BULL. AUSTRAL. MATH. SOC. MOS 40-00, 40A99
VOL. 6 (1972), 75-81.

Sequences defined as minima of two Fibonacci-type relations

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If $\{L_n\}$ is a sequence defined by

$$L_n = \min\{L_{n-a}+L_{n-b},\ L_{n-c}+L_{n-d}\} ,$$

with a, b, c, d positive integers, then one can ask if necessarily $L_n = L_{n-a} + L_{n-b}$ for all sufficiently large n. The answer is yes if a and b are relatively prime, $L_n > 0$ initially, and $\lambda < \mu$, where $\lambda^{-a} + \lambda^{-b} = 1$, $\mu^{-c} + \mu^{-d} = 1$. The answer is no if instead a and b have greatest common divisor $k \geq 2$, with $c \equiv 0 \pmod{k}$, $d \not\equiv 0 \pmod{k}$.

Introduction. Much is known about the properties of sequences defined by a recurrence of the type $L_n = L_{n-a} + L_{n-b}$, where a and b are fixed positive integers. In this note, we produce conditions on a, b, c and d, such that if

$$L_n = \min\{L_{n-a}+L_{n-b},\ L_{n-c}+L_{n-d}\} \tag{1}$$

then

$$L_n = L_{n-a} + L_{n-b} \tag{2}$$

for all sufficiently large n. We concern ourselves only with the case in which all initial values are positive, so that L_n is then positive for all n. For a situation in which this problem arises, see [1].

Received 7 September 1971.

It is well known that $L_n = L_{n-a} + L_{n-b}$ implies $L_n = O(\lambda^n)$, where λ is the positive root of

(3) $$\lambda^{-a} + \lambda^{-b} = 1 .$$

Hence, if (2) holds, we must have $\lambda \le \mu$ where μ is the positive root of

(4) $$\mu^{-c} + \mu^{-d} = 1 .$$

There are examples however, to show that this condition is not sufficient. One such example is

$$L_n = \min\{2L_{n-3}, L_{n-2}+L_{n-4}\} , \quad n \ge 5 ,$$

with the initial conditions $L_1 = L_2 = L_3 = L_4 = 1$.

THEOREM 1. *Suppose* a, b, c *and* d *are positive integers, and* $L_1, L_2, \ldots, L_e$ *are given positive real numbers, where* $e = \max\{a, b, c, d\}$. *Define*

(1) $$L_n = \min\{L_{n-a}+L_{n-b}, L_{n-c}+L_{n-d}\}$$

for $n > e$, *and define* $\lambda > 1$ *and* $\mu > 1$ *by* (3) *and* (4). *If* $\lambda < \mu$, *and if* a *and* b *are relatively prime, then there exists an integer* n_0 *such that*

(2) $$L_n = L_{n-a} + L_{n-b}$$

for all $n \ge n_0$.

Proof. Suppose N is an integer, $N \ge e + 1$. Define

(5) $$c_N = \max_{1 \le k \le e} \left\{L_{N-k}/\lambda^{N-k}\right\} ,$$

(6) $$d_N = \min_{1 \le k \le e} \left\{L_{N-k}/\lambda^{N-k}\right\} .$$

Since

https://doi.org/10.1017/S0004972700044282 Published online by Cambridge University Press

$$\begin{aligned} L_N &\le L_{N-a} + L_{N-b} \\ &\le \lambda^{N-a}c_N + \lambda^{N-b}c_N \\ &= \lambda^N c_N\left(\lambda^{-a}+\lambda^{-b}\right) \\ &= \lambda^N c_N , \end{aligned}$$

it follows that $c_{N+1} \le c_N$, and hence the sequence $\{c_N\}$ is decreasing.

On the other hand

$$\begin{aligned} L_{N-a} + L_{N-b} &\ge d_N\lambda^{N-a} + d_N\lambda^{N-b} \\ &= d_N\lambda^N \end{aligned}$$

and

$$\begin{aligned} L_{N-c} + L_{N-d} &\ge d_N\lambda^{N-c} + d_N\lambda^{N-d} \\ &= d_N\lambda^N\left(\lambda^{-c}+\lambda^{-d}\right) \\ &> d_N\lambda^N\left(\mu^{-c}+\mu^{-d}\right) \\ &= d_N\lambda^N . \end{aligned}$$

Hence, by (1), $L_N \ge d_N\lambda^N$, so that $d_{N+1} \ge d_N$, and the sequence $\{d_N\}$ is increasing.

Since a and b are relatively prime, the set S consisting of all integers of the form $sa + tb$, where s and t are positive integers, contains a smallest element with the property that all greater integers also belong to S. Denote this smallest element by f.

Suppose $0 < \varepsilon < 1$, and r is an integer, $r \ge N - 1 + f$. We claim that

(7) $$L_r/\lambda^r \ge (1-\varepsilon)c_N$$

implies

(8) $$L_{r-q}/\lambda^{r-q} \ge \left(1-\varepsilon\lambda^q\right)c_N$$

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for all q in S , $q < r$.

For, (7) implies that

$$\begin{aligned}(1-\varepsilon)c_N &\le (L_{r-a}+L_{r-b})/\lambda^r \\ &= \lambda^{-a}\{L_{r-a}/\lambda^{r-a}\} + \lambda^{-b}\{L_{r-b}/\lambda^{r-b}\} \\ &\le \lambda^{-a}\{L_{r-a}/\lambda^{r-a}\} + \lambda^{-b}c_N\end{aligned}$$

so that $\lambda^a(1-\varepsilon-\lambda^{-b})c_N \le L_{r-a}/\lambda^{r-a}$ or $(1-\varepsilon\lambda^a)c_N \le L_{r-a}/\lambda^{r-a}$ by (3). Similarly

$$(1-\varepsilon\lambda^b)c_N \le L_{r-b}/\lambda^{r-b} .$$

Successively repeating the argument yields (8).

Since $r \ge N - 1 + f$, each member of the set $\{N-1, N-2, \ldots, N-e\}$ is of the form $r - q$ for q in S . Thus by (6) and (8), the inequality (7) implies $d_N \ge \inf_q (1-\varepsilon\lambda^q)c_N$, where the infimum is taken over those q for which $N - 1 \ge r - q \ge N - e$; that is, $r + 1 - N \le q \le r + e - N$. Thus (7) implies

$$d_N \ge (1-\varepsilon\lambda^{r+e-N})c_N . \tag{9}$$

By reversing the argument, if ε is now chosen such that

$$(1-\varepsilon\lambda^{r+e-N}) > d_N/c_N ,$$

then

$$L_r/\lambda^r < (1-\varepsilon)c_N .$$

It follows, since this implication is valid for all r in $R = \{r : N-1+f \le r \le N+f+e-2\}$, that

$$1 - \varepsilon\lambda^{f+2e-2} > d_N/c_N \tag{10}$$

implies

$$\sup_{r\in R} L_r/\lambda^r < (1-\varepsilon)c_N ,$$

that is, (10) implies

(11) $$c_{N+f+e-2} < (1-\varepsilon)c_N .$$

Put $\phi_N = c_N/d_N$, and choose $\varepsilon = \left(1-\phi_N^{-1}\right)\lambda^{-f-2e+2}/2$ so that (10) holds. It follows from (11), with this choice of ε , and the fact that d_N is increasing, that

$$\phi_{N+f+e-2} < \left[1-\left(1-\phi_N^{-1}\right)\lambda^{-f-2e+2}/2\right]\phi_N ,$$

whence

$$\phi_{N+f+e-2} - 1 < \left[1-\lambda^{-f-2e+2}/2\right]\left(\phi_N-1\right) .$$

Since $\{\phi_N\}$ is decreasing, and the factor in the square brackets is a fixed constant between 0 and 1 , we have

(12) $$\lim_{N\to\infty} \phi_N = 1 .$$

To complete the proof, suppose

$$L_{n-a} + L_{n-b} > L_{n-c} + L_{n-d}$$

for some $n > \max\{N+a, N+b\}$. Then, since

$$\lambda^n d_N \le L_n \le \lambda^n c_N ,$$

we have

$$c_N\lambda^{n-a} + c_N\lambda^{n-b} > d_N\lambda^{n-c} + d_N\lambda^{n-d}$$

or

$$\phi_N\left(\lambda^{-a}+\lambda^{-b}\right) > \lambda^{-c} + \lambda^{-d}$$

or

$$\phi_N > \lambda^{-c} + \lambda^{-d} > 1 .$$

This contradicts (12) if N is big enough.

We consider briefly what can happen if a and b are not relatively

https://doi.org/10.1017/S0004972700044282 Published online by Cambridge University Press

prime. Let k be the highest common factor of a and b. It is immediate, by considering the subsequences of the form L_{n_0+mk}, that the result of Theorem 1 still holds if $c \equiv 0 \pmod{k}$ and $d \equiv 0 \pmod{k}$.

THEOREM 2. *If* $\lambda < \mu$, *if* k *is the greatest common divisor of* a *and* b, *with* $k \geq 2$, *if* $c \equiv 0 \pmod{k}$, *and* $d \not\equiv 0 \pmod{k}$, *then there is a set of positive values for* L_n, $1 \leq n \leq e$, *such that* (1) *holds for* $n > e$, *and* $L_n < L_{n-a} + L_{n-b}$ *for an infinite set of integers* n.

Proof. Define (for convenience) $L_{\gamma k} = 1$ for integer γ, $0 \leq \gamma k < \max(a, b)$. This determines L_n for all $n \equiv 0 \pmod{k}$ by $L_n = L_{n-a} + L_{n-b}$. Next define L_n for $n \equiv -d \pmod{k}$ by the equation $L_n = L_{n-c} + L_{n-d}$ for $n \equiv 0 \pmod{k}$, that is, $L_n \equiv L_{n+d} - L_{n+d-c}$ for $n \equiv -d \pmod{k}$. It is easy to check that one then has $L_n = L_{n-a} + L_{n-b}$ for $n \equiv -d \pmod{k}$, at least for $n \geq c - d + \max(a, b)$. In a similar manner define L_n successively for $n \equiv -2d, n \equiv -3d, n \equiv -4d, \ldots, n \equiv -(k-2)d$. L_n is then determined for all n larger than some fixed integer n_0, $n \not\equiv d \pmod{k}$, and, for such n, $L_n = L_{n-a} + L_{n-b} = L_{n-c} + L_{n-d}$.

Now define $L_n = L_{n-c} + L_{n-d}$ for $n \equiv d$. Since then $n - d \equiv 0$, $L_{n-d} = L_{n-a-d} + L_{n-b-d}$, so the equation $\left(L_n - L_{n-a} - L_{n-b}\right) = \left(L_{n-c} - L_{n-a-c} - L_{n-b-c}\right)$ holds for all $n \equiv d$. Thus, suitable initial conditions can ensure that if this value is initially a negative constant, then by induction,

$$L_n = L_{n-c} + L_{n-d} < L_{n-a} + L_{n-b}$$

for all $n \equiv d \pmod{k}$.

The author has been unable to obtain similar general results for the case when $k \geq 2$ and both $c \not\equiv 0$ and $d \not\equiv 0 \pmod{k}$. We cite two examples to show what may or may not occur.

If $a = b = k = 3$, $c = 1$, and $d = 4$, then $L_n = 2L_{n-3}$ for all sufficiently large n. It is worth noting that this result cannot be established by the method of proof of Theorem 1, since the quotient c_N/d_N need not converge. The proof however is straightforward after observing that

(a) one cannot have $L_n = L_{n-1} + L_{n-4}$ for three consecutive values of n;

(b) if $L_n = 2L_{n-3}$ for four consecutive values of n, then $L_n = 2L_{n-3}$ for all larger n.

On the other hand, if $a = b$, $k = 3$, $c = 1$ and $d = 5$, and if L_1, L_2, L_3, L_4, L_5 respectively equal 16, 16, 11, 6, 1; then

$$L_n = 2L_{n-3} \quad \text{if} \quad n \equiv 0, 1, 2 \text{ or } 5 \pmod 6$$

$$L_n = L_{n-1} + L_{n-5} < 2L_{n-3} \quad \text{if} \quad n \equiv 3 \text{ or } 4 \pmod 6 .$$

Theorems 1 and 2 generalize immediately to sequences of the form

$$L_n = \min_{1 \le i \le m} \left\{ L_{n-a_i} + L_{n-b_i} \right\} .$$

Clearly too, one can establish analogous results for maxima.

Reference

[1] R.S. Booth, "Location of zeros of derivatives. II", *SIAM J. Appl. Math.* **17** (1969), 409-415.

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