J. Austral. Math. Soc. (*Series A*) 36 (1984), 357–364

EPIMORPHISMS PRESERVING PERFECT RADICALS

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(Received 16 May 1982)

Communicated by T. M. Gagen

Abstract

This is an investigation of whether a group epimorphism maps the maximal perfect subgroup of its domain onto that of its image. It is shown how the question arises naturally from considerations of algebraic K-theory and Quillen's plus-construction. Some sufficient conditions are obtained; these relate to the upper central series, or alternatively the derived series, of the domain. By means of topological/homological techniques, the results are then sharpened to provide, in certain circumstances, conditions which are necessary as well as sufficient.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): Primary 20 F 26; Secondary 20 F 14, 20 J 05, 18 F 25.

The purpose of this note is to draw attention to a class of group homomorphisms which has gained prominence from recent developments in algebraic K-theory [2], [5].

1. Definition

To define the class, recall that a group P is perfect if equal to its commutator subgroup $[P, P]$, which is to say that it has trivial abelianization $H_1(P) = 0$ (trivial integer coefficients).

1.1. *The homomorphic image of a perfect group is again perfect.*

The class of perfect subgroups of a given group G is therefore closed under automorphisms of G. It is also evidently closed under group union, because if

https://doi.org/10.1017/S1446788700025416 Published online by Cambridge University Press

each subgroup H_α is generated by its commutators, then so is the subgroup the H_α's generate. Thus the class admits a maximal element, the *perfect radial* $\mathcal{P}G$ of G, which must be a characteristic subgroup. The construction is functorial because, from 1.1,

1.2. *If* $\phi: G \to H$ *is a homomorphism, then* $\phi(\mathcal{P}G) \leqslant \mathcal{P}(\phi G) \leqslant \mathcal{P}H$.

We inquire under what conditions equality holds in (1.2). Since it is to the image ϕG which we wish to restrict consideration, suppose ϕ to be an epimorphism. We seek hypotheses to ensure that $\phi(\mathcal{P}G) = \mathcal{P}(\phi G)$, in other words, that ϕ is EP^{2}R—an Epimorphism Preserving Perfect Radicals.

For an extreme example of an epimorphism which is not EP^{2}R, let the exact sequence $R \rightarrowtail F \xrightarrow{\phi} R$ correspond to a free presentation of a perfect group P. Thus F, being free, has only free non-trivial subgroups, and no free group can be perfect. So although $\mathcal{P}F = 1$, making $\phi\mathcal{P}F = 1$, we have $\mathcal{P}P = P$. On the other hand, there are no examples from finite group theory: we shall see that any surjection of finite groups is EP^{2}R.

Note that solubility of G forces triviality of $\mathcal{P}G$, since among nth derived groups we must have $(\mathcal{P}G)^{(n)} \leqslant G^{(n)}$. In fact,

1.3. *The following three conditions are equivalent*:
(i) G *is soluble*;
(ii) $\mathcal{P}G = 1$ *and, for some* i, $G^{(i)}$ *is finite*;
(iii) $\mathcal{P}G = 1$ *and, for some* j, $G^{(j)}/Z(G^{(j)})$ *is finite*.

The proof is an easy exercise, save for a lemma of Schur to the effect that $G^{(j)}/Z(G^{(j)})$ finite implies $G^{(j+1)}$ finite.

2. Motivation

The preeminent example of a perfect radical is offered by the Whitehead lemma on the general linear group over a ring A. This identifies $\mathcal{P}$GLA as EA, the subgroup generated by elementary matrices. Moreover, EA = [GLA, GLA] with the quotient GLA/$\mathcal{P}$GLA = K_1A. This fact prompted Quillen's definition of K_iA ($i \geqslant 1$) as $\pi_i(B\mathrm{GLA}^+)$. Here B is a classifying space functor Group → Top and the *plus-construction* is a functor from Top to itself such that, for any space X,

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there is a cofibration $q_X\colon X \to X^+$ which is characterized by its inducing

(a) an isomorphism on all homology groups (with local, abelian coefficients) and

(b) an epimorphism $\pi_1(q_X)\colon \pi_1(X) \to \pi_1(X^+)$ of fundamental groups whose kernel is $\mathcal{P}\pi_1(X)$.

Clearly it is important to know how the plus-construction behaves in familiar topological situations. Here are two examples. (Recall that a map f is said to be 0-connected if its homotopy fibre F_f is, or equivalently, if $\pi_1(f)$ is onto. In both examples below we assume that $f\colon X \to Y$ satisfies this condition.)

2.1. *The commuting diagram*

$$\begin{array}{ccc} X & \xrightarrow{q_X} & X^+ \\ f\downarrow & & f^+\downarrow \\ Y & \xrightarrow{q_Y} & Y^+ \end{array}$$

is co-Cartesian ($Y^+ = Y \cup_X X^+$) *if and only if* $\pi_1(f)$ *is* $\mathrm{EP}^2\mathrm{R}$.

Proof. We have to demonstrate that the cofibration $q'_X\colon Y \to Y \cup_X X^+$ determined by pushing-out under q_X satisfies both (a) and (b). For the former, let $\mathcal{Q}$ be a local coefficient system of abelian groups on $Y \cup_X X^+$. Let $f'\colon X^+ \to Y \cup_X X^+$ be the push-out of f. The maps f, f' induce a homomorphism between the homology exact sequences of the pairs $(X^+, q_X(X))$ and $(Y \cup_X X^+, q'_X(Y))$.

$$\begin{array}{ccccccc} \cdots \to H_n(X; q_X^* f'^*\mathcal{Q}) & \xrightarrow{q_{X*}} & H_n(X^+; f'^*\mathcal{Q}) & \to & H_n(X^+, q_X(X); f'^*\mathcal{Q}) & \to \cdots \\ \downarrow f_* & & \downarrow f'_* & & \downarrow & \\ \cdots \to H_n(Y; q'^*_X\, \mathcal{Q}) & \xrightarrow{q'_{X*}} & H_n(Y \cup_X X^+; \mathcal{Q}) & \to & H_n(Y \cup_X X^+, q'_X(Y); \mathcal{Q}) & \to \cdots \end{array}$$

By construction, the right-hand vertical homomorphism is an excision isomorphism. On the other hand q_{X*} is an isomorphism because q_X is acyclic. So q'_{X*} must indeed be an isomorphism.

To check (b), argue via the Steifert-van Kampen theorem. Thus

$$\begin{aligned} \pi_1(Y \cup_X X^+) &= \pi_1(Y) *_{\pi_1(X)} \pi_1(X^+) \\ &= \pi_1(Y)/\pi_1(f)(\mathcal{P}\pi_1(X)), \end{aligned}$$

making $\pi_1(q'_X)$ an epimorphism with kernel $\pi_1(f)(\mathcal{P}\pi_1(X))$.

It should be remarked that there are important examples where the diagram of (2.1) is co-Cartesian yet $\pi_1(f)$ fails to be surjective, notably the inclusion-induced maps $f\colon \mathrm{BGL}_3\mathbf{Z} \to \mathrm{BGLA}$ [6] and $f\colon B\mathfrak{A}_5 \to \mathrm{BGLA}$ [7] (where $\mathfrak{A}_5$ denotes the alternating group).

2.2. *Given a plus-constructive homotopy fibration*

$$F_f \to X \xrightarrow{f} Y$$

(*in the sense that the induced map* $(F_f)^+ \to F_{f^+}$ *is a homotopy equivalence*), *then* $\pi_1(f)$ *is* $\mathrm{EP}^2\mathrm{R}$.

PROOF. After (1.2), $\pi_1(f)(\mathcal{P}\pi_1(X)) \leqslant \mathcal{P}\pi_1(Y)$. So there is a commuting diagram arising from the homotopy exact sequences of $F_f \to X \to Y$ and $F_{f^+} \to X^+ \to Y^+$. From (b) above the vertical sequences are also exact.

$$\begin{array}{ccccc} & & \mathcal{P}\pi_1(X) & \xrightarrow{\pi_1(f)|} & \mathcal{P}\pi_1(Y) \\ & & \downarrow & & \downarrow \\ \pi_1(F_f) & \to & \pi_1(X) & \xrightarrow{\pi_1(f)} & \pi_1(Y) \\ \downarrow & & \downarrow \pi_1(q_X) & & \downarrow \pi_1(q_Y) \\ \pi_1(F_{f^+}) & \to & \pi_1(X^+) & \xrightarrow{\pi_1(f^+)} & \pi_1(Y^+) \end{array}$$

By assumption the left-hand vertical homomorphism is (up to isomorphism) $\pi_1(q_{F_f})$ and thereby onto, as is $\pi_1(f)$. A diagram chase is now all that is needed.

This simple result is crucial to (3.4) below and to [4], where it is used to create fibrations which are not plus-constructive.

In view of the fact that (2.2) makes only mild use of the plus-construction characterization (with property (a) quite ignored), it is not surprising to learn that the necessary condition, that $\pi_1(f)$ be $\mathrm{EP}^2\mathrm{R}$, is far from sufficient. For example, let M be the Poincaré homology 3-sphere $SO(3)/\mathfrak{A}_5$ contained in another space M', constructed in [2] so as to have the same fundamental group and homology (but only when coefficients are trivial) as M. So the inclusion $i\colon M \hookrightarrow M'$ certainly has $\pi_1(i)\,\mathrm{EP}^2\mathrm{R}$. Moreover, from the homotopy exact sequence

$$\pi_2(M') \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\cong} \pi_1(M'),$$

$\pi_1(F_i)$ is seen to be abelian, making $\mathcal{P}\pi_1(F_i) = 1$ and $(F_i)^+ = F_i$. However both M^+ and M'^+ are (standard) 3-spheres, leaving F_{i^+} as a contractible space. Yet F_i cannot be contractible since M and M' are by no means homotopy equivalent.

Another example is due to Quillen [10], who showed that for any ring A the ring projection homomorphism

$$p\colon \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

induces an integral homology equivalence on general linear groups. Now, by the Whitehead Lemma, $\mathrm{GLR}^{(1)} = \mathcal{P}\mathrm{GLR}$, so that general linear groups satisfy hypotheses (iii) of (3.1) below, and $\mathrm{GL}p$ is therefore $\mathrm{EP^2R}$. Moreover, $\mathrm{BGL}p^+$ is a homotopy equivalence [2]. On the other hand, the fibre of $\mathrm{BGL}p$ is BMA, where MA is the additive group of all finite matrices over A. Because MA is abelian, $\mathrm{BMA}^+ = \mathrm{BMA}$ and hence cannot be the (contractible) fibre of $\mathrm{BGL}p^+$.

3. Results

Let $N \rightarrowtail G \overset{\phi}{\twoheadrightarrow} Q$ be a group extension. We now present conditions on G, N sufficient to ensure that ϕ is $\mathrm{EP^2R}$. In (3.1, ii) below, $Z_n(G)$ is the nth center of G defined by the upper central series $Z_0(G) = 1$, $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Hence, if $H \leqslant G$, then

$$[HZ_{i+1}(G), HZ_{i+1}(G)] \leqslant [H, H]Z_i(G),$$

which leads to $(HZ_j(G))^{(j)} = H^{(j)}$, the jth derived group. On the other hand, if $P \trianglelefteq G$ is perfect, then $[HP, HP] = [H, H]P$.

3.1. $\phi\colon G \to Q$ *is* $\mathrm{EP^2R}$ *if either*
(i) ϕ *is a split extension*,
(ii) *the kernel N satisfies, for some n,*

$$N.\mathcal{P}(G)/\mathcal{P}(G) \leqslant Z_n(G/\mathcal{P}(G)),$$

or
(iii) $G^{(n)} \leqslant N.\mathcal{P}G$ *for some n.*

Of course, after (1.2) in order to prove these results one has only to verify that $\mathcal{P}Q \leqslant \phi\mathcal{P}G$. In case (i), let $\psi\colon Q \rightarrowtail G$ be a section for ϕ; then by (1.2) again $\psi\mathcal{P}Q \leqslant \mathcal{P}G$. Therefore $\mathcal{P}Q = \phi\psi\mathcal{P}Q \leqslant \phi\mathcal{P}G$.

It is convenient to look at (3.1, ii) first when $n = 0$, which is to say $N \leqslant \mathcal{P}G$. On setting $J = \phi^{-1}\mathcal{P}Q$, we have

$$J = [J, J]N \leqslant [J, J]\mathcal{P}G,$$

whence

$$J.\mathcal{P}G \leqslant [J, J]\mathcal{P}G = [J.\mathcal{P}G, J.\mathcal{P}G].$$

So $J.\mathcal{P}G$ is perfect and hence a subgroup of $\mathcal{P}G$. This forces $J \leqslant \mathcal{P}G$ too, and thus $\mathcal{P}Q = \phi J \leqslant \phi\mathcal{P}G$, as required.

An immediate consequence of this case is that any perfect extension is EP^{2}R. (Such extensions are discussed in [1, Section 11]; however, note the falsity of Lemma 11.2 there, for example, $\mathfrak{A}_3 \rightarrowtail \mathfrak{S}_3 \twoheadrightarrow \mathbf{Z}/2$ satisfies the homology conditions because $\mathfrak{A}_3 = [\mathfrak{A}_3, \mathfrak{S}_3]$, but is evidently not a perfect extension.) The extreme example of a perfect extension is when $N = \mathcal{P}G$. It follows that $\mathcal{P}(G/\mathcal{P}G) = 1$. Armed with this fact we may continue the proof of (3.1, ii). We pass to the reduced extension $\bar{N} \rightarrowtail \bar{G} \overset{\bar{\phi}}{\twoheadrightarrow} \bar{Q}$ obtained as the push-out of ϕ by $G \twoheadrightarrow \bar{G} = G/\mathcal{P}G$. Thus $\bar{N} = N.\mathcal{P}G/\mathcal{P}G$ and $\bar{Q} = Q/\phi\mathcal{P}G \cong G/N.\mathcal{P}G$. Our justification for this step is...

3.2. ϕ: $G \twoheadrightarrow Q$ *is* EP^{2}R *if and only if* $\bar{\phi}$: $\bar{G} \twoheadrightarrow \bar{Q}$ *is* EP^{2}R.

The proof is a simple exercise, using the triviality of $\mathcal{P}(G/\mathcal{P}G)$ and $\mathcal{P}(Q/\mathcal{P}Q)$. Moreover, since $\mathcal{P}\bar{G} = 1$, $\bar{\phi}$ is EP^{2}R precisely when $\mathcal{P}\bar{Q} = 1$. Given this, we now complete the proof for (3.1, ii), reformulated as $\bar{N} \leqslant Z_n(\bar{G})$. Like before, write $\bar{J} = \bar{\phi}^{-1}\mathcal{P}\bar{Q}$; then, for all $i \geqslant 0$, $\bar{J} = \bar{J}^{(i)}\bar{N}$. In particular, taking $i = 1$ gives $\bar{J} \leqslant \bar{J}^{(1)}.Z_n(\bar{G})$, whence $\bar{J}^{(n)} \leqslant \bar{J}^{(n+1)}$. In other words $\bar{J}^{(n)}$ is perfect, hence trivial; this means that $\bar{J} = \bar{J}^{(n)}.\bar{N} = \bar{N}$. So $\mathcal{P}\bar{Q} = \bar{\phi}\bar{J}$ is trivial after all. It is worth remarking that this result yields that any central extension is EP^{2}R.

Finally, when expressed in these terms, condition (3.1, iii) implies that $\bar{Q}$ is soluble, whence (1.3) applies. This situation obtains whenever the derived sequence of G terminates (after finitely many steps). In particular all finite groups are covered by this condition, as are knot groups with Alexander polynomial 1, and the general linear group GLA referred to in Section 2 above.

Before progressing to a strengthening of (3.1), we note some equivalent versions of (3.1, ii).

3.3. *Suppose that* $\bar{N}$ *is nilpotent. Then the following statements are equivalent*:
(a) $\bar{N} \leqslant Z_n(\bar{G})$ *for some* n;
(b) $\bar{Q}$ *acts nilpotently on the abelianization* $\bar{N}_{ab}$;
(c) $\bar{Q}$ *acts nilpotently on* $\bar{N}$;
(d) $\bar{Q}$ *acts nilpotently on the homology groups* $H_*(\bar{N})$.

Here, homology is taken with trivial integer coefficients; nilpotent actions are analysed in, for example, [9]. The result would be no less valid with all bars deleted; the above format is for purposes of comparison with (3.4) to come. Since (a) is conveniently taken as the definition for (c) (in the special context of an extension), only the equivalence of (b) and (d) with (c) ought to be checked. This is an immediate application of [8, Theorem 2.1] to the nilpotent classifying space $B\bar{N}$. For it is shown there that $\bar{Q}$ acts nilpotently on $\pi_i(B\bar{N})$ for all $i \leq k$ if and only if on $H_i(B\bar{N})$ for the same values of i. Meanwhile, one knows that $\pi_i(B\bar{N}) = \bar{N}$ in case $i = 1$, and 0 otherwise, while $H_i(B\bar{N}) = H_i(\bar{N})$ with $H_1(\bar{N}) = \bar{N}_{ab}$. So the two cases $k = 1$ and k infinite give the two equivalences.

The key observation leading to our final result involves (2.2) above. For, conditions guaranteeing that the classifying space fibration

$$BN \to BF \overset{B\phi}{\to} BQ$$

induces another

$$BN^+ \to BG^+ \overset{B\phi^+}{\to} BQ^+$$

further ensure that $\phi = \pi_1(B\phi)$ is EP^{2}R. (Of course we may equally well discuss instead the reduced epimorphism $\bar{\phi}$.) Such conditions are given in [2], [3]. However, as the examples following (2.2) are intended to suggest, they can be weakened considerably and yet still force the EP^{2}R conclusion.

3.4. *Suppose that $\bar{N}$ is nilpotent. Then the following statements are equivalent*:
(a) ϕ: $G \twoheadrightarrow Q$ *is* EP^{2}R;
(b) $\mathcal{P}\bar{Q}$ *acts trivially on* $\bar{N}_{ab}$;
(c) $\mathcal{P}\bar{Q}$ *acts trivially on* $\bar{N}$;
(d) $\mathcal{P}\bar{Q}$ *acts trivially on* $H_*(\bar{N})$.

Observe that the conditions of (3.4) are weaker than those of (3.3). In the case of statements (a), this is precisely the content of (3.1, ii) ($Z_n(\bar{G})$, and hence its subgroups, being nilpotent). For the other statements this is a direct consequence of the following fact [3 (1.2)].

3.5. *A perfect group acts nilpotently on a group if and only if the action is trivial.*

Combination of (3.5) with our proof of (3.3) establishes the equivalence of (b), (c) and (d) above. Given (3.2) and the triviality of $\mathcal{P}\bar{Q}$ when $\bar{\phi}$ is EP^{2}R, these conditions are evidently consequences of condition (a). So it remains to confirm that (d) in turn implies (a). However, this is an application of (2.2) above, for by [3 (1.1)] the fibration

$$B\bar{N} \to B\bar{G} \xrightarrow{B\bar{\phi}} B\bar{Q}$$

is plus-constructive; so (3.2) again clinches the result. (Note that here $B\bar{\phi}$ plus-constructive need not mean that $B\phi$ is.)

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