ON DIVISIBILITY AND INJECTIVITY

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1. Introduction. In the category of abelian groups the concepts of divisible group and injective group coincide. In **(4)** this is generalized to modules over an integral domain and it is proved for a (commutative) integral domain that the concepts of divisible module and injective module coincide if and only if the ring is hereditary if and only if the ring is a Dedekind domain.

In **(8)** the assumption of commutativity is dropped and the ring is assumed to have a one-sided field of quotients. The result obtained is essentially the same as in the commutative case; see the theorem following 6.2. In **(13)** the requirement that the ring have no zero-divisors is also dropped and the ring is assumed to possess what we have called an Ore ring (see the definition following 6.2). The result obtained is the equivalent of parts (a) and (b) of our 6.13.

Basic to these results are the concepts of "quotient ring" and "invertible ideal" and the relation of "invertible" to "projective." We replace the concept of "quotient ring" by the generalized localizations of **(7)** and approach the concept of "invertible ideal" from two directions, that of a "projective basis" (see 2.5) and that of an "effacement" (see 5.1).

Another problem is the definition of "divisible module" for general rings. In **(4, 8, 11,** and **13)** a (right) A-module M is called divisible if

$$\operatorname{Ext}_A{}^1(A/I, M) = 0$$

for a certain set of right ideals I. In **(4, 8,** and **11)** this is the set of principal right ideals and in **(13)** it is the set of principal right ideals generated by a non-zero divisor. It follows that the set of divisible modules is closed under taking quotient modules and infinite direct sums. In §3 we find the sets of right ideals that give such a result; see 3.2, the proof of which is adapted from **(11** and **1)**.

In §4 we associate a concept of torsion and divisibility to every general localization and give some results (4.4, 4.5, and 4.6). Since the definitions of **(11** and **13)** do not fall under those of §4, we treat them in §6 after having derived a formula for the purpose in §5 which we call the "effacement condition."

We list some notations which will be constantly used. Wherever possible we refer to **(7)**. A shorter development of the generalized localization may be found in **(2**, pp. 157–166**)**.

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Throughout, A will be a ring with a unit element and mod A will denote the category of right A-modules and A-module homomorphisms. The symbols "$M \in \text{mod } A$" and "ϕ: $M \to N \in \text{mod } A$" mean that M is a right A-module and ϕ is a homomorphism of right A-modules. If F is a topologizing and idempotent set of right ideals of A (**7** or **2**), we let $\mathbf{F}$ denote the corresponding localizing subcategory of mod A and $M \rightarrow M_F$ the corresponding localization functor. The symbol FM denotes the unique largest submodule of M that belongs to $\mathbf{F}$. It is the kernel of the canonical map j_M: $M \to M_F \in \text{mod } A$. Given F, F' and $\mathbf{F}'$ refer to the localization induced by F on the category mod A_F, i.e. $\mathbf{F}' = \mathbf{F} \cap \text{mod } A_F$. We shall often make use of the following known result, the proof of which may be found in (**14**) or, more generally, in (**17**).

1.1. Proposition. *Let F be a topologizing and idempotent set of right ideals of the ring A. The following statements are equivalent:*

(1) *The functor $M \rightarrow M_F$ is naturally equivalent to the functor $M \rightarrow M \otimes_A A_F$.*

(2) $\mathbf{F}'$ *contains only the zero module.*

(3) *All right A_F-modules are F-closed* (*i.e. localized*).

(4) *The localization functor is exact and commutes with infinite direct sums.*

By *Goldie's theorem* we mean Theorems 4.1 and 4.4 of (**9**). This usage differs from that of (**7** and **2**).

2. Injectivity of F-closed modules. In this section we investigate what happens when all localized A-modules are injective as A-modules and give a result on right hereditary and right noetherian rings. By (**7**, p. 413, Corollaire), this is equivalent to all localized A-modules being injective as A_F-modules.

2.1. Proposition. *Assume that M_F is injective for all $M \in$* mod A. *Then the localization functor is exact.*

Proof. By (**7**, p. 377, Prop. 7), it is sufficient to show that if $I \in \text{mod } A$ is an injective such that $FI = 0$ and N is an F-closed submodule of I, then I/N is an F-closed module. Now N F-closed means N is isomorphic to N_F and so N is injective. Thus I is isomorphic to $N \oplus I/N$, which shows that I/N is injective and $F(I/N) = 0$. Thus N is F-closed by (**7**, p. 370, Lemma 1).

Recall that right ideal I of A is called *large* if the only right ideal J of A such that $I \cap J = 0$ is the zero ideal. We always denote the set of large right ideals of A by Λ.

2.2. Proposition. *The following statements are equivalent:*

(1) *M_F is injective for all $M \in$* mod A.

(2) *A_F has no large F-closed right ideals.*

(3) *Every F-closed right ideal of A_F is a direct summand of A_F considered as a right A_F-module.*

(4) *F' is the set of large right ideals of A_F.*

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Proof. Clearly (1) implies (3) and (3) implies (2). Assume that (2) holds. Let $I = I_F$ be an F-closed right ideal of A_F. Let J be a complement of I in A_F (i.e. J is maximal with respect to the property $I \cap J = 0$; see (**7** or **9**)). Then $I + J = I \oplus J$ is large and hence $I + J \subset (I \oplus J)_F = I_F \oplus J_F = A_F$ since $I_F + J_F$ is large. Thus (3) holds.

If (3) holds, let $f: I \to M_F \in \text{mod } A_F$. Since $FA_F = 0$ and the functor is left exact, we have $I \subset I_F$. Thus $f_F: I_F \to M_F \in \text{mod } A_F$ extends f. Since I_F is a summand of A_F, this f_F extends to a mapping $A_F \to M_F \in \text{mod } A_F$. Thus M_F is A_F-injective and hence A is injective. Thus (1), (2), and (3) are equivalent.

Again assume that (2) holds. Let I be a large right ideal of A_F. Then $I_F \supset I$ is also a large right ideal and hence $I_F = A_F$ and thus $I \in F'$. Conversely, $I \in F'$ implies $I_F = A_F$ and, since I_F is an essential extension of I, I is large. Thus (2) implies (4). Conversely, suppose (4) holds. Let $I \in F'$; then if I is F-closed we have $I = I_F = A_F$ and so I is not a proper ideal.

It is well known (**4**, p. 11) that if all right modules of a ring are injective, then the ring is semi-simple. However, not all A_F-modules are localized A-modules and so the conditions of 2.2 do not imply that A_F is semi-simple. The proof of (**7**, p. 417, Lemma 4) shows that the conditions of 2.2 do imply, however, that A_F is a self-injective von Neumann ring.

2.3. Proposition.

(1) *Assume that M_F is injective for all $M \in$ mod A. Then A_F is semi-simple if and only if the localization functor is isomorphic to the functor $M \to M \otimes_A A_F$.*

(2) *Assume that A_F is semi-simple; then the localization functor is isomorphic to the functor $M \to M \otimes_A A_F$ and M_F is injective for all $M \in$ mod A.*

Proof. This follows easily from 1.1.

Recall that $M \in \text{mod } A$ is projective if and only if every generating set of M defines a *projective basis*. That is, a generating set $\{m_\beta : \beta \in B\}$ together with a set of A-homomorphisms $\phi_\beta : M \to A$ such that all $m \in M$ the set $\{\phi_\beta(m) : \beta \in B\}$ contains only a finite number of non-zero elements and $m = \sum_\beta m_\beta \phi_\beta(m)$; see (**4**, p. 132).

The next proposition will be used in §6 to generalize a theorem of Gentile. The corollary indicates why hereditary rings are often noetherian (e.g. integral domains) and thus why our results state "hereditary *and* noetherian" where the familiar results merely state "noetherian"; cf. (**4**, p. 132).

2.4. Proposition. *Assume that $FA = 0$. Then $I \in F$ is projective if and only if given $\{x_\beta : \beta \in B\}$, a set of generators of I, there is a set $\{q_\beta : \beta \in B\} \subset A_F$ such that $q_\beta I \subset A$ for all $\beta \in B$ and, for $x \in I$, the set $\{q_\beta x : \beta \in B\}$ contains only a finite number of non-zero elements and $x = \sum_\beta x_\beta q_\beta x$;* cf. (**4** *and* **13**).

Proof. Suppose the latter condition holds. Take $\phi_\beta x = q_\beta x$. Then $\{x_\beta, \phi_\beta\}$ gives a projective basis for I. Conversely, assume that I is projective. Then for each $\beta \in B$ we have an A-module mapping $\phi_\beta: I \to A$ such that

$$x = \sum_\beta x_\beta \phi_\beta(x)$$

for all $x \in I$. Then, since $I \in F$ implies $I_F = A_F$, we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & A_F \\ & & \Big\downarrow \phi_\beta & & \Big\downarrow (\phi_\beta)_F \\ 0 & \longrightarrow & A & \longrightarrow & A_F \end{array}$$

Thus let $q_\beta = (\phi_\beta)_F(1)$.

2.5. COROLLARY. *Assume that $FA = 0$ and $I \in F$.*

(1) *I is projective and finitely generated if and only if there are $x_1, \ldots, x_n \in I$ and $q_1, \ldots, q_n \in A_F$ such that $q_i I \subset A$ for $i = 1, \ldots, n$ and $1 = \sum x_i q_i$.*

(2) *If I is projective and contains an element that is not a right zero divisor in A_F, then I is finitely generated.*

Proof. (1) Assume I projective and finitely generated. Then, by 2.4, we have the x_i and q_i such that $x = \sum_1^n x_i q_i x$ for all $x \in I$. Then

$$(1 - \sum x_i q_i)I = 0.$$

Thus $1 - \sum x_i q_i \in FA_F = 0$. The converse is clear.

(2) Let $y \in I$ be such that $a \in A_F$ and $ay = 0$ implies $a = 0$. Then if I is projective, $y = \sum_\beta x_\beta q_\beta y$ shows that

$$1 = \sum_1^n x_{\beta_i} q_{\beta_i}$$

where the β_i's are such that $q_{\beta_i} y \neq 0$.

Now let Λ denote the set of large right ideals of the ring A and let

$$Z_r(A) = \{a \in A: (0: a) \in \Lambda\}.$$

The two-sided ideal $Z_r(A)$ is called the right singular ideal of A and it is well known that of $Z_r(A) = 0$, then Λ is topologizing and idempotent; see (**7**, p. 416). In (**11**, Lemma 2) it is shown that a principal right ideal aA of A is projective if and only if there is an idempotent $e_a \in A$ such that $ae_a = a$ and $(0: a) = (0: e_a) = (1 - e_a)A$. Combining these facts, we obtain

2.6. LEMMA. *Let A be a ring for which every principal right ideal is projective. Then the set of large right ideals of A is topologizing and idempotent.*

Proof. We need only show that $Z_r(A) = 0$. Notice that if $e^2 = e \in Z_r(A)$, then $(0: e) = (1 - e)A \in \Lambda$. But $(1 - e)A$ is a direct summand of A. Thus $(0: e) = A$ and so $e = 0$. Now assume that $a \in Z_r(A)$. Then there is an idempotent e_a such that $ae_a = a$ and $(0: e_a) = (0: a) \in \Lambda$. Thus $e_a \in Z_r(A)$, which implies that $e_a = 0$ and hence $a = 0$.

Hattori **(11)** has studied such rings in connection with his theory of divisibility.

In **(7**, pp. 416 ff.**)** it is proved that if A is a ring such that Λ is topologizing and idempotent, then every Λ-closed right ideal of A is a direct summand of A. Thus by 2.2, M_Λ is injective for all $M \in \text{mod } A$. Combining this with 2.3, 2.5, and 2.6, we obtain

2.7. THEOREM. *Let A be a right hereditary and right noetherian ring. Then the set Λ of large right ideals of A is topologizing and idempotent. The ring A is semi-simple; M_Λ is injective for all $M \in$ mod A; the localization may be identified with the functor $M \to M \otimes_A A$; and if $I \in \Lambda$ is generated by $x_1, \ldots, x_n$, then there are $q_1, \ldots, q_n \in A$ such that $q_i I \subset A$ for $i = 1, \ldots, n$ and*

$$1 = \sum_1^n x_i q_i.$$

3. Restricted injectivity and divisibility.

Definitions. (1) Let $\mathfrak{A}$ be a set of right ideals of a ring A. A module $M \in$ mod A is called $\mathfrak{A}$-*injective* if for each $I \in \mathfrak{A}$ and each $f: I \to M \in$ mod A, f may be extended to a map $A \to M \in$ mod A.

(2) A monomorphism $j: M \to N \in$ mod A is called an $\mathfrak{A}$-*effacement* if for $I \in A$ and every $f: I \to M \in$ mod A we may find a $g: A \to N \in$ mod A such that $g/I = j \cdot f$. When $\mathfrak{A}$ is the set of all right ideals of A an $\mathfrak{A}$-effacement is called an *injective effacement*; cf. **(10**, p. 135**)**.

Such restricted types of injectivity have been used by Hattori **(11)** and Levy **(13)** as definitions of "divisible module." Hattori takes $\mathfrak{A}$ to be the set of principal right ideals and Levy takes $\mathfrak{A}$ to be the set of all right ideals generated by a regular element (i.e. neither a left nor a right zero divisor).

The usual arguments show that a direct product of modules is $\mathfrak{A}$-injective if and only if each factor is $\mathfrak{A}$-injective; $M \in$ mod A is $\mathfrak{A}$-injective if and only if $\text{Ext}_A{}^1 (A/I, M) = 0$ for all $I \in \mathfrak{A}$; and a monomorphism $i: M \to N$ is an $\mathfrak{A}$-effacement if and only if $\text{Ext}_A{}^1 (A/I, i) = 0$ for all $I \in \mathfrak{A}$. The construction of **(4**, p. 9**)** also shows that every $M \in$ mod A may be embedded in an $\mathfrak{A}$-injective module. For later use we single out a portion of this statement.

3.1. PROPOSITION. *Let $\mathfrak{A}$ be a set of right ideals of the ring A and let $M \in$ mod A. Then there is an $\mathfrak{A}$-effacement $i: M \to \mathfrak{A}(M) \in$ mod A.*

Proof. Let $\Phi = \{(I,f): I \in \mathfrak{A}, f: I \to M \in \text{mod } A\}$. Let F be the free right A-module on Φ. Let $\mathfrak{A}(M)$ be the module $M \oplus F$ modulo the submodule consisting of all finite sums of elements of the form $(f(a), -(I,f)a)$ where $(I,f) \in \Phi$ and $a \in I$. Let $i: m \to \overline{(m, 0)}$ be the composite of the natural maps $M \to M \oplus F \to \mathfrak{A}(M)$. Now i is a monomorphism, for assume that

$$i(m) = 0 \in \mathfrak{A}(M).$$

Then in $M \oplus F$, $(m, 0) = \sum (f_i a_i, -(I_i, f_i)a_i)$. Thus $\sum (I_i, f_i)a_i = 0 \in F$. Since Φ is a free basis for F, this shows that $m = \sum f_i a_i = \sum f_i(0) = 0$.

Further, $i: M \to \mathfrak{A}(M)$ is an $\mathfrak{A}$-effacement; let $f: I \to M$ where $I \in \mathfrak{A}$. If $a \in I$, then

$$i \cdot f(a) = \overline{(fa, 0)} = \overline{(0, (I, f)a)} = \overline{(0, (I, f))}a.$$

Thus define $g: A \to \mathfrak{A}(M)$ by $g(b) = \overline{(0, (I, f))}b$. Then $g/I = i \cdot f$, which completes the proof.

If $\mathfrak{A}$ is actually a topologizing and idempotent set of right ideals of A, we may imitate the usual results for injective modules (indeed, we even have an "$\mathfrak{A}$-injective hull"; see **(17)**.) The usual proofs require only slight modification. We shall need the following statements from **(7)**.

3.2. PROPOSITION. *Let F be a topologizing and idempotent set of right ideals of A. For $M \in$ mod A the following statements are equivalent:*

(a) *M is F-injective.*

(b) *If $\alpha: P \to Q \in$ mod A is a monomorphism such that* coker $\alpha \in \mathbf{F}$ *and if $f: P \to M \in$ mod A, then there is a $g: Q \to M \in$ mod A such that $f = g \cdot \alpha$.*

(c) *Every exact sequence $0 \to M \to N \to P \to 0$ in* mod A *with $P \in \mathbf{F}$ splits.*

In integral domains a divisible module is an $\mathfrak{A}$-injective module where $\mathfrak{A}$ consists of all principal ideals. The class of divisible modules, however, is closed under arbitrary direct sums and epimorphisms. This is not generally true of injective modules. We wish to define "divisibility" as $\mathfrak{A}$-injectivity in such a manner that the "divisible" modules are closed under arbitrary direct sums and epimorphisms. The next proposition shows how this may be accomplished. The divisibilities thus defined include those used in **(4, 11**, and **13)** but not that of **(14)** nor the "h-divisibility" of **(15)**.

3.3. PROPOSITION. *Let $\mathfrak{A}$ be a set of right ideals of the ring A. Let $\mathfrak{C}$ be the class of $\mathfrak{A}$-injective right A-modules.*

(1) *$\mathfrak{C}$ is closed under epimorphisms if and only if every element of $\mathfrak{A}$ is projective.*

(2) *$\mathfrak{C}$ is closed under epimorphisms and arbitrary direct sums if and only if every element of $\mathfrak{A}$ is projective and finitely generated.*

Proof. In **(11**, Prop. 7**)** Hattori proves (1) for the set $\mathfrak{A}$ of principal right ideals. The proof for arbitrary sets $\mathfrak{A}$ is the same. Thus we assume (1) and prove (2). This proof is an adaptation of a proof in **(1)**.

Suppose $\mathfrak{A}$ consists of projective and finitely generated ideals. Then $\mathfrak{C}$ is closed under epimorphisms. Let $M_\lambda \in \Lambda$ for each λ in some index set Λ. Let $M = \oplus M_\lambda$. Let $I \in \mathfrak{A}$ be generated by $x_1, \ldots, x_n$ and let $\phi: I \to M \in$ mod A. Now, for each i, $\phi(x_i)$ is contained in the sum of a finite number of components of M. Thus $\phi(I)$ is contained in the sum of a finite number of components of M. This latter sum is $\mathfrak{A}$-injective since $\mathfrak{C}$ is always closed under direct products. Thus ϕ may be extended to a map $A \to M$.

Conversely, assume that $\mathfrak{C}$ is closed under arbitrary direct sums. Let $I \in \mathfrak{A}$ and let J be a countably generated direct summand of I. We claim that J must be finitely generated. For let $x_1, x_2, \ldots$ be a countable generating set for J. Let Q_i be an injective module containing J modulo the right ideal generated by $x_1, \ldots, x_i$. Then $Q = \oplus_{i=1}^{\infty} Q_i$ is $\mathfrak{A}$-injective, and thus is clearly $\{J\}$-injective. Define $\phi: J \rightarrow Q$ as follows. Let ϕ_i be the composition

$$J \rightarrow J/\sum_{k=1}^{i} x_k A \rightarrow Q_i.$$

Now if $x \in J$, then there is an integer n such that

$$x \in \sum_{k=1}^{n} x_k A.$$

Thus $\phi_i(x) = 0$ for $i > n$. Hence the map $\phi(x) = (\phi_1(x), \phi_2(x), \ldots)$ is a well-defined map $J \rightarrow Q$ and thus it extends to a map $\tilde{\phi}: A \rightarrow Q$. Now there is an integer s such that $\tilde{\phi}(A) = \tilde{\phi}(1)A \subset Q_1 \oplus \ldots \oplus Q_s$. Hence

$$\phi(J) \subset Q_1 \oplus \ldots \oplus Q_s,$$

which shows that J is generated by $x_1, \ldots, x_s$.

Now assume that $\mathfrak{A}$ is also closed under epimorphisms. Then I is projective and a result of Kaplansky **(12)** shows that $I = \oplus_\Lambda J_\lambda$ where each J_λ is countably generated. Thus each J_λ is finitely generated. We claim that Λ is a finite index set. For suppose Λ is infinite. Then there is a proper subset $B \subset \Lambda$ such that B is countably infinite. Then $J = \oplus_B J_\lambda$ is a countably generated direct summand of I that is not finitely generated, which is impossible.

Definition. Let $\mathfrak{A}$ be a set of right ideals of A. We say that $M \in \text{mod } A$ is $\mathfrak{A}$-*divisible* if M is $\mathfrak{A}'$-injective where $\mathfrak{A}'$ is the set of projective and finitely generated elements of $\mathfrak{A}$.

4. F-divisibility and F-injectivity. Throughout this section F will denote a topologizing and idempotent set of right ideals of the ring A.

Definition. We say that right A-module M is *F-torsion free* if $FM = 0$ and *F-torsion* if $FM = M$.

Note that if A is an integral domain and F is the set of all non-zero ideals of A, then F-injective, F-divisible, F-torsion, and F-torsion-free become the usual notions of injective, divisible, torsion, and torsion free; see **(4**, p. 133, Prop. 3.4**)**.

3.2, of course, gives the following statement.

4.1. THEOREM. *Let F be a topologizing and idempotent set of right ideals of the ring A. Then every F-divisible right A-module is F-injective if and only if the elements of F are all projective and finitely generated.*

We next show how the localized modules are characterized by our definitions.

4.2. PROPOSITION. *Let $M \in \operatorname{mod} A$. Then M is F-closed if and only if M is F-torsion-free and F-injective.*

Proof. By (**7**, p. 370, lemma 1) an F-closed module is F-torsion-free and F-injective. Conversely, if M is F-torsion-free and F-injective, then (**7**, p. 370, lemma 1) again shows that M is F-closed since the extension given by 3.2(b) must then be unique.

4.3. LEMMA. *Arbitrary direct sums of F-closed modules are F-closed if and only if the functor $M \to M_F$ commutes with infinite direct sums.*

Proof. Assume that the direct sum of F-closed modules is again F-closed. Let $\{M_\lambda : \lambda \in \Lambda\}$ be a collection of A-modules. For each $\lambda \in \Lambda$ we have the canonical exact sequence

$$0 \to FM_\lambda \to M_\lambda \xrightarrow{j_\lambda} (M_\lambda)_F.$$

This gives the exact sequence

$$0 \to \oplus_\Lambda FM_\lambda \to \oplus_\Lambda M_\lambda \xrightarrow{\sum j_\lambda} \oplus_\Lambda (M_\lambda)_F$$

and hence

$$0 \to (\oplus_\Lambda M_\lambda)_F \to \oplus_\Lambda (M_\lambda)_F$$

where the map is $(\sum j_\lambda)_F$, To show that this map is onto, we find an inverse map. The diagram

$$\begin{tikzcd} M_\lambda \arrow[r, "i_\lambda"] \arrow[d, "j_\lambda"'] & \oplus_\Lambda M_\lambda \arrow[d, "(\sum j_\lambda)_F"] \\ (M_\lambda)_F \arrow[r, "(i_\lambda)_F"] & (\oplus_\Lambda M_\lambda)_F \end{tikzcd}$$

where i_λ is the canonical inclusion commutes and thus we have a map

$$\sum (i_\lambda)_F : \oplus_\Lambda (M_\lambda)_F \to (\oplus_\Lambda M_\lambda)_F.$$

Since $M \to M_F$ is a functor, we have

$$(\textstyle\sum_\lambda j_\lambda)_F \cdot (\sum_\mu (i_\mu)_F) = \sum_\mu [(\sum_\lambda j_\lambda)_F \cdot (i_\mu)_F] = \sum_\mu (\sum_\lambda j_\lambda \cdot i_\mu)_F = \sum_\mu (j_\mu)_F$$

which is the identity on $\oplus_\Lambda (M_\lambda)_F$. Thus the localization commutes with direct sums.

Conversely, let $\{M_\lambda : \lambda \in \Lambda\}$ be a collection of F-closed modules. Then

$$(\oplus M_\lambda)_F \simeq \oplus (M_\lambda)_F \simeq \oplus M_\lambda.$$

Thus $\oplus M_\lambda$ is F-closed.

If A is an integral domain, then an A-module is torsion free and divisible if and only if it is a vector space over the quotient field of A. The next result shows when this is true for general localizations.

4.4. THEOREM. *Let F be a topologizing and idempotent set of right ideals of the ring A. The following statements are equivalent:*

(a) *Every F-torsion-free and F-divisible right A-module is an F-injective module.*

(b) *The localization functor may be taken to be $M \to M \otimes_A A_F$ and if M is F-torsion-free and F-divisible, then $M \in \text{mod } A_F$.*

(c) *M is F-torsion-free and F-divisible if and only if $M \in \text{mod } A_F$.*

Proof. We first show that (a) implies (b). To show that the localization functor may be taken to be $M \to M \otimes_A A_F$, it is sufficient to show that it is exact and commutes with direct sums by 1.1. Let $\{M_\lambda : \lambda \in \Lambda\}$ be a collection of F-closed modules. Then $\oplus M_\lambda$ is F-divisible and F-torsion-free and hence is F-torsion-free and F-injective, i.e. F-closed. Thus by 4.3, the functor commutes with direct sums. To show that the functor is exact, it is sufficient to show that every exact sequence

$$0 \to N \to M \xrightarrow{p} M/N \to 0$$

with N and M F-closed has M/N F-closed; see (**7**, p. 376, Prop. 7). We first show that M/N is F-torsion-free. Let $N' = p^{-1}[F(M/N)]$. Since N is F-injective, 3.2 shows that the sequence

$$0 \to N \to N' \to F(M/N) \to 0$$

splits. Thus $F(M/N)$ is isomorphic to a submodule of M and so $F(M/N) = 0$. Now M is F-divisible and hence M/N is F-divisible. Thus, by (a), M/N is F-closed. If M is F-closed, then M is an A_F-module by means of the isomorphism $j_M : M \to M_F$.

Now assume that (b) is true. Since the localization functor is $M \to M \otimes_A A_F$, 1.1 shows that $\mathbf{F}'$ contains only the zero module. Thus mod A_F contains only F-closed modules which are, of course, F-torsion-free and F-divisible.

On the other hand, if (c) is true, then $\mathbf{F}'$ contains only the zero module and (b) is immediate. If (b) is true, then an F-torsion-free and F-divisible module is an A_F-module and thus, since $\mathbf{F}'$ contains only the zero module, is F-closed.

The next two results generalize the fact that over integral domains the only localizations that always give injective modules are $M \to M \otimes_A Q$ (Q the quotient field of A) and $M \to (0)$ for all A-modules M.

4.5. PROPOSITION. *Let F be a topologizing and idempotent set of right ideals of A. Every F-torsion-free and F-divisible module is injective if and only if every F-torsion free and F-divisible module is F-injective and A_F is a semi-simple ring.*

Proof. Assume that every F-torsion-free and F-divisible module is injective. Then 4.4 and 2.3 show that A_F is semi-simple. Conversely, if every F-torsion-free and F-divisible module is F-injective, then 4.4(c) shows that every

F-torsion-free and F-divisible module is A_F-injective. Since such a module is also F-closed by 4.2, we see that it is A-injective (see the remarks beginning §2),

4.6. Theorem. *Let F be a topologizing and idempotent set of right ideals of the ring A. Then every F-divisible right A-module is injective if and only if A is a right hereditary, right noetherian ring and every large right ideal of A is an element of F. Furthermore, when these conditions hold, the set Λ of large right ideals is topologizing and idempotent, A_Λ is a semi-simple ring, and if $A_\Lambda = S_1 \oplus \ldots \oplus S_n$ is the decomposition of A_Λ by minimal two-sided ideals, then there is a subset J of $\{1, \ldots, n\}$ such that $A_F = \oplus_J S_J$. (J may be empty.)*

Proof. Assume that every F-divisible module is injective. By 3.2 with $\mathfrak{A}$ taken to be the set of all right ideals of A, we see that A is right hereditary and right noetherian. The statements about A_Λ then follow from 2.7. Let $F^* = F \cap \Lambda$. Then F^* is topologizing and idempotent. $F^*A \subset \Lambda A = 0$ and so $A \subset A_{F^*} \subset A_\Lambda$ as rings. Now every F^*-injective module M is F-injective and hence injective, for if $I \in F$ and ϕ: $I \to M \in \text{mod } A$, let J be a complement of I in A. Then $I + J = I \oplus J \in F^*$ and so ϕ extends to $I + J$ and hence to A whenever M is F^*-injective. In particular, A_{F^*} is injective as a right A-module and hence $A_{F^*} = A_\Lambda$ since A_Λ is an essential extension of A. By 2.3 the F^* localization may be taken to be $M \rightarrow M \otimes_A A_{F^*}$. Thus F^* and Λ define the same localization and thus $F^* = \Lambda$.

Assume that A is right hereditary and right noetherian and that $\Lambda \subset F$. Then by 4.1 F-divisible implies F-injective and the above argument on complements shows that F-injectivity implies injectivity.

Only the statement about minimal ideals remains to be shown. By (**3**, p. 46, Prop. 10) we need only find a ring homomorphism of A_Λ onto A_F. Let j_Λ: $A \to A_\Lambda$ and j_F: $A \to A_F$ be the canonical maps. Since A_F is injective and j_Λ is one-to-one, there is a homomorphism of right A-modules g: $A \to A_F$ such that $g \cdot j = j_F$. Since A is hereditary, Im g is an injective A-module containing $j_F[A]$. But A_F is an essential extension of $j_F[A]$. Thus g is onto. We must show that g is a ring homomorphism. First notice that any two

$$h, h'\colon A \to A_F \in \text{mod } A$$

such that $h \cdot j_\Lambda = h' \cdot j_\Lambda$ are equal. For $h - h'$ induces a map: coker $j_\Lambda \to A_F$. Now $\Lambda \subset F$ shows that coker $j_\Lambda \in \mathbf{F}$. Thus any image of coker j_Λ is F-torsion. But A_F is F-torsion-free and so $h = h'$. Now let $y \in A_\Lambda$. Define h_y, h'_y: $A \to A_F$ by $h_y(x) = g(yx)$, $h'_y(x) = g(y)g(x)$. These extend to A-mappings of A_Λ to A_F and agree on $j_\Lambda[A]$. Thus they have the same unique extension and so $g(yz) = g(y)g(z)$ for all y and z elements of A_Λ. This completes the proof of 4.6.

5. The effacement condition. In this section we derive a condition that is useful for analysing $\mathfrak{A}$-injectivity when $\mathfrak{A}$ is taken to be an arbitrary set of right ideals. First we give a preliminary definition.

Definition. Let $\mathfrak{A}$ be a set of right ideals of the ring A. We shall say that A satisfies the *effacement condition for* $\mathfrak{A}$ if every $\mathfrak{A}$-effacement is an injective effacement. (See the beginning of §3 for definitions.)

Notice that if A satisfies the effacement condition for $\mathfrak{A}$, then every $\mathfrak{A}$-injective $M \in \operatorname{mod} A$ is injective since $1_M: M \to M$ is an $\mathfrak{A}$-effacement. Although the results of this paper give the converse for a few special $\mathfrak{A}$'s, we have no general results on when the converse holds.

All our results on the effacement condition stem from the following statement.

5.1. PROPOSITION. *Let $\mathfrak{A}$ be a set of right ideals of the ring A and J a right ideal of A. The following two statements are equivalent:*

(a) *Every $\mathfrak{A}$-effacement is a $\{J\}$-effacement.*

(b) *There exist an $m \in J$, $\lambda_1, \ldots, \lambda_n$ elements of A, $I_1, \ldots, I_n$ elements of $\mathfrak{A}$, and A-homomorphisms $\phi_i: I_i \to J$, $i = 1, \ldots, n$, such that $\lambda_i J \subset I_i$, $i = 1, \ldots, n$ and $p = mp + \sum_1^n \phi_i(\lambda_i p)$ for all $p \in J$.*

Since statement (b) is so complex we shall discuss some consequences of 5.1 before we give the proof.

(1) First consider the case where $\mathfrak{A}$ is empty. Then the effacement condition asserts that every right A-module is injective and thus is equivalent to asserting that A is a semi-simple ring (**4**, p. 11). Let J be any right ideal of A. Then (b) asserts that there is an $m \in J$ such that $p = mp$ for every $p \in J$. In particular, $m = m^2$. Thus every right ideal is generated by an idempotent and thus is a direct summand of A. But this is again equivalent to saying that A is semi-simple. Conversely, if $A = J \oplus J'$, J and J' right ideals of A, then every monomorphism is an $\mathfrak{A}$-effacement.

(2) Suppose that $J \oplus J' \in \mathfrak{A}$ where J, J' are right ideals such that $J \cap J' = 0$; then clearly (a) is satisfied. In this case we have $n = 1$, $m = 0$, $\lambda_1 = 1$, and $\phi: J \oplus J' \to J$ the natural projection.

(3) Condition (b) may be restated as follows: The mapping

$$J \to A \oplus I_1 \oplus \ldots \oplus I_n$$

given by $p \to (p, \lambda_1 p, \ldots, \lambda_n p)$ is an isomorphism of J onto a direct summand of $A \oplus I_1 \oplus \ldots \oplus I_n$. Suppose A is a (commutative) integral domain and $\mathfrak{A}$ is the set of principal ideals of A. Let J be an ideal of A and assume that (a) holds. Then $A \oplus I_1 \oplus \ldots \oplus I_n$ is a finitely generated free module and hence J is a finitely generated projective module. Conversely, suppose that the ideal J is finitely generated and projective (actually a projective ideal is automatically finitely generated; see 2.5 for $F = \Lambda$), say, $J = a_1 A + \ldots + a_n A$. Then there are elements $q_1, \ldots, q_n$ of the quotient field Q of A $(Q = A_\Lambda)$ such that

$$1 = \sum_1^n q_i a_i$$

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and $q_i J \subset A$ for $i = 1, \ldots, n$. Let $I_i = a_i A$, $m = 0$, $\lambda_i = q_i a_i \in A$, and ϕ_i: $I_i \to J$ be the natural inclusion. Then

$$p = \sum_1^n \phi_i(\lambda_i p)$$

for all $p \in J$ and so (b) holds. Thus for integral domains and $\mathfrak{A}$ the set of principal ideals, we have that A satisfies the effacement condition if and only if A is a Dedekind ring and condition (b) reduces to the concept of a projective basis; see (**4**, p. 133).

Proof of 5.1. We use the notation of 3.1f. Assume (a) and consider J embedded in $\mathfrak{A}(J)$, the particular effacement constructed in 3.1. The identity map 1_J: $J \to J$ then extends to a map f: $A \to \mathfrak{A}(J)$ and $f/J = 1_J$. Thus, for $a \in A$, we have $f(a) = \bar{x}a$, where $\bar{x} = f(1) \in \mathfrak{A}(J)$, and for $p \in J$ we have $p = \bar{x}p$. Let $x = (m, \sum_\alpha (I_\alpha, f_\alpha)\lambda_\alpha)$ be a representative of $\bar{x}$ in $J \oplus F$. Then for $p \in J$ we have

$$(p, 0) = (mp, \textstyle\sum_\alpha (I_\alpha, f_\alpha)\lambda_\alpha p) + \sum_\beta (\phi_\beta a, -(J_\beta, \phi_\beta)a_\beta)$$

where $J_\beta \in \mathfrak{A}$, ϕ_β: $J_\beta \to J$, and $a_\beta \in J_\beta$. This may be written as

$$(p, 0) = (mp + \textstyle\sum_\beta \phi_\beta a_\beta, \sum_\alpha (I_\alpha, f_\alpha)\lambda_\alpha p - \sum_\beta (J_\beta, \phi_\beta)a_\beta).$$

Thus $p = mp + \sum_\beta \phi_\beta(a_\beta)$ and $\sum_\beta (J_\beta, \phi_\beta)a_\beta = \sum_\alpha (I_\alpha, f_\alpha)\lambda_\alpha p$. Since the pairs (I, f), where $I \in \mathfrak{A}$ and f: $I \to J$, are free generators of F, we see that $p = mp + \sum_\alpha \phi_\alpha(\lambda_\alpha p)$ where $\lambda_\alpha p \in I_\alpha$, ϕ_α: $I_\alpha \to J$, and $m \in J$. This proves (b).

Conversely, suppose (b) is true and let g: $J \to M \in \text{mod } A$ where i: $M \to N$ is an $\mathfrak{A}$-effacement. Using (b) we have, for each $p \in J$,

$$p = mp + \sum_1^n \phi_i(\lambda_i p)$$

where ϕ_i: $I_i \to J$. Thus

$$g(p) = g(mp) + \sum_1^n g\cdot\phi_i(\lambda_i p)$$

where ϕ_i: $I_i \to J$. Thus

$$g(p) = g(mp) + \textstyle\sum g\cdot\phi_i(\lambda_i p) = g(m)p + \sum g\cdot\phi_i(\lambda_i p).$$

Now $g\cdot\phi_i$: $I_i \to M$. Thus it extends to a map $\overline{g\cdot\phi_i}$: $A \to N$. Hence

$$g\cdot\phi_i(\lambda_i p) = \overline{g\cdot\phi_i}(\lambda_i p) = \overline{g\cdot\phi_i}(\lambda_i)p$$

and

$$g(p) = \left[g(m) + \sum_1^n \overline{g\cdot\phi_i}(\lambda_i)\right]p$$

and so the map $\bar{g}$: $A \to M$ given by $\bar{g}(1) = g(m) + \overline{g\cdot\phi_i}(\lambda_i)$ extends g. Thus i: $M \to N$ is a $\{J\}$-effacement.

6. Theorems of Gentile and Levy. In this section we have some results on two other definitions of divisibility. We obtain some generalizations of theorems due to Gentile **(8)** and Levy **(13)**.

In **(11)** Hattori defines divisibility to be $\mathfrak{A}$-injectivity where $\mathfrak{A}$ is the set of principal right ideals of the ring A. Then, because of 3.2(1), he restricts his attention to rings for which every principal right ideal is projective. We take a slightly different approach.

Definition. We say that a right A-module M is *Hattori-divisible* if it is $\mathfrak{A}$-injective where $\mathfrak{A}$ is the set of projective principal right ideals of A.

The next statement is our only result on Hattori-divisibility. It provides the most direct generalization of the integral domain case.

6.1 THEOREM. *For a ring A the following two statements are equivalent:*
(a) *Every Hattori divisible right A-module is injective.*
(b) *A is a right hereditary and right noetherian ring.*

Proof. If (a) is true, then the class of injectives in mod A is closed under epimorphisms and arbitrary direct sums. Thus (b) follows from 3.2 applied to the set of all right ideals of A.

Conversely, assume that (b) is true. It is sufficient to show that A satisfies the effacement condition with respect to the set $\mathfrak{A}$ of principal right ideals of A. By arguing on complements we may restrict our attention to large right ideals. By 2.7, Λ, the set of large right ideals of A, is topologizing and idempotent and if

$$J = \sum_{1}^{n} x_i A \qquad (x_i \in A)$$

is an element of Λ, then there are $q_1, \ldots, q_n \in A_\Lambda$ such that $q_i J \subset A$ for all i and

$$1 = \sum_{1}^{n} x_i q_i.$$

Using the notation of 5.1, this gives the effacement condition as follows: take $m = 0$, $I_i = x_i A$, $\lambda_i = x_i q_i$, and ϕ_i: $I_i \to J$ is the inclusion. This proves (b).

6.2. COROLLARY. *Let $\mathfrak{A}$ be the set of principal right ideals of A. Then A satisfies the effacement condition for $\mathfrak{A}$ if and only if every Hattori-divisible right A-module is injective.*

Definition. Let A be a ring. A *right Ore ring A_D* for A is an overring of A such that if $d \in A$ is neither a right nor a left zero divisor, then $d^{-1} \in A_D$ and, further, every element of A_D may be written in the form ad^{-1} where $a, d \in A$.

In **(8)** Gentile states the following.

THEOREM. *Let A be a ring without zero divisors that has a right Ore ring which is a division ring. Then every (Hattori-)divisible right A-module is injective if and only if A is right hereditary.*

This is an immediate consequence of 6.1 and 2.5. We point out that Goldie's theorem (see the last sentence of §1) shows that one hypothesis is unnecessary.

6.3. THEOREM. *Let A be a ring without zero divisors. Then the following two statements are equivalent:*

(a) *Every Hattori-divisible right A-module is injective.*

(b) *A is right hereditary.*

Further, when (a) *and* (b) *hold, A has a right Ore ring which is a division ring.*

Proof. This is just the statement that A_Λ is then a right Ore ring for A. This follows from the proof of **(7**, p. 419**)**.

We now consider the concepts of torsion-free and divisible studied by Levy **(13)**.

Definitions. (a) A non-zero element of the ring A is called *regular* if it is not a zero-divisor.

(b) $M \in \operatorname{mod} A$ is called *r-divisible* if $Md = M$ for every regular $d \in A$.

(c) Let $M \in \operatorname{mod} A$ and $m \in M$. The element m is called an *r-torsion element* if there is some regular $d \in A$ such that $md = 0$. M is called *r-torsion-free* if it contains no non-zero r-torsion elements.

Note that r-divisibility is $\mathfrak{A}$-divisibility for $\mathfrak{A} = \{dA : d \in A \text{ is regular}\}$.

6.4. LEMMA. *For any ring A the following statements are equivalent:*

(a) *A possesses a right Ore ring.*

(b) *For each $M \in \operatorname{mod} A$ the set of r-torsion elements of M is a submodule of M.*

(c) *If $d \in A$ is regular, then A/dA consists of r-torsion elements.*

Proof. The equivalence of (a) and (b) is proved in **(13)**. Assume that (b) is true. Let $d \in A$ be a regular element. Then $1 + dA$ is an r-torsion element of A/dA and generates A/dA. Now assume that (c) holds and let $a \in A$, $d \in A$ with d regular. Then $a + dA$ is an r-torsion element of A/dA. Thus there is a regular $d' \in A$ such that $ad' \in dA$. Thus conditions (*) and (**) of **(7**, p. 415**)** hold and thus A has a right Ore ring.

We next find conditions under which every r-torsion-free and r-divisible module is injective. To this end we define a special localization.

Definition. Let $\mathbf{T}$ be the class of all right A-modules M such that $\operatorname{Hom}_A(M, X) = 0$ for all r-torsion-free and injective $X \in \operatorname{mod} A$. It is easy to see that $\mathbf{T}$ is a localizing subcategory of mod A. Let T denote the corresponding set of topologizing and idempotent right ideals of A.

6.5. LEMMA. *If $M \in \operatorname{mod} A$ is generated by r-torsion elements, then $M \in \mathbf{T}$. In particular, $dA \in T$ for all d regular in A.*

Proof. Let X be any r-torsion-free A-module and let $f: M \to X \in \operatorname{mod} A$. Then if $m \in M$ is r-torsion, we have $f(m) = 0$. Thus f is zero on a set of generators and hence is the zero map. Thus $M \in \mathbf{T}$.

Now $dA \in T$ if and only if $A/dA \in \mathbf{T}$ and if d is regular, then

$$1 + dA \in A/dA$$

is an r-torsion element.

6.6. Proposition. *Let d be a regular element of A. Then $j_A(d)$ is invertible in A_T where j_A: $A \to A_T$ is the canonical map.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\phi} & A & \longrightarrow & A/dA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle j_A} & & \downarrow & & \\
0 & \longrightarrow & A_T & \xrightarrow{\phi_T} & A_T & \longrightarrow & 0 = (A/dA)_T & &
\end{array}$$

where the rows are exact and $\phi(a) = da$ for all $a \in A$. By the universal mapping property of the functor $M \to M_T$, ϕ_T is the unique map that renders the left-hand square commutative. Thus ϕ_T is multiplication by $j_A(d)$ and $j_A(d)^{-1} = \phi_T^{-1}(j_A(1))$.

6.7. Corollary. *Let $M \in$ mod A_T. Then, considered as an A-module, M is r-torsion-free and r-divisible. In particular, every T-closed module is r-torsion-free and r-divisible.*

Definition **(6)**. (1) Let D, $B \in$ mod A. A *partial homomorphism* ϕ: $B \to D$ is a map ϕ: $B' \to D \in$ mod A where B' is a submodule of B.

(2) Let D, B, $C \in$ mod A such that $C \subset B$ and let ρ: $B \to B/C$ be canonical. We write $C \leqslant B(D)$ if the only partial homomorphism ϕ: $B \to D$ for which there exists a partial homomorphism ψ: $B/C \to D$ such that $\phi = \psi\rho$ is the zero homomorphism.

(3) A right ideal I of the ring A is called *dense* if $I \leqslant A(A)$. We denote the set of dense right ideals of A by Δ.

For details concerning the relation $C \leqslant B(D)$, see **(6)**. We remark that, if $E(A) \in$ mod A is the injective envelope of A, then $I \in \Delta$ if and only if $\mathrm{Hom}_A(A/I, E(A)) = 0$. Thus Δ is topologizing and idempotent by the same reasoning used for T.

6.8. Proposition. *If $M \in$ mod A, let $E(M)$ denote the injective envelope of M. Then*

(1) *$E(M)$ is r-torsion-free if and only if the canonical map j_M: $M \to M_T$ is a monomorphism.*

(2) *The following statements are equivalent:*

(a) *$E(A)$ is r-torsion-free.*

(b) *Every element of T is dense.*

(c) *dA is dense for all regular $d \in A$.*

Proof. (1) Assume $E(M)$ is r-torsion-free. Then every submodule of M has a non-zero map into an r-torsion-free injective. Thus $TM = 0$. But $TM = \ker j_M$. Conversely, assume $E(M)$ is not r-torsion-free. Then there is a non-zero x in $E(M)$ and a regular d in A such that $0 = xd$. Thus, by 6.5, $0 \neq xA \subset TE(M)$ and so $0 \neq TE(M) \cap M \subset TM = \ker j_M$.

(2) Suppose $E(A)$ is r-torsion-free and $I \in T$. Let $f: J \to E(A) \in \text{mod } A$ where J is a right ideal of A and $\ker f \supset I$. Then the induced map $J/I \to E(A)$ must be zero since $J/I \in \mathbf{T}$. Thus (a) implies (b). Clearly (b) implies (c), so assume that (c) is true, i.e., assume that $dA \leqslant A(E(A))$ for all d regular in A. Let $x \in E(A)$ and d be a regular element such that $0 = xd$. Define $\phi: A \to E(A)$ by $\phi(a) = xa$. Then ϕ factors through A/dA and hence $\phi = 0$. Thus $x = 0$ and $E(A)$ is r-torsion-free.

6.9. Theorem. *The statements below are connected as follows:* (a) *is equivalent to* (b), (a) *implies* (c), *and if* M *r-torsion-free implies* $TM = 0$, *then* (c) *implies* (a).

(a) *Every r-torsion-free and r-divisible right A-module is injective.*

(b) A_T *is semi-simple and* $M \in \text{mod } A$ *is r-torsion-free and r-divisible if and only if* $M \in \text{mod } A_T$.

(c) A_T *is semi-simple and, if* $j_A: A \to A_T$ *is the canonical map,* A_T *is generated (as a ring) by* $j_A[A]$ *and* $\{j(d)^{-1}: d \in A$ *is regular*$\}$.

Proof. First assume that (a) is true. Then, by 6.7, every $M \in \text{mod } A_T$ is r-torsion-free and injective as an A-module. Thus $TM = 0$ and so $\mathbf{T}'$ $(=\mathbf{T} \cap \text{mod } A_T)$ contains only the zero module. Thus by 1.1 the T localization is $M \to M \oplus {}_AA_T$ and hence by 2.3 A_T is semi-simple. Let $M \in \text{mod } A$ be r-torsion-free and r-divisible. Thus $\text{Hom}_A(TM, M) = 0$ and hence $TM = 0$. Thus M is T-closed and hence is an A_T-module. This proves (b). Clearly (b) gives (a).

Again assume that (a) is true. Then the subring generated by $j_A[A]$ and $\{j_A(d)^{-1}: d$ is regular in $A\}$ is clearly r-torsion-free and r-divisible as an A-module. But A_T is the injective envelope of $j_A[A]$. Thus this ring is all of A_T.

Conversely, if the hypothesis holds, we may take j_A to be inclusion since $TA = 0$. If M is r-torsion-free and r-divisible, then division by regular elements gives a unique result and so, since A_T has this special form, M is an A_T-module and so is injective as an A-module (since it is T-closed).

We now give the connection between A_T and the Ore ring. Let D be the multiplicative set of regular elements of A. Let F_D be the topologizing and idempotent set of right ideals of A defined by D; see (**7**, p. 414). When the Ore ring exists, F_D becomes the set of right ideals of A which contain an element of D and, for $M \in \text{mod } A$, $F_D M$ becomes the set of r-torsion elements of M, and, further, $F_D \subset T$ by 6.5.

6.10. Proposition. *Let* A *be a ring for which the right Ore ring exists. Then* $F_D = T$ *and so* A_T *is the Ore ring.*

Proof. Since $F_D \subset T$, we have $F_D M \subset TM$ for all $M \in \text{mod } A$. We wish to show equality. First notice that if $F_D = 0$, then $TM = 0$. For $F_D M = 0$ means that M is r-torsion-free and implies that $F_D(E(M)) = 0$. Thus $E(M)$ is r-torsion-free and so, by 6.8, $TM = 0$. Then $TM/F_D M \subset M/F_D M$ shows that $F_D(TM/F_D M) = 0$ and hence $TM/F_D M = T(TM/F_D M) = 0$, i.e. $TM = F_D M$ for all $M \in \text{mod } A$, i.e. $\mathbf{F}_D = \mathbf{T}$, and hence $F_D = T$.

Thus 6.9 gives the following result of Levy **(13)**.

THEOREM. *Let A be a ring possessing a right Ore ring. Then every r-torsion-free and r-divisible right A-module is injective if and only if the right Ore ring of A is semi-simple.*

We now give a result on r-divisibility and prove a theorem of Levy.

Definition. We say a monomorphism in mod A is an *r-effacement* if it is an $\mathfrak{A}$-effacement where $\mathfrak{A} = \{dA : d \text{ is regular in } A\}$.

6.11. PROPOSITION. *For any ring A, the following two statements are equivalent:*

(a) *Every r-effacement is an injective effacement.*

(b) *The set Λ of large right ideals is topologizing and idempotent and every right ideal I has a projective basis of the following form: there is a generating set $p_0, \ldots, p_n$ in I and $q_0, \ldots, q_n$ in A such that for $p \in I$ we have*

$$p = \sum_1^n p_i q_i p$$

where $q_i I \subset A$ and $q_i = d_i^{-1}\lambda_i$ where d_i is a regular element of A and λ_i is an element of A.

Proof. Assume that A satisfies the r-effacement condition. Then every r-divisible module is injective. Thus A is right hereditary and right noetherian by 3.2, and so by 2.7 A_Λ exists. By 5.1, there exist an $m \in I$, $d_1, \ldots, d_n$ regular elements of A, mappings $\phi_i : d_i A \to I$ and $\mu_1, \ldots, \mu_n$ in A such that for all $p \in I$, $\mu_i p \in d_i A$ and

$$p = mp + \sum_1^n \phi_i(\mu_i p).$$

Let $\mu_0 = 1$, $\phi_0 : A \to I$ by $\phi_0(a) = ma$, $d_0 = 1$, and $I_0 = A$. Then we may write this as

$$p = \sum_0^n \phi_i(\mu_i p).$$

Now $d_i A \in \Lambda$ (since A is noetherian) and so considering ϕ_i to be a map from $d_i A$ to A_Λ, we see that ϕ_i has an extension $q'_i : A \to A_\Lambda$ where $q'_i(x) = q'_i x$ where $q'_i \in A_\Lambda$. Let $p_i = \phi_i(d_i) = q'_i d_i \in I$. Then $q'_i = p_i d_i^{-1} \in A_\Lambda$. Thus we have the equation

$$p = \sum_0^n \phi_i(\mu_i p) = \sum_0^n q'_i \mu_i p = \sum_0^n p_i d_i^{-1} \mu_i p$$

for all $p \in I$. Now $\mu_i I \subset d_i A$ and so, given $p \in I$, there is an $a \in A$ such that $\mu_i p = d_i a$ and so $d_i^{-1}\mu_i p = d_i^{-1}d_i a = a \in A$. Thus if we put $q_i = d_i^{-1}\mu_i$, we obtain (b). Conversely, if (b) is true, we merely interpret this as 5.1 (b).

6.12. LEMMA. *Let A be a ring for which every r-divisible right A-module M is injective. Then $T = \Lambda$.*

Proof. 5.1 shows that A is right hereditary and right noetherian. Thus the Λ localization is $M \to M \otimes_A A_\Lambda$ by 2.7. If $d \in A$ is regular, then $dA \in \Lambda$, which, in this case, is equal to Δ, the set of dense ideals. Thus by 6.8 $TA = 0$. Since A_T is r-divisible, we have $A_T = A_\Lambda$ (as rings) and so by 2.7 A_T is semi-simple. Thus by 2.3, the T localization is $M \to M \otimes_A A_T$. Thus T and Λ define the same localization and hence $T = \Lambda$.

In the next theorem the equivalence of (a) and (b) is the theorem of Levy already mentioned.

6.13. THEOREM. *Let A be a ring possessing a two-sided Ore ring* (i.e., A has a right Ore ring every element of which may also be written in the form $d^{-1}a$ where $d, a \in A$ and d is regular). *The following statements are equivalent:*

(a) *Every r-divisible right A-module is injective.*

(b) *A is right hereditary and the Ore ring is semi-simple.*

(c) *A satisfies the r-effacement condition.*

Proof. For the equivalence of (a) and (b), see (**13**, p. 141, Theorem 3.4). Assume that (b) is true. Then, since the Ore ring is semi-simple (**9**, Theorem 4.4 and **7**, p. 419, theoreme 2) show that A_Λ is the Ore ring. Since every element of Λ contains a regular element (**7**, *ibid.*), 2.5 shows that A is noetherian and that there are $x_1, \ldots, x_n$ generating each $I \in \Lambda$ and $q_1, \ldots, q_n \in A_\Lambda$ such that

$$p = \sum_0^n x_i q_i p$$

where $q_i I \subset A$. Since A_Λ is a two-sided Ore ring, it follows by the proof of 6.11 that every r-effacement is a Λ-effacement. But every right ideal is a direct summand of a large right ideal. Thus (c) is true. Clearly (c) implies (a).

Added in proof. It has come to my notice that Theorem 1 is actually a consequence of a result (obtained by other methods and stated in a different context) in Y. Hinohara, *Note on non-commutative semi-local rings*, Nagoya Math. J., *17* (1960), 161–166.

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