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BOUNDED LINEAR OPERATORS ON SPACES IN NORMED DUALITY

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Abstract. Let T be a bounded linear operator on a Banach space W, assume W and Y are in normed duality, and assume that T has adjoint $T^\dagger$ relative to Y. In this paper, conditions are given that imply that for all $\lambda \neq 0$, $\lambda - T$ and $\lambda - T^\dagger$ maintain important standard operator relationships. For example, under the conditions given, $\lambda - T$ has closed range if, and only if, $\lambda - T^\dagger$ has closed range.

These general results are shown to apply to certain classes of integral operators acting on spaces of continuous functions.

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1. Preliminaries. We use fairly standard notation from operator theory: $B(X)$ denotes the algebra of all bounded linear operators on a Banach space X; for $S \in B(X)$, $\mathbf{R}(S)$ and $\mathbf{N}(S)$ are the range and null space of S, respectively; the operator S^* is the usual adjoint of S on X^*; for $V \subseteq X$, $V \neq \phi$, $V^\perp = \{\alpha \in X^* : \alpha(V) = \{0\}\}$; for $\Gamma \subseteq X^*$, $\Gamma \neq \phi$, ${}^\perp\Gamma = \{x \in X : \alpha(x) = 0 \text{ for all } \alpha \in \Gamma\}$. Assume that W is a subspace of X which is a Banach space. The space W is continuously embedded in X if there exists $c > 0$ such that $c\|w\|_W \geq \|w\|_X$ for all $w \in W$.

Two Banach spaces, W and Y, are in normed duality if there is a nondegenerate bilinear form on $W \times Y$, $\langle w, y\rangle$, and a constant $c > 0$ such that $|\langle w, y\rangle| \leq c\|w\|_W \cdot \|y\|_Y$ for all $w \in W$ and $y \in Y$. For nonemply subsets $R \subseteq W$ and $S \subseteq Y$, we set $R^\triangle = \{y \in Y : \langle w, y\rangle = 0 \text{ for all } w \in R\}$, and ${}^\triangle S = \{w \in W : \langle w, y\rangle = 0 \text{ for all } y \in S\}$. We let $A_{W,Y} = \{T \in B(W) : \exists T^\dagger \in B(Y) \text{ such that } \langle Tw, y\rangle = \langle w, T^\dagger y\rangle \text{ for all } w \in W, y \in Y\}$.

Two good sources for the theory of linear operators on spaces in normed duality are the books [**7**] and [**9**]. In [**7**] there is an extensive Fredholm theory of such operators. This Fredholm theory is further studied in [**2**].

Assume that $T \in A_{W,Y}$. In this paper we give conditions that imply that the operators $\lambda - T$ and $\lambda - T^\dagger$, for all $\lambda \neq 0$, have the same important operator properties and relationships as those of the operators $\lambda - T \in B(X)$ and its adjoint $\lambda - T^* \in B(X^*)$. There are many important relationships between $\lambda - T$ and $\lambda - T^*$; for example:

(1) $\mathbf{R}(\lambda - T)$ is closed $\Longleftrightarrow$ $\mathbf{R}(\lambda - T^*)$ is closed [part of the Closed Range Theorem];
(2) $\overline{\mathbf{R}(\lambda - T)}^X = {}^\perp\mathbf{N}(\lambda - T^*)$ (here, $\overline{\mathbf{R}(\lambda - T)}^X$ is the closure of $\mathbf{R}(\lambda - T)$ in X) [**8**, Theorem 8.4, p. 232];
(3) $\lambda - T$ is Fredholm $\Longleftrightarrow$ $\lambda - T^*$ is Fredholm; and when $\lambda - T$ is Fredholm, $\operatorname{ind}(\lambda - T) = -\operatorname{ind}(\lambda - T^*)$ [**7**, Corollary 3, p. 91].

The conditions we give on an operator $T \in A_{W,Y}$ imply that for all $\lambda \neq 0$:

(1′) $\mathbf{R}(\lambda - T)$ is closed $\Longleftrightarrow \mathbf{R}(\lambda - T^{\dagger})$ is closed;

(2′) $\overline{\mathbf{R}(\lambda - T)}^{W} = {}^{\triangle}\mathbf{N}(\lambda - T^{\dagger})$;

(3′) $\lambda - T$ is Fredholm on $W \Longleftrightarrow \lambda - T^{\dagger}$ is Fredholm on Y; and when $\lambda - T$ is Fredholm, $\mathrm{ind}(\lambda - T) = -\mathrm{ind}(\lambda - T^{\dagger})$.

(1′)–(3′) are samples of the kind of results that we prove under reasonable conditions on T and $T^{\dagger}$. We prove a complete version of the important Closed Range Theorem in Section IV.

At this point, it is convenient to describe for future reference two standard setups that occur in this paper (setup I is the setting for the general Closed Range Theorem).

The two standard setups.

I. Assume that X is a Banach space, and let X^* be the dual space of X. Assume that W and Y are Banach spaces, W is a subspace of X which is continuously embedded in X, and Y is a subspace of X^* which is continuously embedded in X^*. Also assume that the natural bilinear form on $X \times X^*$ when restricted to $W \times Y$ is nondegenerate. Then W and Y are in normed duality with respect to this form (note that this bilinear form is bounded on $W \times Y$ by the assumptions that the embeddings are continuous). In what follows we will consider properties of certain operators in $A_{W,Y}$.

II. For the second setup, assume that X is a Hilbert space, and that W is a subspace of X which is a Banach space continuously embedded in X. In this situation, the inner product restricted to W is a bounded inner product on W.

As an example of setup I, assume that M is a locally compact, σ-compact, T_2 space. Fix a regular Borel measure μ defined on the Borel subsets of M, and assume that μ is strictly positive [for U open and nonempty, $\mu(U) > 0$]. We use $BC(M)$ to denote the space of all bounded continuous $\mathbf{C}$-valued functions on M. Take $X = L^1(M, \mu)$, $X^* = L^{\infty}(M, \mu)$, $W = BC(M) \cap L^1(M, \mu)$, and $Y = BC(M)$. The condition that μ be strictly positive implies that $Y = BC(M)$ is a closed subspace of $L^{\infty}(M, \mu)$. The natural complete norm on W is $\|f\|_W = \|f\|_1 + \|f\|_u$ (here $\|f\|_u$ is the sup-norm of f). The spaces W and Y are in normed duality with respect to the bilinear form inherited from $L^1(M, \mu) \times L^{\infty}(M, \mu)$:

$$\langle f, g\rangle = \int_M f(x)g(x)\, d\mu(x), \quad f \in L^1(M, \mu), g \in L^{\infty}(M, \mu).$$

This setup is used in paragraph 10 of [**7**].

For an example of setup II, let M and μ be as above. Take X to be the Hilbert space $L^2(M, \mu)$, and let $W = BC(M) \cap L^2(M, \mu)$ with complete norm $\|f\|_W = \|f\|_2 + \|f\|_u$.

These setups are used in the examples below.

2. Two examples involving integral operators. In the case of setup I, where $T \in B(X)$, the conditions we make on T and T^* have the form: for some $n \geq 1$ and $m \geq 1$, $T^n(X) \subseteq W$ and $(T^*)^m(X^*) \subseteq Y$.

In Example 1 below, we present a large class of integral operators for which these conditions hold (with $n = 1$ and $m = 1$).

For both examples, let M and μ be as above. For convenience we suppress reference to the measure μ, for example writing $L^1(M)$ in place of $L^1(M, \mu)$, *a.e.* for μ-*a.e.*, and

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dy for $d\mu(y)$. Let $J(x, y)$ be a kernel defined on $M \times M$. For $g(y)$ a **C**-valued function on M, let

$T_J(g)(x) = \int_M J(x, y)g(y)\,dy$, $x \in M$, whenever this integral is defined.

In both examples below, we assume that

(1) $K(x, y) \in BC(M \times M)$.

EXAMPLE 1. In setup I, let $X = L^1(M)$, $X^* = L^\infty(M)$, $W = L^1(M) \cap BC(M)$, and $Y = BC(M)$. Assume that

(2) $y \to K(\cdot, y)$ is a continuous bounded function from M into $L^1(M)$.

The condition in (2) plays an important role in [**7**]; see for example (12.7) (a), p. 303 and Theorem 12.5, p. 315.

Setting $K^t(y, x) = K(x, y)$ for all $x, y \in M$, we have by property (2) that $T_{K^t} \in B(L^\infty(M))$ that $T_{K^t}(L^\infty(M)) \subseteq BC(M)$, and that $BC(M)$ is T_{K^t}-invariant. These assertions are easy to verify.

CLAIM A. $T_K \in B(L^1(M))$.

Proof. Assume that $f \in L^1$ and $g \in L^\infty$. Using Fubini's Theorem, we have

$$\left|\int_M \left[\int_M K(x, y)f(y)\,dy\right] g(x)\,dx\right| = \left|\int_M f(y)\left[\int_M K(x, y)g(x)\,dx\right] dy\right| \\ \leq J\|f\|_1\|g\|_\infty$$

where J is the operator norm of T_{K^t} on L^∞. Now take the sup over all $g \in L^\infty$, $\|g\|_\infty \leq 1$. Using the standard converse of Holder's Inequality [F1, p. 181], we have that $\| \int_M K(x, y) f(y)\,dy \|_1 \leq J\|f\|_1$. □

CLAIM B. $T_K(L^1(M)) \subseteq BC(M) \cap L^1(M)$.

Proof. Assume that $f \in L^1$. First note that $|T_K(f)(x)| \leq \int_M |K(x, y)|\,|f(y)|\,dy \leq \|K\|_u\|f\|_1$. Thus, $\|T_K(f)\|_u < \infty$. Now let $\{x_n\} \subseteq M$ be an arbitrary convergent sequence, $x_n \to x_0$. The sequence of functions $K(x_n, y)f(y) \to K(x_0, y)f(y)$ for *a.a.* y, and $|K(x_n, y)f(y)| \leq \|K\|_u|f(y)|$ for all $n \geq 1$. It follows that $T_K(f)(x_n) \to T_K(f)(x_0)$ by Lebesgue's Dominated Convergence Theorem. □

Now set $T = T_K$ and $T^\dagger = T_{K^t}$. Then $T \in A_{W,Y}$ with $\langle Tf, g\rangle = \langle f, T^\dagger g\rangle$ for all $f \in W$ and all $g \in Y$ [Fubini's Theorem].

Concerning the kernels $K(x, y)$ with the properties (1) and (2) in this example, note that the kernel $|K(x, y)|$ also satisfies (1) and (2). Also, if $H(x, y) \in B(M \times M)$, then the pointwise product, $H(x, y)K(x, y)$ satisfies (1) and (2).

EXAMPLE 2. In setup II, set $X = L^2(M)$, and let $W = L^2(M) \cap BC(M)$. In addition to (1), assume that

(2) $T_K \in B(L^2)$;

(3) $\sup_{x\in M}[\int_M |K(x, y)|^2\,dy]^{\frac{1}{2}} \equiv P < \infty$, and $\sup_{y\in M}[\int_M |K(x, y)|^2\,dx]^{\frac{1}{2}} \equiv Q < \infty$.

The condition in (3) is a type of bi-Carleman condition.

CLAIM C. $T_K(L^2) \subseteq BC(M) \cap L^2(M)$.

Proof. First note that by (3), for $f \in L^2$, $|T_K(f)(x)| \leq \int_M |K(x, y)|\;|f(y)|\;dy \leq P\|f\|_2$ by Cauchy-Schwarz. Therefore, $T_K(f) \in L^\infty$. Now assume that $f \in L^1 \cap L^2$. The argument given in Claim B shows that $T_K(f)$ is continuous on M. When $f \in L^2$, choose a sequence $\{f_n\} \subseteq L^1 \cap L^2$ such that $\|f_n - f\|_2 \to 0$. Using Property (3) above and Cauchy-Schwarz, we have for all $x \in M$, $|T_K(f_n)(x) - T_K(f)(x)| \leq \int_M |K(x, y)| \, |f_n(y) - f(y)| \, dy \leq P\|f_n - f\|_2 \to 0$.

Thus, $T_K(f_n) \to T_K(f)$ uniformly on M. Therefore, since the functions $T_K(f_n) \in BC(M)$, it follows that $T_K(f) \in BC(M)$. □

For $x, y \in M$, let $K^*(x, y) = \overline{K(y, x)}$.

CLAIM D. (1) $T_{K^*} = (T_K)^*$; (2) $(T_K)(L^2) \subseteq L^2 \cap BC(M)$ and $(T_K^*)(L^2) \subseteq L^2 \cap BC(M)$.

Proof. That $T_{K^*} = (T_K)^*$ is a straightforward application of Fubini's Theorem. Also, $T_{K^*}(L^2) \subseteq BC(M) \cap L^2(M)$ as in Claim C. □

These two examples provide a large class of integral operators to which the results of this paper apply. Another large class of examples to which the results apply are certain convolution operators; this is shown in the last section of this paper.

3. Annihilators and the closure of the range. When $T \in B(X)$ and W is a T-invariant subspace of X, we denote the restriction of T to W by T_W.

Throughout, we use two continuity properties of linear operators. Both of these properties follow in a straightforward way from the Closed Graph Theorem.

◦ Assume that $T \in B(X)$, W is a Banach space which is a subspace of X, and W is continuously embedded in X. If W is T-invariant, then $T_W \in B(W)$.

◦ Assume that $T \in B(X)$, W is a Banach space which is a subspace of X, and W is continuously embedded in X. If $T(X) \subseteq W$, then $T \in B(X, W)$, the space of all bounded linear maps from X into W.

For the convenience of the reader, we state the usual relations involving annihilators that hold for an operator $S \in B(X)$; These relations can be found in [**8**, Theorems 8.4 and 8.5, p. 232]:

(i) $\overline{\mathbf{R}(S)}^{\perp} = \mathbf{R}(S)^{\perp} = \mathbf{N}(S^*)$; (ii) $\overline{\mathbf{R}(S)} = {}^{\perp}\mathbf{N}(S^*)$; (iii) ${}^{\perp}\mathbf{R}(S^*) = \mathbf{N}(S)$; (iv) $\overline{\mathbf{R}(S^*)} \subseteq \mathbf{N}(S)^{\perp}$.

A version of some of these properties hold for any operator in $A_{W,Y}$. We verify these elementary relationships first.

NOTE 3. Assume that $S \in A_{W,Y}$.

(1) ${}^{\triangle}\mathbf{R}(S^{\dagger}) = \mathbf{N}(S)$; and
(2) $\mathbf{R}(S)^{\triangle} = \mathbf{N}(S^{\dagger})$.
(3) $\overline{\mathbf{R}(S^{\dagger})}^{Y} \subseteq \mathbf{N}(S)^{\triangle}$.

Proof. We verify (1), and (2) follows in the same way [since the roles of S and $S^{\dagger}$ can be interchanged]. Assume that $w \in \mathbf{N}(S)$. Then for all $y \in Y$, $\langle w, S^{\dagger}y\rangle = \langle Sw, y\rangle = 0$. Therefore, $w \in {}^{\triangle}\mathbf{R}(S^{\dagger})$. Conversely suppose $v \in {}^{\triangle}\mathbf{R}(S^{\dagger})$. Then $0 = \langle v, S^{\dagger}y\rangle = \langle Sv, y\rangle$ for all $y \in Y$. Therefore, by the fact that the form is nondegenerate, $v \in \mathbf{N}(S)$.

Now it follows from (1) that $\mathbf{R}(S^{\dagger}) \subseteq \mathbf{N}(S)^{\triangle}$. Then since $\mathbf{N}(S)^{\triangle}$ is $Y-$closed, (3) holds. □

In what follows, we will often make the assumptions listed below in (#).

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(#) Assume that X and W are Banach spaces with W a subspace of X and such that W is continuously embedded in X. Assume that $T \in B(X)$, W is T-invariant, and $T^n(X) \subseteq W$ for some $n \geq 1$.

In (#), note that when $n = 1$, W is automatically T-invariant.

Assume that (#) holds. The following construction provides a means by which a proof for the case $n = 1$ (the case where $T(X) \subseteq W$) can be induced to give a proof in the situation where $n > 1$. This construction was introduced in [**3**]. First, consider the case when $n = 2$, so $T^2(X) \subseteq W$. Set $V = T^{-1}[W]$, so $W \subseteq V \subseteq X$. Define a norm on V by $\|v\|_V = \|v\|_X + \|Tv\|_W$.

(a) $\|v\|_V$ is a complete norm on V;
(b) W is continuously embedded in V, and V is continuously embedded in X;
(c) V is T-invariant, $T(V) \subseteq W$, and $T(X) \subseteq V$.

This was stated in [**3**, Lemma 5], and a proof of part (a) was provided there. The proofs of (a), (b), and (c) are all straightforward.

Now suppose that $T^n(X) \subseteq W$ for some $n > 2$. Just as in the case where $n = 2$, a finite sequence of T-invariant Banach subspaces can be constructed which allows the reduction of proofs to the the case $n = 1$. Set $V_0 = W$ and $\|w\|_0 = \|w\|_W$ for $w \in W$. Let $V_k = \{x \in X : T^k(X) \subseteq W\}$, $1 \leq k \leq n-1$, with norm $\|v\|_k = \|v\|_X + \|Tv\|_{k-1}$. As in (a), (b) and (c) above, we have:

(1) $(V_k, \|v\|_k)$ is a Banach space, $1 \leq k \leq n-1$;
(2) $W = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots . \subseteq V_{n-1} \subseteq V_n = X$, and each of the embeddings are continuous;
(3) Each V_k is T-invariant, and $T(V_k) \subseteq V_{k-1}$ for $1 \leq k \leq n$.

THEOREM 4. *Assume (#) holds. Then for* $\lambda \neq 0$, $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W$.

Proof. First we do the case where $n = 1$, so we make the assumption that $T(X) \subseteq W$. As a first step, we show that $\overline{\mathbf{R}(\lambda - T)}^X \cap W = \overline{(\lambda - T)(W)}^X \cap W$. The inclusion "$\supseteq$" between these two sets is obvious. Now assume that $w_0 \in \overline{\mathbf{R}(\lambda - T)}^X \cap W$. Then there exists $\{x_n\} \subseteq X$ with $\|(\lambda - T)x_n - w_0\|_X \to 0$ as $n \to \infty$. Set $w_n = \lambda^{-1}(Tx_n + w_0)$, and note that by hypothesis, $w_n \in W$. Then $\|x_n - w_n\|_X \to 0$, and this implies that $\|(\lambda - T)w_n - w_0\|_X \to 0$.

We have that $\mathbf{R}(\lambda - T_W) \subseteq \overline{\mathbf{R}(\lambda - T)}^X \cap W$, and note that because W is continuously embedded in X, $\overline{\mathbf{R}(\lambda - T)}^X \cap W$ is closed in W. Thus the inclusion, $\overline{\mathbf{R}(\lambda - T_W)}^W \subseteq \overline{\mathbf{R}(\lambda - T)}^X \cap W$, holds. Suppose that $\overline{\mathbf{R}(\lambda - T_W)}^W \neq \overline{\mathbf{R}(\lambda - T)}^X \cap W$. Now $\overline{\mathbf{R}(\lambda - T_W)}^W = {}^{\perp}\mathbf{N}(\lambda - T_W^*)$, so there exist $\alpha \in \mathbf{N}(\lambda - T_W^*)$ and $z \in \overline{\mathbf{R}(\lambda - T)}^X \cap W$ such that $\langle z, \alpha\rangle = \alpha(z) \neq 0$. Now $\alpha = \lambda^{-1}T_W^*(\alpha)$, so $0 \neq \langle z, \alpha\rangle = \langle z, \lambda^{-1}T_W^*(\alpha)\rangle = \lambda^{-1}\langle T_W(z), \alpha\rangle = \lambda^{-1}(\alpha \circ T)(z)$. Note that since $T \in B(X, W)$, we have $\alpha \circ T \in X^*$. Also, $(\alpha \circ T)((\lambda - T)(W)) = \{0\}$. It follows that $(\alpha \circ T)(\overline{(\lambda - T)(W)}^X \cap W) = \{0\}$. But $z \in \overline{\mathbf{R}(\lambda - T)}^X \cap W = \overline{(\lambda - T)(W)}^X \cap W$ (as shown in the first paragraph). Therefore, $(\alpha \circ T)(z) = 0$, a contradiction. The contradiction implies that

$$\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W.$$

Now assume that $T^n(X) \subseteq W$ for some $n \geq 2$. Then as outlined just prior to the theorem, we construct the finite sequence of spaces $\{V_k\}$ with properties (1), (2), and (3). We do the case $n = 2$, and from this the proof for larger n will be clear. With $n = 2$, we have spaces $W \subseteq V \subseteq X$ with properties (a), (b), and (c) above. Now since $T(X) \subseteq V$,

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the result for case $n = 1$ implies that $\overline{\mathbf{R}(\lambda - T_V)}^V = \overline{\mathbf{R}(\lambda - T)}^X \cap V$. Again, $T(V) \subseteq W$, so applying the result for case $n = 1$ implies that $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^V \cap W$. Thus, $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^V \cap W = [\overline{\mathbf{R}(\lambda - T)}^X \cap V] \cap W = \overline{\mathbf{R}(\lambda - T)}^X \cap W$. □

As a corollary to Theorem 4, we recover a form of the important annihilator relation in (ii) above for $\lambda - T_W$ when $T \in A_{W,Y}$, $\lambda \in \mathbf{C}$, $\lambda \neq 0$, and condition (#) holds.

Corollary 5. *Assume that (#) holds. Also assume that Y is a subspace of X^*, Y is a Banach space, Y is T^*-invariant, that W and Y are in normed duality, and $T \in A_{W,Y}$. Suppose $\lambda \neq 0$ and $\mathbf{N}(\lambda - T^*) = \mathbf{N}(\lambda - T^\dagger)$. Then*

$$\overline{\mathbf{R}(\lambda - T_W)}^W = {}^{\triangle}\mathbf{N}(\lambda - T^\dagger).$$

Proof. From Theorem 4, we have, $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W$. Also, $\overline{\mathbf{R}(\lambda - T)}^X = {}^{\perp}\mathbf{N}(\lambda - T^*)$. By hypothesis, $\mathbf{N}(\lambda - T^*) = \mathbf{N}(\lambda - T^\dagger)$. Therefore, $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W = {}^{\perp}\mathbf{N}(\lambda - T^*) \cap W = {}^{\perp}\mathbf{N}(\lambda - T^\dagger) \cap W = {}^{\triangle}\mathbf{N}(\lambda - T^\dagger)$. □

4. States of operators; a closed range theorem. The states of an operator (from [**8**, p. 237]): Assume that $S \in B(X)$. Consider the list of possible basic proerties of S: I. $\mathbf{R}(S) = X$; II. $\overline{\mathbf{R}(S)} = X$, but $\mathbf{R}(S) \neq X$; III. $\overline{\mathbf{R}(S)} \neq X$. 1. $\mathbf{R}(S)$ is closed and $\mathbf{N}(S) = \{0\}$; 2. $\mathbf{R}(S)$ is not closed and $\mathbf{N}(S) = \{0\}$; 3. $\mathbf{N}(S) \neq \{0\}$.

Then, for example, S is said to be in state II_3 when $\overline{\mathbf{R}(S)} = X$, $\mathbf{R}(S) \neq X$, and $\mathbf{N}(S) \neq \{0\}$ (of course, some states are impossible, such as I_2).

Theorem 6. *Assume that (#) holds. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$.*

(1) $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T_W)$;

(2) $\mathbf{R}(\lambda - T) = X \Longleftrightarrow \mathbf{R}(\lambda - T_W) = W$;

(3) $\overline{\mathbf{R}(\lambda - T)}^X = X \Longleftrightarrow \overline{\mathbf{R}(\lambda - T_W)}^W = W$;

(4) $\mathbf{R}(\lambda - T)$ *is closed in* $X \Longleftrightarrow \mathbf{R}(\lambda - T_W)$ *is closed in* W.

It follows from (1), (2), (3), *and* (4), *that $\lambda - T$ on X is in exactly the same state as $\lambda - T_W$ on W.*

Proof. Clearly, $\mathbf{N}(\lambda - T_W) \subseteq \mathbf{N}(\lambda - T)$. If $x \in \mathbf{N}(\lambda - T)$, then $Tx = \lambda x$, and $T^n x = \lambda^n x$. Therefore, $x = \lambda^{-n} T^n x \in W$. This verifies (1).

(2): From [**3**, Propositions 2 and 3], we have $\mathbf{R}(\lambda - T_W) = \mathbf{R}(\lambda - T) \cap W$. Therefore, if $\mathbf{R}(\lambda - T) = X$, then $\mathbf{R}(\lambda - T_W) = W$. Conversely, suppose $\mathbf{R}(\lambda - T_W) = W$. We that $\mathbf{R}(\lambda - T) = X$ in the case $n = 1$; the general case where $n \geq 2$ follows from this using the finite sequence of subspaces $\{V_k\}$ as before. Let $y \in X$. Since $Ty \in W$, there exists $w \in W$ such that $(\lambda - T)w = Ty$. Then $(\lambda - T)(\lambda^{-1}(y + w)) = y + \lambda^{-1}[-Ty + (\lambda - T)w] = y$. This proves (2).

(3): From Theorem 4, we have $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W$. If $\overline{\mathbf{R}(\lambda - T)}^X = X$, then $\overline{\mathbf{R}(\lambda - T)}^W = X \cap W = W$. Again, in the proof of the converse, we assume that $n = 1$. Now suppose that $\overline{\mathbf{R}(\lambda - T_W)}^W = W$. Let $y \in X$. Since $Ty \in W$, there exists $\{w_n\} \subseteq W$ such that $\|(\lambda - T)(w_n) - T(y)\|_W \to 0$. Then

$$\|(\lambda - T)(\lambda^{-1}(y + w_n)) - y\|_X = \| - \lambda^{-1} T(y) + \lambda^{-1}(\lambda - T)(w_n)\|_X \to 0.$$

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Therefore, $y \in \overline{\mathbf{R}(\lambda - T)}^X$. This proves (3).

(4): This follows from [**3**, Theorem 6].

That $\lambda - T$ on X is in exactly the same state as $\lambda - T_W$ on W follows directly from (1), (2), (3), and (4). □

7. A Closed Range Theorem Assume the standard setup I. Assume that $T \in B(X)$ has the properties:

(i) W is T-invariant and Y is T^*-invariant;

(ii) There exist $n \geq 1$ and $m \geq 1$ such that $T^n(X) \subseteq W$ and $(T^*)^m(X^*) \subseteq Y$.

Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$. The following are equivalent:

(1) $\mathbf{R}(\lambda - T_W)$ is closed in W;

(2) $\mathbf{R}(\lambda - (T_W)^\dagger)$ is closed in Y [note that $(T_W)^\dagger = (T^*)_Y$];

(3) $\mathbf{R}(\lambda - T_W) = {}^\triangle\mathbf{N}(\lambda - (T_W)^\dagger)$;

(4) $\mathbf{R}(\lambda - (T_W)^\dagger) = \mathbf{N}(\lambda - T_W)^\triangle$;

(5) $\mathbf{R}(\lambda - T)$ is closed in X;

(6) $\mathbf{R}(\lambda - T^*)$ is closed in X^*.

Proof. It is clear that (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2). Also, (5) $\Longleftrightarrow$ (6) by the usual Closed Range Theorem [**8**, Theorem 10.1, p. 240]. From [**3**, Theorem 6], using (i) and (ii), we have (1) $\Longleftrightarrow$ (5) and (2) $\Longleftrightarrow$ (6).

Now assume that (1) holds. By Corollary 5, $\overline{\mathbf{R}(\lambda - T_W)}^W = {}^\triangle\mathbf{N}(\lambda - T^\dagger)$. Thus in this case, (1) $\Longrightarrow$ (3). The same argument verifies that (2) $\Longrightarrow$ (4). This proves the equivalence of the statements (1)–(6). □

Assume that $S \in B(X)$, so $S^* \in B(X^*)$. The state of the pair (S, S^*) is the pair (state of S, state of S^*). See [**8**, pp. 237–8] for basic information concerning possible states of the pair (S, S^*). For example, $(\mathrm{III}_1, \mathrm{I}_3)$ is a possible state of the pair (S, S^*). In this case, S is in state III_1 [meaning that $\mathbf{R}(S)$ is closed, $\mathbf{R}(S) \neq X$, and $\mathbf{N}(S) = \{0\}$] and S^* is in state I_3 [meaning that $\mathbf{R}(S^*) = X^*$ and $\mathbf{N}(S^*) \neq \{0\}$].

Corollary 8. *Assume the standard setup I. Assume that $T \in B(X)$, and the following conditions hold:*

(i) *W is T-invariant and Y is T^*-invariant;*

(ii) *For some $n \geq 1$, $m \geq 1$, $T^n(X) \subseteq W$ and $(T^*)^m(X^*) \subseteq Y$.*

Then for all $\lambda \neq 0$, the state of the pair $(\lambda - T, \lambda - T^*)$ is the same as the state of the pair $(\lambda - T_W, \lambda - T_Y^*)$.

5. Fredholm properties. With the same setup and hypotheses on T and T^* as in the Closed Range Theorem in Section IV, in this section we prove that the Fredholm properties of $\lambda - T_W$ and $\lambda - T^\dagger$ are the same as those of $\lambda - T$ and $\lambda - T^*$.

Notation: For $S \in B(X)$, $\mathrm{nul}(S) = \dim(\mathbf{N}(S))$; $\mathrm{def}(S) = \dim(X/\mathbf{R}(S))$; $S \in \Phi_+(X)$ if $\mathrm{nul}(S) < \infty$ and $\mathbf{R}(S)$ is closed; $S \in \Phi_-(X)$ if $\mathrm{def}(S) < \infty$; $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$; when $S \in \Phi(X)$, $\mathrm{ind}(S) = \mathrm{nul}(S) - \mathrm{def}(S)$. Note that by [**1**, Cor. 2.17, p. 76], when $\mathrm{def}(S) < \infty$, the $\mathbf{R}(S)$ is closed.

Proposition 9. *Assume that (#) holds. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$.*

(1) *If* $\mathrm{def}(\lambda - T) = k < \infty$, *then* $\mathrm{def}(\lambda - T_W) \leq k$.

(2) *Assume in addition that W and Y are in normed duality, Y is a subspace of X^*, $T \in A_{W,Y}$, and* $\mathbf{N}(\lambda - T^*) = \mathbf{N}(\lambda - T^\dagger)$. *Then* $\mathrm{def}(\lambda - T) = k < \infty \Longleftrightarrow \mathrm{def}(\lambda - T_W) = k < \infty$.

Proof. We need only do the proof in the case where $n = 1$. Suppose $\mathrm{def}(\lambda - T) = k < \infty$. Then $X/\mathbf{R}(\lambda - T)$ has dimension k. Let $\{w_1, w_2, \ldots, w_p\} \subseteq W$ be linearly independent (l.i.) modulo $\mathbf{R}(\lambda - T_W)$.

Claim: $\{w_1, w_2, \ldots, w_p\}$ is l.i. modulo $\mathbf{R}(\lambda - T)$. For suppose that for some scalars, λ_j, and some $x \in X$, $\lambda_1 w_1 + \lambda_2 w_2 + \cdots \lambda_p w_p = (\lambda - T)x$. Since $Tx \in W$, it follows that $x \in W$. Thus, $\lambda_1 w_1 + \lambda_2 w_2 + \cdots \lambda_p w_p \in \mathbf{R}(\lambda - T_W)$, and so by assumption, $\lambda_j = 0$ for $1 \leq j \leq p$. This verifies the Claim.

The Claim implies that $p \leq k$, so that $\mathrm{def}(\lambda - T_W) \leq k$.

Now assume the hypotheses in (2). Suppose that $\mathrm{def}\,(\lambda - T_W) = k < \infty$, so that $W/\mathbf{R}(\lambda - T_W)$ is k-dimensional. For $y \in \mathbf{N}(\lambda - T^\dagger)$, define $\widehat{y}$ on $W/\mathbf{R}(\lambda - T_W)$ by $\widehat{y}(w + \mathbf{R}(\lambda - T_W)) = \langle w, y\rangle$. Since $\widehat{y} = 0$ implies $y = 0$, we conclude that $y \to \widehat{y}$ is a $1 - 1$ linear map of $\mathbf{N}(\lambda - T^\dagger)$ into the dual of $W/\mathbf{R}(\lambda - T_W)$. It follows that $\dim(\mathbf{N}(\lambda - T^\dagger)) \leq k$. By hypothesis, $\mathbf{N}(\lambda - T^*) = \mathbf{N}(\lambda - T^\dagger)$, so $\dim(\mathbf{N}(\lambda - T^*)) \leq k$. Note that $\mathbf{R}(\lambda - T_W)$ is closed, and so, $\mathbf{R}(\lambda - T)$ is closed by [B2, Theorem 6]. Then $\mathbf{R}(\lambda - T) = {}^\perp\mathbf{N}(\lambda - T^*)$, and it follows that $\mathrm{def}\,(\lambda - T) \leq k$. This proves: $\mathrm{def}\,(\lambda - T_W) = k < \infty \Longrightarrow \mathrm{def}\,(\lambda - T_W) \leq k$. This implication together with the implication established in (1), proves that $\mathrm{def}\,(\lambda - T) = k < \infty \Longleftrightarrow \mathrm{def}\,(\lambda - T_W) = k < \infty$. $\square$

Corollary 10. *Assume that (#) holds. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$.*

(1) $\mathrm{nul}(\lambda - T) = \mathrm{nul}(\lambda - T_W)$; $(\lambda - T) \in \Phi_+(X) \Longleftrightarrow (\lambda - T_W) \in \Phi_+(W)$;

Assume in addition to (#) that W and Y are in normed duality, Y is a subspace of X^*, $T \in A_{W,Y}$, and $\mathbf{N}(\lambda - T^*) = \mathbf{N}(\lambda - T^\dagger)$.

(2) $\mathrm{def}\,(\lambda - T) = \mathrm{def}\,(\lambda - T_W)$; $(\lambda - T) \in \Phi_-(X) \Longleftrightarrow (\lambda - T_W) \in \Phi_-(W)$.

Proof. By [**3**, Theorem 6], $\mathbf{R}(\lambda - T_W)$ is closed in $W \Longleftrightarrow \mathbf{R}(\lambda - T)$ is closed in X. As noted in part (1) of Theorem 6, $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T_W)$, so clearly, $\mathrm{nul}(\lambda - T) = \mathrm{nul}(\lambda - T_W)$. Also, from part (2) of Proposition 9, $\mathrm{def}\,(\lambda - T) = \mathrm{def}\,(\lambda - T_W)$. These facts imply (1) and (2) of the corollary. $\square$

Theorem 11. *Assume that Setup I holds, and that $T \in A_{W,Y}$. Assume that there exist $n \geq 1$ and $m \geq 1$ such that $T^n(X) \subseteq W$ and $(T^*)^m(X^*) \subseteq Y$. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$. The following are equivalent:* (1) $(\lambda - T) \in \Phi(X)$; (2) $(\lambda - T_W) \in \Phi(W)$; (3) $(\lambda - T^*) \in \Phi(X^*)$; (4) $(\lambda - T^\dagger) \in \Phi(Y)$.

Moreover, when any one of these four conditions hold, then

$$\mathrm{ind}(\lambda - T) = \mathrm{ind}(\lambda - T_W) = -\mathrm{ind}(\lambda - T^\dagger) = -\mathrm{ind}(\lambda - T^*).$$

Proof. By the Closed Range Theorem 7, the range of any one of the four operators in (1)–(4) is closed implies the ranges of all the four operators are closed. As shown in Corollary 10, $\mathrm{nul}(\lambda - T) = \mathrm{nul}(\lambda - T_W)$ and $\mathrm{def}\,(\lambda - T) = \mathrm{def}\,(\lambda - T_W)$. Also, since $T^\dagger = (T^*)_Y$, Corollary 10 applies to these adjoint operators. Therefore, $\mathrm{nul}(\lambda - T^*) = \mathrm{nul}(\lambda - T^\dagger)$ and $\mathrm{def}\,(\lambda - T^*) = \mathrm{def}\,(\lambda - T^\dagger)$. Finally, it is a standard fact from Fredholm theory that $\mathrm{ind}(\lambda - T) = -\,\mathrm{ind}(\lambda - T^*)$. Together these facts prove the result. $\square$

In [**7**], a very successful Fredholm theory is developed for operators in $A_{W,Y}$. The key hypothesis there on an operator $S \in A_{W,Y}$ is that $S \in \Phi(W)$, $S^\dagger \in \Phi(Y)$, and $\mathrm{ind}(S) = -\,\mathrm{ind}(S^\dagger)$. Therefore, this Fredholm theory applies to the operators $\lambda - T_W$ and $\lambda - T^\dagger$ in the setting of the last theorem; see [**7**, 5.8].

6. Linear operators on spaces with a bounded inner product. Throughout this short section, X is a Hilbert space, W is a subspace of X, and W is a Banach space which is continuously embedded in X. For V a subset of X, we use the usual meaning of $V^{\perp}$, and when $V \subseteq W$, we set $V^{\triangle} = V^{\perp} \cap W$. Consider the following condition:

($\flat$) $T \in B(X)$, W is both T and T^* invariant, and there exist $n \geq 1, m \geq 1$ such that $T^n(X) \subseteq W$ and $(T^*)^m(X) \subseteq W$.

It is clear that when ($\flat$) holds, then the results of the previous sections hold for T and T^* with the form of the result adjusted to the context. For example, there is the following theorem.

12. A Closed Range Theorem. Assume that ($\flat$) holds.

For $\lambda \in \mathbf{C}$, $\lambda \neq 0$. The following are equivalent:
(1) $\mathbf{R}(\lambda - T_W)$ is closed in W;
(2) $\mathbf{R}(\lambda - T_W^*)$ is closed in W;
(3) $\mathbf{R}(\lambda - T_W) = \mathbf{N}(\lambda - T_W^*)^{\triangle}$;
(4) $\mathbf{R}(\lambda - T_W^*) = \mathbf{N}(\lambda - T_W)^{\triangle}$;
(5) $\mathbf{R}(\lambda - T)$ is closed in X;
(6) $\mathbf{R}(\lambda - T^*)$ is closed in X.

NOTE 13. It follows from a result of R. Douglas, that if $S \in B(X)$ and S is normal, then $\mathbf{R}(S) = \mathbf{R}(S^*)$ [**4**, Theorem 1]. Now assume that T is normal and $T^n(X) \subseteq W$ for some $n \geq 1$. Then as T^n is normal, $\mathbf{R}(T^n) = \mathbf{R}((T^*)^n)$. Thus the condition $(T^*)^n(X) \subseteq W$ (part of condition ($\flat$)) will automatically hold in this case.

PROPOSITION 14. *Assume $T \in B(X)$ and that* ($\flat$) *holds.* $W = \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\bar{\lambda} - T_W^*)$ *and* $\overline{\mathbf{R}(\lambda - T_W)}^W = \mathbf{N}(\bar{\lambda} - T_W^*)^{\triangle}$.

When T is normal, $W = \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\lambda - T_W)$.

Proof. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$. Since X is a Hilbert space, we have

$$X = \overline{\mathbf{R}(\lambda - T)}^X \oplus \mathbf{N}(\bar{\lambda} - T^*) \text{ and } \overline{\mathbf{R}(\lambda - T)}^X = \mathbf{N}(\bar{\lambda} - T^*)^{\perp}. \tag{1}$$

It suffices to consider the case $n = 1$. Thus we assume that $T(X) \subseteq W$. We prove in this case that since (1) holds, then $W = \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\bar{\lambda} - T_W^*)$ and $\overline{\mathbf{R}(\lambda - T_W)}^W = \mathbf{N}(\bar{\lambda} - T_W^*)^{\triangle}$.

Assume that $w \in W$. Then by (1), $w = v + z$ where $v \in \overline{\mathbf{R}(\lambda - T)}^X$ and $z \in \mathbf{N}(\bar{\lambda} - T^*)$. Now $z \in \mathbf{N}(\bar{\lambda} - T^*) = \mathbf{N}(\bar{\lambda} - T_W^*) \subseteq W$. It follows that $v \in \overline{\mathbf{R}(\lambda - T)}^X \cap W$. By Theorem 4, $\overline{\mathbf{R}(\lambda - T_W)}^W = \overline{\mathbf{R}(\lambda - T)}^X \cap W$. Therefore, $v \in \overline{\mathbf{R}(\lambda - T_W)}^W$. This proves that $W \subseteq \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\bar{\lambda} - T_W^*)$, so that $W = \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\bar{\lambda} - T_W^*)$. That $\overline{\mathbf{R}(\lambda - T_W)}^W = \mathbf{N}(\bar{\lambda} - T_W^*)^{\triangle}$ follows from Corollary 5.

When T is normal, it is easy to see that $\mathbf{N}(\lambda - T_W) = \mathbf{N}(\bar{\lambda} - T_W^*)$, so in this case,

$$W = \overline{\mathbf{R}(\lambda - T_W)}^W \oplus \mathbf{N}(\lambda - T_W). \qquad \square$$

7. Convolution operators acting on spaces of continuous functions. In this last section, we look at examples involving certain convolution operators, T_f. We show that the results of this paper apply to the operators $\lambda - T_f$ and $\lambda - T_{\tilde{f}}$, $\lambda \neq 0$, acting on the spaces of continuous functions $L^1(G) \cap BC(G)$ and $BC(G)$, respectively. Part of this example appears in [**3**].

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EXAMPLE 15. Let G be a locally compact, T_2, topological group, and assume that G is unimodular. Fix a Haar measure on G. In what follows, we omit G in the notation of the various spaces involved; eg., $L^1 = L^1(G)$ and $BC = BC(G)$. For $f \in L^1$, $g \in L^p$ for some p, $1 \leq p \leq \infty$, we set $T_f(g) = f * g$. The operator T_f is the usual left convolution operator on the various spaces.

First in Setup I, take $X = L^1$, $W = L^1 \cap BC$, $X^* = L^\infty$, and $Y = BC$. From Folland's book [**6**], Propositions (2.39) and (2.40), we have the following facts:

(1) For $f \in L^1 \cap L^2$, $f * f \in L^1 \cap BC$, and this implies, $(T_f)^2(L^1) = T_{f*f}(L^1) \subseteq L^1 \cap BC$;

(2) For $f \in L^1$, $T_f(L^\infty) \subseteq BC$.

For $f \in L^1$, let $\widetilde{f}(x) = f(x^{-1})$, $x \in G$. Then $\widetilde{f} \in L^1$ and $T_f \in B(L^1)$ has adjoint $(T_f)^* = T_{\widetilde{f}} \in B(L^\infty)$. Therefore, we have the result:

16. For $f \in L^1 \cap L^2$, $(T_f)^2(L^1) \subseteq L^1 \cap BC$ and $(T_f)^*(L^\infty) \subseteq BC$.

Thus, the results of the previous sections apply to the convolution operators given the setup above.

Make the same assumptions as in the first paragraph in this example. Now we consider Setup II with $L^2 = L^2(G)$ and $W = L^2 \cap BC$. We use the same results from Folland [**6**] to yield:

(3) For $f \in L^1 \cap L^2$, we have $T_f(L^2) \subseteq L^2 \cap BC$.

In this case, $T_f \in B(L^2)$ has adjoint $(T_f)^* = T_{f^*}$ [here $f^*(x) = \overline{f(x^{-1})}$, $x \in G$] These observations lead to a second result:

17. For $f \in L^1 \cap L^2$, $T_f(L^2) \subseteq L^2 \cap BC$ and $(T_f)^*(L^2) \subseteq L^2 \cap BC$.

It follows that condition ($\flat$) holds when $X = L^2$, $W = L^2 \cap BC$, and $T = T_f$, $f \in L^1 \cap L^2$.

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