_n$ the class of p-groups of order p^n possessing a cyclic maximal subgroup. This contain the cyclic p-group $\mathbb{Z}_{p^n}$, the abelian p-group $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$, as well as the non-abelian p-groups described exhaustively in [10, Theorem 4.1], II:

- the modular p-group

$$M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, \ y^{-1}xy = x^{p^{n-2}+1} \rangle,$$

  where $n \geq 4$ for $p = 2$,
- the dihedral group

$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, \ yxy = x^{-1} \rangle,$$

- the generalized quaternion group

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, \ yxy^{-1} = x^{2^{n-1}-1} \rangle,$$

- the quasi-dihedral group

$$QD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, \ yxy = x^{2^{n-2}-1} \rangle,$$

  where $n \geq 4$.

Notice that for every non-cyclic p-group $G \in \mathcal{G}_n$ we have $|G : \Phi(G)| = p^2$. Moreover, all maximal subgroups of a group in $\mathcal{G}_n$ belong to $\mathcal{G}_{n-1}$.

We will prove by induction on n that $\mathcal{G}_n$ is included in $\mathcal{D}$. Clearly, this holds for $n = 3$. Assume now that n is a minimal positive integer such that $\mathcal{G}_n \nsubseteq \mathcal{D}$ and take $G \in \mathcal{G}_n \smallsetminus \mathcal{D}$. Let $\{H_1, K_1, H_2, K_2, \ldots, H_m, K_m\}$ be a crown of order $2m$ in $L(G)$. By our assumption, we infer that there is no maximal subgroup M of G such that $K_i \subseteq M$ for all $i = 1, 2, \ldots, m$. Thus there exist at least two maximal subgroups M_1 and M_2 of G containing subgroups in the set $\mathcal{K} = \{K_1, K_2, \ldots, K_m\}$. Since every H_i is included in the intersection of two subgroups in $\mathcal{K}$, it follows that we can choose $H_{i_1} \neq H_{i_2}$, which are contained in $M_1 \cap M_2 = \Phi(G)$. In other words, the cyclic p-group $\Phi(G)$ contains two incomparable subgroups, a contradiction. ∎

An important class of finite nilpotent groups is constituted by hamiltonian groups, that is non-abelian groups all of whose subgroups are normal. The structure of such a group H is well known, namely $H \cong Q_8 \times \mathbb{Z}_2^n \times A$, where Q_8 is the quaternion group, $n \in \mathbb{N}$ and A is a finite abelian group of odd order. By Theorem 2.5 we easily infer that if H belongs to $\mathcal{D}$, then $n = 0$ and A is trivial. In this way, a nice characterization of Q_8 is obtained.

Corollary 2.7 *The quaternion Q_8 is the unique hamiltonian group contained in $\mathcal{D}$.*

In Theorem 1.3(iv) and (v), two types of semidirect products of orders pq^m and p^2q, respectively, have been presented. These belong to $\mathcal{D}$ because their subgroup lattices are planar. We end our note by studying the containment to $\mathcal{D}$ for another remarkable class of semidirect products, namely dihedral groups (notice that these are not of type (iv) or (v)). Recall that the dihedral group D_{2n} is the symmetry group of a regular polygon with n sides and it has the order $2n$. The most convenient abstract

description of D_{2n} is obtained by using its generators: a rotation x of order n and a reflection y of order 2. Under these notations, we have

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, \ yxy = x^{-1} \rangle.$$

The subgroup structure of D_{2n} is precisely known: for every divisor d or n, D_{2n} possesses a subgroup isomorphic to $\mathbb{Z}_d$, namely $\langle x^{\frac{n}{d}} \rangle$, and $\frac{n}{d}$ subgroups isomorphic to D_{2d}, namely $\langle x^{\frac{n}{d}}, x^{i-1}y \rangle$, $i = 1, 2, \ldots, \frac{n}{d}$.

Theorem 2.8   *The dihedral group D_{2n} is contained in $\mathcal{D}$ if and only if n is of type p^m, where p is a prime and $m \in \mathbb{N}$.*

Proof   Assume first that $D_{2n} \in \mathcal{D}$. By Lemma 2.1 we infer that $n = 2^k p^m$ with p an odd prime and $k, m \in \mathbb{N}$. If $m = 0$, then n is of the desired type. For $m \geq 1$ we will prove that $k = 0$. Suppose $k \geq 1$. Then it is easy to see that the subgroups $H_1 = \langle x^{2^{k-1}p^m} \rangle$, $K_1 = \langle x^{2^{k-1}p^m}, y \rangle$, $H_2 = \langle y \rangle$, $K_2 = \langle x^{2^k}, y \rangle$, $H_3 = \langle x^{2^k} \rangle$, and $K_3 = \langle x \rangle$ form a crown of order 6 in $L(D_{2n})$, a contradiction.

Conversely, we already know that $D_{2^{m+1}}$ belongs to $\mathcal{D}$ by Theorem 2.6. Therefore we can assume that p is odd. We will prove by induction on m that D_{2p^m} is also contained in $\mathcal{D}$. This is obviously true for $m = 1$ since D_{2p} and $\mathbb{Z}_p \times \mathbb{Z}_p$ are lattice-isomorphic. Let m be minimal with the property $D_{2p^m} \notin \mathcal{D}$ and take a crown of order $2r$ in $L(D_{2p^m})$, say $\{H_1, K_1, H_2, K_2, \ldots, H_r, K_r\}$. We observe that the maximal subgroups of D_{2p^m} are $\langle x \rangle \cong \mathbb{Z}_{p^m}$ and $\langle x^p, x^{i-1}y \rangle \cong D_{2p^{m-1}}$, $i = 1, 2, \ldots, p$. Since by inductive hypothesis all K_i's cannot be contained in the same maximal subgroup of D_{2p^m} and by using a similar argument as in the proof of Theorem 2.6, one obtains that there are $H_i \neq H_j$ included in $\Phi(D_{2p^m})$. But $\Phi(D_{2p^m}) = \langle x^p \rangle \cong \mathbb{Z}_{p^{m-1}}$ is cyclic, a contradiction. Hence $D_{2p^m} \in \mathcal{D}$.   ∎

Finally, we indicate a natural open problem concerning the above study.

Open problem   Characterize *arbitrary* finite groups G contained in $\mathcal{D}$. Is it true that all these groups are metacyclic?

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