Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 372–375

ULTRAFUNCTORS

ANDY R. MAGID

Let G be a functor from commutative rings to abelian groups and let $\{R_i : i \in S\}$ be a family of commutative rings indexed by the set S. Let $\mathscr{F}$ be an ultrafilter on S, and let $\prod R_i/\mathscr{F}$ denote the ultraproduct of the R_i with respect to $\mathscr{F}$. This paper studies the problem of computing $G(\prod R_i/\mathscr{F})$ from the $G(R_j)$ via the map

(*) $\quad G(\prod R_i/\mathscr{F}) \to \prod G(R_i)/\mathscr{F}$.

The functors studied are Pic = Picard group, Br = Brauer group, U = units, and the functors K_0, K_1, SK_1, K_2 of Algebraic K-Theory. For $G = Pic$, U, K_1 and SK_1, (*) is always a monomorphism. An example is given to show that even if all the R_i are finite fields the map (*) has a kernel for $G = K_2$. For the remaining functors, a necessary and sufficient condition on the family $\{R_i\}$ is given under which (*) is a monomorphism is given, but we do not know if there are families which do not satisfy the condition.

Ultraproducts have been used to construct non-standard models of the integers and the real numbers, and our techniques are applied to compute the values of the above functors on these models.

We begin by recalling the definition of ultraproducts (see, for example [**1**, p. 179]): let S be a set, $\mathscr{F}$ an ultrafilter on S and $\{A_i : i \in S\}$ a family of rings or abelian groups indexed by S. If T is a subset of S, let $\prod^T A_i$ be the product of the A_i's for i in T. If $S \supset T \supset V$, we have a projection $\prod^T A_i \to \prod^V A_i$. $\mathscr{F}$ is directed by containment, and $\prod A_i/\mathscr{F} = \text{dir lim} \prod^F A_i$, where the maps in the directed system are the above projections, F ranging over $\mathscr{F}$.

Definition. Let $\mathscr{C}$ be a class of commutative rings. A functor G from commutative rings to abelian groups is called a *$\mathscr{C}$-ultrafunctor* if

(a) G commutes with arbitrary direct limits and finite products; and

(b) $G(\prod R_i) \to \prod G(R_i)$ is injective for all families $\{R_i\}$ of commutative rings contained in $\mathscr{C}$.

THEOREM 1. *Let $\mathscr{C}$ be a class of commutative rings and G a functor from commutative rings to abelian groups satisfying condition* (a). *Then the map* (*) *above is injective for all ultraproducts of rings in $\mathscr{C}$ if and only if G is a $\mathscr{C}$-ultrafunctor.*

Proof. Let $\{R_i : i \in S\}$ be a family in $\mathscr{C}$ and $\mathscr{F}$ an ultrafilter on S. Then

$$G(\prod R_i/\mathscr{F}) = G(\text{dir lim} \prod^F R_i) = \text{dir lim}\, G(\prod^F R_i),$$

Received September 17, 1973. This research was partially supported by NSF Grant GP-37051.

and hence (*) is the direct limit of the maps $G(\prod^F R_i) \to \prod^F G(R_i)$. If G is a $\mathscr{C}$-ultrafunctor, these maps are all injective, and hence so is their direct limit (*). Now suppose G is not a $\mathscr{C}$-ultrafunctor and that $x \neq 0$ is in the kernel of $G(\prod^S R_i) \to \prod^S G(R_i)$. Let $\mathscr{F}_0 = \{ F \subset S$: the image of x in $G(\prod^{S-F} R_i)$ is zero$\}$. We will show that $\mathscr{F}_0$ is a filter: clearly $\emptyset$,t he empty set, is not in $\mathscr{F}_0$. Suppose H and K are in $\mathscr{F}_0$. Then

$$T = S - (H \cap K) = (S - H) \cup (S - K) = P \cup Q \cup R$$

where $P \subset S - H$, $R \subset S - K$, $Q = (S - H) \cap (S - K)$ and P, Q, R are pairwise disjoint. Then the images of x in $G(\prod^P R_i)$, $G(\prod^Q R_i)$ and $G(\prod^R R_i)$ are all zero, being restrictions of images of x in $G(\prod^{S-H} R_i)$ and $G(\prod^{S-K} R_i)$; since G commutes with finite products, this means the image of x in

$$G(\textstyle\prod^T R_i) = G(\prod^P R_i) \times G(\prod^Q R_i) \times G(\prod^R R_i)$$

is zero, so $H \cap K \in \mathscr{F}_0$. Finally, suppose that $F \in \mathscr{F}_0$ and $L \supset F$. The image of x under the composite $G(\prod^S R_i) \to G(\prod^{S-F} R_i) \to G(\prod^{S-L} R_i)$ is zero, so $L \in \mathscr{F}_0$. Thus $\mathscr{F}_0$ is a filter. Expand $\mathscr{F}_0$ to an ultrafilter $\mathscr{F}$. Let $F \in \mathscr{F}$, and suppose the image of x in $G(\prod^F R_i)$ is zero. Then $S - F \in \mathscr{F}$, and this is impossible. Thus for all $F \in \mathscr{F}$, the image of x is non-zero in $G(\prod^F R_i)$. Thus x is non-zero in dir lim $G(\prod^F R_i) = G(\prod R_i/\mathscr{F})$, but belongs to the kernel of (*) since it is already zero in $\prod^S G(R_i)$. Thus if G is not a $\mathscr{C}$-ultrafunctor, (*) is not always injective.

We note that the first part of the proof requires only that G satisfy (b) and the first part of (a).

We record two consequences of Theorem 1 for later use:

COROLLARY 2. *Let G be a $\mathscr{C}$-ultrafunctor and $\{R_i : i \in S\}$ a family of commutative rings in $\mathscr{C}$. Suppose $G(R_i) = 0$ for all i. Then for any ultrafilter $\mathscr{F}$ on S, $G(\prod R_i/\mathscr{F}) = 0$.*

COROLLARY 3. *Let G be a $\mathscr{C}$-ultrafunctor and $\{R_i : i \in S\}$ a family of commutative rings in $\mathscr{C}$. Suppose there is a commutative ring R_0 and ring homomorphisms $R_0 \to R_i$ for each $i \in S$ inducing isomorphisms $G(R_0) \to G(R_i)$ and suppose further that $G(R_0)$ is finite. Then for any ultrafilter $\mathscr{F}$ on S, $G(\prod R_i/\mathscr{F}) = G(R_0)$.*

Proof. The corollary follows from the fact that the map $G(R_0) \to \prod G(R_i)/\mathscr{F}$ induced by the diagonal is an isomorphism (which can be seen, for example, by characterizing the finite abelian group $G(R_0)$ be a finite set of equations).

The second corollary applies, in particular, to the case where all R_i equal R_0. Next, we examine which of the special functors are ultrafunctors:

PROPOSITION 4. *Pic, K_1, SK_1 and U are ultrafunctors for any class of commutative rings.*

https://doi.org/10.4153/CJM-1975-045-9 Published online by Cambridge University Press

Proof. For Pic, we use that fact that if $R = \prod R_i$ and P is a finitely generated projective R-module then $P = \prod (P \otimes_R R_i)$. Thus if $[I]$ is in the kernel of $Pic\ (R) \to \prod Pic\ (R_i)$, $I \otimes_R R_i = R_i$ for all i, so $[I] = [R]$ is trivial. For K_1, we note that $GL(R) = \prod GL(R_i)$ (GL = general linear group) and hence the same formula applies to the abelianization of both sides. The argument for U and SK_1 is similar.

Example 5. K_2 is not an ultrafunctor for any class containing all finite fields: let $\{R_i\}$ be a collection of finite fields such that, for a suitable ultrafilter $\mathscr{F}$ on the index set, $\prod R_i/\mathscr{F}$ is uncountable (such collections exist by [**3**, Theorem 5.1.1, p. 387]). By [**2**, Theorem 11.10, p. 107], $K_2(\prod R_i/\mathscr{F})$ is also countable, but by [**2**, Corollary 9.9, p. 75], each $K_2(R_i)$, and hence $\prod K_2(R_i)/\mathscr{F}$, is zero. Thus (*) is not injective.

To study the remaining functors, we need to recall that if $\{P_i\}$ is a collection of finitely generated projective modules over corresponding rings $\{R_i\}$, then $P = \prod P_i$ is a finitely generated projective $R = \prod R_i$-module if and only if for some n each P_i can be generated by n elements.

The next definition is designed to make possible the propositions which follow:

Definition 6. A class $\mathscr{C}$ of commutative rings if *K_0-bounded* if there is a function f from positive integers to positive integers such that for any R in $\mathscr{C}$ a stably free R-module on n generators has a free complement of rank $f(n)$. $\mathscr{C}$ is *Br-bounded* if there is such a function f such that for any R in $\mathscr{C}$ and any finitely generated projective R-module P, if $\mathrm{END}_R(P)$ can be generated by n elements there is a finitely generated projective R-module Q with $f(n)$ generators such that $\mathrm{END}_R(P) = \mathrm{END}_R(Q)$.

Note that there are bounded classes: for example, for each d, the class of Noetherian rings of dimension at most d, also, the class of local rings.

PROPOSITION 7. *Let $\mathscr{C}$ be a K_0-bounded class of commutative rings. Then K_0 is a $\mathscr{C}$-ultrafunctor.*

Proof. Let $\{R_i\}$ be a family in $\mathscr{C}$, $R = \prod R_i$. Suppose $[P] - [R^r]$ is in the kernel of $K_0(R) \to \prod K_0(R_i)$. This means that, for each i, $P_i = P \otimes_R R_i$ is stably isomorphic to R_i^r. Suppose P, which has rank r, can be generated by n elements, and that $k = f(n)$, where f is the bounding function for $\mathscr{C}$. Then $P_i \otimes R_i^k = R_i^{r+k}$ for each i, so $P \oplus R^k = R^{r+k}$. Thus P and R^r are also stably isomorphic.

We note that the proof shows that all elements of the kernel are of constant rank zero, and hence, if $\tilde{K}_0$ denotes the subgroup of K_0 consisting of classes of rank 0, we have:

COROLLARY 8. *Let $\mathscr{C}$ be a K_0-bounded class of commutative rings. Then $\tilde{K}_0$ is a $\mathscr{C}$-ultrafunctor.*

PROPOSITION 9. *Let $\mathscr{C}$ be a Br-bounded class of commutative rings. Then Br is a $\mathscr{C}$-ultrafunctor.*

Proof. Let $\{R_i\}$ be a family in $\mathscr{C}$, $R = \prod R_i$ and $[A]$ in the kernel of $Br(R) \to \prod Br(R_i)$. Then for each i, $A_i = A \otimes_R R_i = \mathrm{END}_{R_i}(P_i)$ for some projective R_i-module P_i. Suppose A has n generators and $k = f(n)$ where f is the bounding function for $\mathscr{C}$. Then $A_i = \mathrm{END}_{R_i}(Q_i)$ where Q_i is projective with k generators. It follows that $Q = \prod Q_i$ is a finitely generated projective R-module and $A = \mathrm{END}_R(Q)$, so $[A] = 0$.

We note some applications: $\tilde{K}_0$, SK_1, Br, and Pic are all trivial for the integers Z. The class consisting of Z alone is K_0- and Br-bounded. Thus if $\hat{Z}$ is a non-standard model of Z constructed by ultraproducts, the theorem and propositions show that the above four functors vanish on $\hat{Z}$.

The class consisting of the real numbers $\mathbf{R}$ above is clearly Br-bounded. Thus if $\hat{\mathbf{R}}$ is a non-standard model of $\mathbf{R}$ constructed by ultraproducts, then by Corollary 3 and Proposition 9, $Br(\hat{\mathbf{R}})$ is cyclic of order 2, so every division algebra over $\hat{\mathbf{R}}$ comes from either $\mathbf{R}$ or the quarternions.

The class of local rings is also clearly Br-bounded. We can use this observation and Proposition 9 to show that if R is a profinite ring, $Br(R)$ is trivial: for by [**4**, Theorem 2, p. 87], $R = \prod R_i$, where each R_i is a complete local ring with finite residue class field. Now $Br(R) \to \prod Br(R_i)$ is one-one, and it is known that $Br(R_i) = 0$.

REFERENCES

1. I. N. Herstein, *Noncommutative rings*, Carus Monograph *15*, Mathematical Association of America, 1968.

2. J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies *72*, Princeton, 1971.

3. R. S. Pierce, *Rings of continuous integer valued functions*, Trans. Amer. Math. Soc. *100* (1961), 371–394.

4. D. Zelinsky, *Linearly compact modules and rings*, Amer. J. Math. *75* (1953), 79–90.

The University of Oklahoma,
Norman, Oklahoma