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REMARKS ON THE INTERSECTION OF FINITELY GENERATED SUBGROUPS OF A FREE GROUP

BY

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ABSTRACT. The first result gives a (modest) improvement of the best general bound known to date for the rank of the intersection $U \cap V$ of two finite-rank subgroups of a free group F in terms of the ranks of U and V. In the second result it is deduced from that bound that if A is a finite-rank subgroup of F and $B \leq F$ is non-cyclic, then the index of $A \cap B$ in B, if finite, is less than $2(\text{rank}(A) - 1)$, whence in particular if rank $(A) = 2$, then $B \leq A$. (This strengthens a lemma of Gersten.) Finally a short proof is given of Stallings' result that if U, V (as above) are such that $U \cap V$ has finite index in both U and V, then it has finite index in their join $\langle U, V\rangle$.

1. **The Howson property**. In [5] A. G. Howson showed that the intersection of any two finitely generated subgroups U, V of a free group F is again finitely generated, and gave a bound for the rank $r(U \cap V)$ of the intersection in terms of the ranks $r(U)$ $(= m$ say) $r(V)$ $(= n$ say) of U and V, subsequently improved by Hanna Neumann [8] to the following (where it is assumed that $m, n > 0$):

$$r(U \cap V) - 1 \leq 2(m - 1)(n - 1).$$

(Note that in contrast with this one can easily show (essentially as in the second remark in [2]) that if either U or V has finite index in F, then we have (assuming $m, n > 0$, $r(F) > 1$) $r(U \cap V) - 1 \leq (m - 1)(n - 1)/(r(F) - 1)$.)

The best general bound to date is that established in [1] (see also [9], [10]), namely

$$(1) \quad r(U \cap V) - 1 \leq 2(m - 1)(n - 1) - \min\{m - 1, n - 1\} \; (m, n > 0).$$

Our first result represents a modest improvement of the latter bound.

PROPOSITION 1. *Let U, V be (non-trivial) subgroups of a free froup F of finite ranks m, n respectively and suppose that A, B are subgroups of F such that U has index i in A, and V has index j in B. Then*

$$(2) \qquad r(U \cap V) - 1 \leq 2(m - 1)(n - 1) - \min\{j(m - 1), i(n - 1)\}.$$

PROOF. The index of $U \cap V$ in $A \cap B$ is at most ij. Denoting the ranks of A and B by a and b, we have by the Schreier index formula

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$$(m - 1) = i(a - 1), (n - 1) = j(b - 1), r(U \cap V) - 1 \leq ij(r(A \cap B) - 1).$$

The desired bound (2) now follows since by (1) applied to $A \cap B$ we have

$$r(A \cap B) - 1 \leq 2(a - 1)(b - 1) - \min\{a - 1, b - 1\}.$$

EXAMPLES. If $m = n = 3$, $i = j = 2$ (whence $a = b = 2$ necessarily), then (2) yields $r(U \cap V) \leq 5$, while (1) yields only $r(U \cap V) \leq 7$. On the other hand if $m = n = 3$, $i = 2$, $j = 1$ (whence $a = 2$, $b = 3$ necessarily), then (2) affords no improvement over (1).

Our second result was prompted by [3, Lemma 5.2].

PROPOSITION 2. *Let A be a finitely generated subgroup and B a non-cyclic subgroup of a free group F. If $A \cap B$ has finite index in B, then*

$$[B : A \cap B] < 2(r(A) - 1), \tag{3}$$

where $[B : A \cap B]$ denotes the index of $A \cap B$ in B.

PROOF. Clearly A cannot be cyclic. Let B_1 be any non-cyclic, finitely generated subgroup of B and write

$$i = [B_1 : A \cap B_1] \ (\leq [B : A \cap B]).$$

By the Schreier index formula

$$r(A \cap B_1) - 1 = i(r(B_1) - 1). \tag{4}$$

On the other hand since neither A nor B_1 is cyclic we have from (1) (with $U = A$, $V = B_1$)

$$r(A \cap B_1) - 1 < 2(r(A) - 1)(r(B_1) - 1). \tag{5}$$

Comparing (4) and (5) we deduce that $i < 2(r(A) - 1)$.

Write $j = [B : A \cap B]$, let $b_1, \ldots, b_j$ form a (right) transversal for $A \cap B$ in B, and now let B_1 be any non-cyclic, finitely generated subgroup of B containing $b_1, \ldots, b_j$. Then $b_1, \ldots, b_j$ determine distinct right cosets of $A \cap B_1$ in B_1, whence by the above $j < 2(r(A) - 1)$ as required.

COROLLARY (cf. Gersten [3, Lemma 5.2]). *Let A be a subgroup of rank 2 of a free group F, and let B be a non-cyclic subgroup of F such that $A \cap B$ has finite index in B. Then B is contained in A.*

PROOF. From (3) with $r(A) = 2$, we obtain $[B : A \cap B] < 2$, whence $A \cap B = B$.

2. **The Stallings–Greenberg property**. We conclude with a simple proof (i.e. simple modulo some more-or-less basic facts) of the following result of Stallings [11], based on results of Greenberg [4]. (We note that a sketch of this proof appeared in Review 20013, Zentralblatt für Math., vol. 521 (1984), and also that a proof along similar lines has been obtained by Akbar Rhemtulla and David Meier (unpublished).)

It would seem likely that, more generally, the property in question is enjoyed by the surface groups.

THEOREM (Stallings [11]). *If U, V are finitely generated subgroups of a free group with the property that $U \cap V$ has finite index in both U and V, then $U \cap V$ has finite index in $\langle U, V\rangle$ (the subgroup generated by U and V).*

PROOF. Write $F = \langle U, V\rangle$. We may suppose neither U nor V trivial since the contrary case is easy. Since every finitely generated subgroup of a free group is a free factor of some subgroup of finite index in F (see [2] or [6]), there exist subgroups A and B of F such that $\langle U, A\rangle = U * A$ with $[F : U * A] < \infty$, and $\langle V, B\rangle = V * B$ with $[F : V * B] < \infty$. By the Kurosh subgroup theorem (or more directly from [7, p. 117, Ex. 32]), since $(U * A) \cap V$ is a subgroup of the free product $U * A$, it will have as a free factor (of itself) its intersection with the free factor U of $U * A$; thus $(U * A) \cap V \cap U = U \cap V$ is a free factor of $(U * A) \cap V$. However since the latter group is contained in V and since $[V : U \cap V] < \infty$, it follows that in fact

$$(U * A) \cap V = U \cap V, \tag{6}$$

and, similarly, that

$$(V * B) \cap U = U \cap V. \tag{7}$$

Now let N be any normal subgroup of finite index in F contained in $(U * A) \cap (V * B)$. Then by (6) and (7)

$$U \cap V \geq N \cap U, N \cap V,$$

whence

$$N \cap U = N \cap V = N \cap U \cap V.$$

Hence $N_1 = N \cap U \cap V$ is normal in both U and V and therefore in $F = \langle U, V\rangle$, and is moreover easily seen to be non-trivial. Furthermore by Howson's theorem (see above), N_1 is finitely generated, being the intersection of three finitely generated subgroups of F. It follows that N_1 must have finite index in F (in view of the result of Schreier that a non-trivial normal subgroup of infinite index in a free group must have infinite rank (see e.g. [2, Corollary 2])). Since $U \cap V \geq N_1$ we deduce that $U \cap V$ has finite index in F, as required.

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