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On K3 Surface Quotients of K3 or Abelian Surfaces

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Abstract. The aim of this paper is to prove that a K3 surface is the minimal model of the quotient of an Abelian surface by a group G (respectively of a K3 surface by an Abelian group G) if and only if a certain lattice is primitively embedded in its Néron–Severi group. This allows one to describe the coarse moduli space of the K3 surfaces that are (rationally) G-covered by Abelian or K3 surfaces (in the latter case G is an Abelian group). When G has order 2 or G is cyclic and acts on an Abelian surface, this result is already known; we extend it to the other cases.

Moreover, we prove that a K3 surface X_G is the minimal model of the quotient of an Abelian surface by a group G if and only if a certain configuration of rational curves is present on X_G. Again, this result was known only in some special cases, in particular, if G has order 2 or 3.

1 Introduction

Thanks to the Torelli theorem for K3 surfaces and to the theory of the lattice polarized K3 surfaces, in order to describe the moduli space of K3 surfaces having a certain geometric property it is useful to express this geometric property in terms of embeddings of certain lattices. In this paper we analyze the geometric property: "a K3 surface is the minimal model of the quotient of an Abelian or a K3 surface by a finite group". Under certain conditions we are able to translate this property to a lattice theoretic property and thus to describe the coarse moduli space of the K3 surfaces that are (rationally) covered by Abelian surfaces or by K3 surfaces. This generalizes several previous results by Nikulin [N1], by Bertin [Be], and by Sarti and the author [GS1].

The first and crucial example is given by the Kummer surfaces. A Kummer surface is a K3 surface obtained as minimal resolution of the quotient A/ι, where A is an Abelian surface and ι is an involution on A. In [N1], Nikulin proved that a K3 surface is a Kummer surface if and only if at least one of the two following equivalent conditions holds:

(a) a certain lattice, called the Kummer lattice, is primitively embedded in the Néron–Severi group of the K3 surface;
(b) there are sixteen disjoint smooth rational curves on the K3 surface.

The first condition is more related to the lattice theory and allows one to describe the coarse moduli space of the K3 surfaces that are Kummer surfaces. The second one is clearly more related to the geometry of the surface.

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In a more general setting, we consider the following situation: Y is either an Abelian or a K3 surface, G is a finite group of automorphisms of Y, and the minimal model of Y/G is a K3 surface X. In this case we say that X is (rationally) G-covered by Y. In view of the results by Nikulin on Kummer surfaces, it is quite natural to pose the following two questions.

Question A *Is the property "a K3 surface X is (rationally) G-covered by a surface Y" equivalent to the condition "there is a certain lattice (depending on G) that is primitively embedded in* $NS(X)$*"?*

We observe that a positive answer to this question immediately provides a description of the coarse moduli space of the K3 surfaces (rationally) covered by Abelian or K3 surfaces.

Question B *Is the property "a K3 surface X is (rationally) G-covered by a surface Y" equivalent to the condition "there is a certain configuration of rational curves on X"?*

The main results of this paper are to give a positive answer to:

- Question A in the case where Y is an Abelian surface (see Theorem 4.4);
- Question B in the case where Y is an Abelian surface (see Theorem 4.7);
- Question A in the case where Y is a K3 surface and G is an Abelian group (see Theorem 5.2).

It is not possible in general to give a positive answer to Question B in the case where Y is a K3 surface. For example, it is known that the answer is negative if we assume that Y is a K3 surface and $G = \mathbb{Z}/2\mathbb{Z}$; *cf.* [GS2]. I do not know if it is possible to extend the positive answer given to Question A in the case where Y is a K3 surface and G is an Abelian group to the weaker hypothesis that Y is a K3 surface, without assumptions (or with different assumptions) on G.

The positive answer to Question A in the case where Y is an Abelian surface was already known if G is a cyclic group; indeed, the classical case of the Kummer surface, *i.e.*, $G = \mathbb{Z}/2\mathbb{Z}$, was considered by Nikulin [N1], as we said above, the other cyclic cases are considered in [Be]. In Theorem 4.4 we address the remaining cases. In order to state and prove this theorem, the first step is to find all the finite groups G acting on an Abelian surface in such a way that A/G desingularizes to a K3 surface. We assume, without loss of generality, that G does not contain translations. The list of these groups is classically known (see [F]) and consists of four cyclic groups and three noncyclic (and non Abelian) groups. One of the noncyclic groups, the quaternion group, can act on two different families of Abelian surfaces and the actions have different sets of points with nontrivial stabilizer. So we have to consider four actions of noncyclic groups on an Abelian surface. The second step is the identification of the lattice that should characterize the K3 surfaces that are (rationally) G-covered by an Abelian surface. This lattice depends on G, and the natural candidate (also in view of the previous results by Nikulin and Bertin) is the minimal primitive sublattice of $NS(X)$, which contains all the curves arising from the desingularization of A/G. We call the lattices constructed in this way lattices of Kummer type, and we denote them by K_G. They have already been determined if G is a cyclic group, and they are computed in the noncyclic case in Section 4.2. The lattices arising in the noncyclic cases were considered in [W], but unfortunately some of the results presented in [W, Proposition 2.1] are not correct, as we show below. The last step is to prove our main result; that is, the primitive embedding of the lattices K_G in the Néron–Severi group of a K3 surface X is equivalent to the fact that X is (rationally) G-covered by an Abelian surface. We combine a result of [Be], which allows us to give a geometric interpretation of the (-2)-classes appearing in the lattice K_G, with classical results on cyclic covers between surfaces, in order to reconstruct the surface A starting from (X, K_G).

The positive answer to Question A in the case where Y is a K3 surface and G is an Abelian group, is contained in Theorem 5.2. The fourteen admissible Abelian groups are listed in [N3, Theorem 4.5]. Only the case $G = \mathbb{Z}/2\mathbb{Z}$ was already known; see [GS1]. The proof of the result is totally analogous to the one given in the case where Y is an Abelian surface, with the advantage that the lattices of Kummer type are substituted by other lattices, denoted by M_G, which were already computed in [N3, Sections 6 and 7] for all the admissible groups G. The extension to the non Abelian groups G seems more complicated: the lattices M_G are known also in the non Abelian cases, [X], but it is not so clear how to reconstruct the surface Y only from the data (X, M_G).

Let us now discuss the more geometric Question B. The positive answer to Question B in case Y is an Abelian surface was already known in cases $G = \mathbb{Z}/2\mathbb{Z}$ and $G = \mathbb{Z}/3\mathbb{Z}$. The case of the involution was considered by Nikulin [N1], as mentioned. The case $G = \mathbb{Z}/3\mathbb{Z}$ is due to Barth [Ba1]. The other groups are considered here. The proof of this very geometric result is essentially based on computations in lattice theory. Indeed, the idea is to prove that if a K3 surface X admits a certain configuration of curves, then the minimal primitive sublattice of the Néron–Severi group containing these curves is in fact K_G. We emphasize that the computations with these lattices are strongly conditioned by the fact that we are considering many curves, which implies that the rank of the lattices that they span is high. This is exactly the hypothesis that fails if we consider the case where Y is a K3 surface (and not an Abelian surface). Indeed, in this case the result cannot be extended (at least without conditions on the Abelian group G).

In Section 2 we recall some known results. In Section 3 we present Proposition 3.2 (based on previous results by Bertin), which is fundamental in the proof of our main theorems. In Section 4.2 we compute the lattices K_G in the case where G is not a cyclic group and in Section 4.3 we give an exhaustive description of the lattices of Kummer type and of their properties. In Section 4.4 we state and prove two of our main results, giving a positive answer to the Questions A and B in case Y is an Abelian surface (see Theorems 4.4 and 4.7). In Section 4.5 we discuss the relation between K3 surfaces that are (rationally) $\mathbb{Z}/3\mathbb{Z}$-covered by Abelian surfaces, K3 surfaces which are (rationally) $(\mathbb{Z}/3\mathbb{Z})^2$-covered by K3 surfaces, and K3 surfaces that are (rational) $(\mathbb{Z}/3\mathbb{Z})^2$-covers of K3 surface. This generalizes a similar result on Kummer surfaces, proved in [GS2].

In Section 5 we concentrate on K3 surfaces covered by K3 surfaces, giving a positive answer to Question A in this setting. Moreover, we give more precise results on the K3 surfaces that are (rationally) $\mathbb{Z}/3\mathbb{Z}$-covered by K3 surfaces, presenting all the possible Néron–Severi groups of a K3 surface with this property and minimal Picard number. This generalizes results proved in [GS1].

Notation

- $\mathbb{D}_{4n}$ is the dicyclic group of order $4n$ (also called the binary dihedral group), which has the following presentation: $\langle a, b, c \mid a^n = b^2 = c^2 = abc \rangle$. (In the case where $n = 2$ it is the quaternion group). The GAP id of $\mathbb{D}_8$ is $(8, 4)$ and that of $\mathbb{D}_{12}$ is $(12, 1)$.
- $\mathbb{T}$ is the binary tetrahedral group: it has order 24 and the following presentation: $\langle r, s, t \mid r^2 = s^3 = t^3 = rst \rangle$. The GAP id of $\mathbb{T}$ is $(24, 3)$.
- $\mathfrak{A}_{3,3}$ is the generalized dihedral group of the abelian group of order 9; it has order 18 and the following presentation: $\langle r, s, t \mid r^2 = s^3 = t^3 = 1,\ tr = rt^2,\ sr = rs^2 \rangle$. The GAP id of $\mathfrak{A}_{3,3}$ is $(18, 4)$

2 Preliminaries

In this section we recall some very well known facts and fix the notation.

2.1 Lattices

Definition 2.1 A lattice is a pair (L, b_L), where $L = \mathbb{Z}^n$, $n \in \mathbb{N}$ and $b_L\colon L \times L \to \mathbb{Z}$ is a symmetric nondegenerate bilinear form taking values in $\mathbb{Z}$. The number n is the rank of L. The signature of (L, b_L) is the signature of the $\mathbb{R}$ linear extension of (L, b_L).

A lattice is said to be *even* if the quadratic form induced by b_L on L only takes values in $2\mathbb{Z}$.

The *discriminant group* of L is $L^\vee/L$, where the dual $L^\vee$ can be identified with the set $\{m \in L \otimes \mathbb{Q} \mid b_L(m, l) \in \mathbb{Z} \text{ for all } l \in \mathbb{Z}\}$ (here we also denote by b_L the $\mathbb{Q}$ linear extension of b_L). The *discriminant form* is the one induced by b_L on the discriminant group. The *length* of a lattice (L, b_L), denoted by $l(L)$, is the minimal number of generators of the discriminant group.

A lattice is said to be *unimodular* if its discriminant group is trivial, *i.e.,* if its length is zero.

The discriminant group of a lattice is a finite free product of cyclic groups. Its order is the determinant of some (and so of any) matrix that represents the form b_L with respect to some basis of L. This number is called the *discriminant* of the lattice L and is denoted by $d(L)$.

In the sequel we are interested in the construction of overlattices of finite index of a given lattice. Let L and M be two lattices with the same rank. Let $L \hookrightarrow M$. Then M is generated by the vectors that generate L plus by some other vectors that are nontrivial in M/L but that necessarily have an integer intersection with all the vectors in L (otherwise the form on M cannot take values in $\mathbb{Z}$). This means that the nontrivial vectors in M/L are nontrivial elements of in the discriminant group of L.

If, moreover, we require that the lattice M is even, then L is automatically even (since it is a sublattice of M) and also the nontrivial classes in M/L have an even self intersection. So, if we have an even lattice L and we want to construct an even overlattice of finite index, we have to add to the generators of L some nontrivial elements in $L^\vee/L$ that have an even self intersection.

More generally, every isotropic subgroup of $L^\vee/L$ (where a subgroup H of $L^\vee/L$ is isotropic if the discriminant form restricted to H is trivial) corresponds to an overlattice of L of finite index , and, vice-versa, every overlattice of L of finite index corresponds to an isotropic subgroup of $L^\vee/L$; see [N4, Section 1].

If M is an overlattice of L with index r, then $d(L)/d(M) = r^2$.

Definition 2.2 Let M and L be two lattices with $\operatorname{rank}(M) \le \operatorname{rank}(L)$. Let $\varphi: M \to L$ be an embedding of M in L. We say that φ is *primitive*, or that M is *primitively embedded* in L, if $L/\varphi(M)$ is torsion free.

Proposition 2.3 ([N4, Proposition 1.6.1]) *Let L be a unimodular lattice, let M be a primitive sublattice of L, and let $M^\perp$ be the orthogonal complement to M in L. The discriminant group of M is isomorphic to the discriminant group of $M^\perp$. In particular, since the length of a lattice is at most the rank of the lattice,*

$$l(M) = l(M^\perp) \le \min\{\operatorname{rank}(M), \operatorname{rank}(L) - \operatorname{rank}(M^\perp)\}.$$

Definition 2.4 A root of the lattice (L, b_L) is a vector $v \in L$ such that $b_L(v,v) = -2$. The root lattice of a given lattice L is the lattice spanned by the set of all the roots in L.

A lattice is called a *root lattice* if it is generated by its roots.

2.2 Covers

Here we recall a very well known and classical result (see [BHPV, Chapter I, § 17]) on covers, which will be essential for our purpose.

Let Y be a connected complex manifold and let B be an effective divisor on Y. Suppose we have a line bundle $\mathcal{L}$ on Y such that

$$\mathcal{O}_Y(B) = \mathcal{L}^{\otimes n} \tag{2.1}$$

and a section $s \in H^0(Y, \mathcal{O}_Y(B))$ vanishing exactly along B. We denote by L the total space of $\mathcal{L}$ and we let $p: L \to Y$ be the bundle projection. If $t \in H^0(L, p^*(\mathcal{L}))$ is the tautological section, then the zero divisor of $p^*s - t$ defines an analytic subspace X in L. The variety X is an n-cover of Y branched along B and determined by $\mathcal{L}$, and the cover map is the restriction of p to X. If both Y and B are smooth and reduced, then X is smooth.

Let us denote by $D \in \operatorname{Pic}(Y)$ the divisor associated with the line bundle $\mathcal{L}$. Condition (2.1) is equivalent to $B = nD$, *i.e.*, $B/n = D \in \operatorname{Pic}(Y)$. For this reason we call B an *n-divisible divisor* in the Picard group. We often call the curves in the support of B an *n-divisible set of curves*. The previous discussion implies that with each effective divisible divisor one can associate a cyclic cover of the variety.

Let us consider a sort of converse. Let $\pi: X \to Y$ be an n-cyclic cover between smooth varieties such that the branch locus is smooth and all its components have codimension 1 in Y. Then π determines a divisor (supported on the branch locus) that is divisible by n. This applies in particular to a special situation that we will now consider. Let X and Y be two surfaces. Let α be an automorphism of X of order 2 or 3 that fixes only isolated points. Then it is possible to construct a blow up $\widetilde{X}$ of X such that α induces an automorphism $\widetilde{\alpha}$ of $\widetilde{X}$ whose fixed locus consists of disjoint curves. So $\widetilde{X}/\widetilde{\alpha}$ is a smooth surface that we denote by Y which is birational to X/α. The quotient map $\widetilde{X} \to \widetilde{X}/\widetilde{\alpha} = Y$ is a $|\alpha|$:1 cover of Y branched along a smooth union of curves. Hence there is an $|\alpha|$-divisible set of curves on Y that is a smooth birational model of X/α. If $|\alpha| = 2^a 3^b$, then the iterated application of the previous procedure to suitable powers of α, produces a suitable $|\alpha|$-divisible set of curves on a surface Y that is a smooth surface birational to X/α.

2.3 K3 Surfaces

We work with smooth projective complex surfaces.

Definition 2.5 A surface Y is called a K3 surface if its canonical bundle is trivial and $h^{1,0}(Y) = 0$.

The second cohomology group of a K3 surface equipped with the cup product is the unique even unimodular lattice of rank 22 and signature $(3, 19)$, and it is denoted by Λ_{K3}. The Néron–Severi group of a K3 surface Y is a primitively embedded sublattice of Λ_{K3} with signature $(1, \rho(Y) - 1)$. Consequently, the transcendental lattice, which is the orthogonal complement to the Néron–Severi group in the second cohomology group, is a primitively embedded sublattice of Λ_{K3} with signature $(2, 20 - \rho(Y))$.

Let $G \subset \mathrm{Aut}(Y)$ be a group of automorphisms of Y. We will say that it acts symplectically if it preserves the symplectic structure of Y, *i.e.*, if its action on $H^{2,0}(Y)$ is trivial. The finite groups acting symplectically on a K3 surface are classified by Nikulin [N3], in the case of the Abelian group, and by Mukai [M] in the other cases. A complete list can be found in [X].

If a finite group G acts symplectically on a K3 surface Y, then Y/G is a singular surface, whose desingularization $\widetilde{Y/G}$ is a K3 surface.

Definition 2.6 Let Y be a K3 surface admitting a symplectic action of a finite group G. Let $\widetilde{Y/G}$ be the minimal model of Y/G. We will denote by E_G the sublattice of $NS(\widetilde{Y/G})$ generated by the curves arising from the desingularization of Y/G. We will denote by M_G the minimal primitive sublattice of $NS(\widetilde{Y/G})$ that contains E_G. We observe that M_G is an overlattice of finite index of E_G.

We now show an explicit and very classic example: let Y be a K3 surface that admits a symplectic action of $\mathbb{Z}/2\mathbb{Z}$. Then $Y/(\mathbb{Z}/2\mathbb{Z})$ has eight singular points of type A_1. The desingularization of Y/G introduces eight rational curves on $\widetilde{Y/(\mathbb{Z}/2\mathbb{Z})}$; let us denote them by M_i, $i = 1, \dots, 8$. The lattice spanned by the curves M_i is clearly isomorphic to A_1^8, so $E_{\mathbb{Z}/2\mathbb{Z}} = A_1^8$. One can also consider a different construction: one blows up Y in the eight fixed points for the action of $\mathbb{Z}/2\mathbb{Z}$. One obtains the surface $\widetilde{Y}$, with eight exceptional curves E_i, $i = 1, \dots, 8$. Then one lifts the action of $\mathbb{Z}/2\mathbb{Z}$ on Y to an action of $\mathbb{Z}/2\mathbb{Z}$ on $\widetilde{Y}$ that fixes the exceptional curves. So one obtains a smooth surface $\widetilde{Y}/(\mathbb{Z}/2\mathbb{Z})$ that is in fact isomorphic to $\widetilde{Y/(\mathbb{Z}/2\mathbb{Z})}$. The 2:1 map $\widetilde{Y} \to \widetilde{Y}/(\mathbb{Z}/2\mathbb{Z})$ is ramified on the union of the curves E_i and so it is branched along the union of the curves M_i. By Section 2.2, it follows that $\sum_i M_i$ is divisible by 2 in $NS(\widetilde{Y/G})$. Hence,

$E_{\mathbb{Z}/2\mathbb{Z}}$ is generated by M_i, $i = 1, \dots, 8$ and is isometric to A_1^8; $M_{\mathbb{Z}/2\mathbb{Z}}$ is generated by the same classes as $E_{\mathbb{Z}/2\mathbb{Z}}$ and by the divisible class $\sum_i M_i/2$.

Similarly one can apply the results of Section 2.2 to the cyclic groups of order 3, 4, 6, and 8 in order to conclude that the K3 surface $\widetilde{Y/G}$ contains a divisible set of rational curves. The same is true also for cyclic groups of order 5 and 7 as proved by Nikulin in [N3]. This shows that for every cyclic group G acting symplectically on a K3 surface Y, there is a $|G|$-divisible set of rational curves on the minimal model of Y/G. The description of this $|G|$-divisible set is given in [N3] and implies the description of the lattice M_G.

Let us assume that the sum of n disjoint rational curves is divisible by 2 in $NS(Y)$. By Section 2.2, there exists a 2:1 cover of Y branched along the union of these curves. The covering surface is not minimal, but one can contract certain curves in order to obtain a minimal model. It was proved by Nikulin that only two possibilities occur: the minimal model of the covering surface is a K3 surface and in this case $n = 8$, or the minimal model of the covering surface is an Abelian surface and in this case $n = 16$. A similar result holds for covers of order 3 and was proved by Barth [Ba1]. We collect these results in Proposition 2.8 after introducing some definitions.

Definition 2.7 An A_k (resp. D_m, $m \geq 4$, E_l, $l = 6, 7, 8$) configuration of curves is a set of k (resp. m, l) irreducible smooth rational curves whose dual diagram is a Dynkin diagram of type A_k (resp. D_m, E_l).

A set of disjoint A_k configurations is *n-divisible* if there is a linear combination of the curves contained in the configuration that can be divided by n in the Néron–Severi group of the surface.

Proposition 2.8 ([Ba1, N1]) *Let Y be a K3 surface that contains a set of m disjoint rational curves (*i.e.*, a set of m disjoint A_1-configurations). If this set is divisible by* 2*, then m is either* 8 *or* 16*. In the first case the cover surface associated to the divisible class is a K3 surface, in the latter it is an Abelian surface.*

Let Y be a K3 surface which contains a set of m disjoint A_2-configurations. If this set is divisible by 3*, then m is either* 6 *or* 9*. In the first case the cover surface associated with the divisible class is a K3 surface, in the latter it is an Abelian surface.*

In case where G is an Abelian group acting symplectically on a K3 surface Y, the type and the number of points with a nontrivial stabilizer is determined by Nikulin in [N3, Section 5]. In the same paper the author determines the lattice E_G and M_G for all the admissible Abelian groups (we will recall this result in Proposition 5.1).

In certain cases the presence of certain configurations of rational curves suffices to conclude that the K3 surface is covered either by an Abelian or by a K3 surface. Since this property will be useful, we summarize the cases where it appears.

Proposition 2.9 *Let Y be a K3 surface that admits sixteen disjoint rational curves. Then it is the desingularization of the quotient of an Abelian surface by the group $\mathbb{Z}/2\mathbb{Z}$. In particular, the set of these sixteen curves is* 2*-divisible* (see [N1, Theorem 1]).

Let Y be a K3 surface that admits fifteen disjoint rational curves. Then it is the desingularization of the quotient of a K3 surface by the group $(\mathbb{Z}/2\mathbb{Z})^4$. In particular, the set

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of these fifteen rational curves contains four independent subsets of eight rational curves that are 2*-divisible* (see [GS2, Theorem 8.6]).

Let Y be a K3 surface that admits fourteen disjoint rational curves. Then it is the desingularization of the quotient of a K3 surface by the group $(\mathbb{Z}/2\mathbb{Z})^3$. In particular, the set of these fourteen rational curves contains 3 *independent subsets of eight rational curves that are* 2*-divisible* (see [GS2, Corollary 8.7]).

Let Y be a K3 surface that admits nine disjoint A_2-configurations of rational curves. Then it is the desingularization of the quotient of an Abelian surface by the group $\mathbb{Z}/3\mathbb{Z}$. In particular, the set of nine disjoint A_2-configurations of rational curves is 3*-divisible* (see [Ba1, Theorem]).

3 A Preliminary and Fundamental Result

In this section we recall a result by Bertin [Be, Lemma 3.1], and we deduce a corollary of this result, Proposition 3.2. These are essential for the sequel.

First we introduce some notation, following [Be]. Let us consider a K3 surface Y. We denote by $\mathcal{C}^+$ the component of the cone $\{v \in NS(Y) \otimes \mathbb{R}$ such that $v^2 > 0\}$ that contains at least one ample class. We observe that the ample cone is contained in $\mathcal{C}^+$. Let us define

$$\Delta := \{\delta \in NS(Y) \text{ such that } \delta^2 = -2\} \text{ and } \Delta^+ := \{\delta \in \Delta \text{ such that } \delta \text{ is effective}\}.$$

Moreover, we let

$$B := \{b \in \Delta^+ \text{ such that } b \text{ is the class of an irreducible curve}\}.$$

We observe that the curves C whose classes are contained in the set B are smooth irreducible rational curves. We pose

$$\mathcal{K} := \{v \in \mathcal{C}^+ \text{ such that } vb > 0 \text{ for all } b \in B\}.$$

The cone $\mathcal{K}$ is the ample cone of S, and so $NS(Y) \cap \overline{\mathcal{K}} \cap \mathcal{C}^+$ is the set of the pseudo ample divisors of Y. This means that if $h \in NS(Y) \cap \overline{\mathcal{K}} \cap \mathcal{C}^+$, then $h^2 > 0$ and $hv \geq 0$ for all the effective classes v.

Lemma 3.1 ([Be, Lemma 3.1]) *Let Y be a projective K3 surface and let $h \in NS(Y) \cap \overline{\mathcal{K}} \cap \mathcal{C}^+$. Let us denote $\Delta_h := \Delta \cap h^\perp$. Then $B \cap h^\perp$ is a basis of Δ_h.*

Proposition 3.2 *Let Y be a K3 surface. Let h be a pseudoample divisor on Y and let $L = h^\perp := \{l \in NS(Y)$ such that $lh = 0\}$ be the orthogonal complement of h in $NS(Y)$. Let us assume that there exists a root lattice R such that:*

(i) *L is an overlattice of finite index of R;*
(ii) *the roots of R and of L coincide.*

Then there exists a basis of R that is supported on smooth irreducible rational curves.

Proof Let us consider the root lattice of L, denoted by $R(L)$. By the definition of Δ_h, $R(L)$ and Δ_h coincide. So there is a basis of $R(L)$ which is supported on smooth irreducible rational curves by [Be, Lemma 3.1] (*i.e.*, by Lemma 3.1). By the hypothesis,

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$R(L)$ is isometric to R and so there exists a basis for R that is supported on smooth irreducible rational curves. ∎

4 K3 Surface Quotients of Abelian Surfaces

In this section we concentrate on K3 surfaces that are constructed as quotients of an Abelian surface by a group of finite order. First, we recall some known results and we compute the lattices associated with this construction in the case where G is not Abelian. The results about these lattices are summarized in Proposition 4.3. Then we state and prove the main results of this section, which are Theorems 4.4 and 4.7.

4.1 Preliminaries and Known Results

Definition 4.1 Let A be an Abelian surface. Let $G \subset \mathrm{Aut}(A)$ be a finite group of automorphisms of A. Let us consider the minimal model of A/G and let us call it X_G.

Let K_i be the curves on X_G arising by the resolution of the singularities of A/G. Let F_G be the lattice spanned by the curves K_i and let K_G be the minimal primitive sublattice of $NS(X_G)$ containing the curves K_i. Clearly, K_G is an overlattice of finite index, r_G, of F_G. We will say that the lattice K_G is a lattice of Kummer type.

The following well known result, due to Fujiki, classifies the group $G \subset \mathrm{Aut}(A)$ such that G does not contain translations and X_G is a K3 surface.

Theorem 4.2 ([F]) *Let G be a group of automorphisms of an Abelian surface A that does not contain translations. If the minimal resolution of A/G is a K3 surface, then $G = \mathbb{Z}/n\mathbb{Z}$, $n = 2, 3, 4, 6$, or $G \in \{\mathbb{D}_8, \mathbb{D}_{12}, \mathbb{T}\}$.*

The requirement that G does not contain translations is not seriously restrictive; indeed, the quotient of an Abelian surface by a finite group of translations produces another Abelian surface. Up to replacing the first Abelian surface by its quotient by translations, we can assume without loss of generality that the group G does not contain translations.

4.2 Non-cyclic Quotients of Abelian Surfaces

The aim of this section is to describe the lattices K_G in case G is not cyclic. These lattices were also computed in [W], but two of the results given in [W, Proposition 2.1] are incorrect. In particular, we prove that the lattices K_G are not the ones given in [W, Proposition 2.1] if $G = \mathbb{D}_8'$ and $G = \mathbb{D}_{12}$.

4.2.1 The Actions of $\mathbb{D}_8$ and $\mathbb{D}_8'$

Let G be the quaternion group. There are two different families of tori on which we can define the action of G in such a way that X_G is a K3 surface, and on these two different families G has different sets of points with nontrivial stabilizer. So the quotients of an Abelian surface by each of these actions produce two different singular surfaces with different sets and types of singularities.

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The group G has the presentation

$$\left\langle \alpha_4, \beta \mid \alpha_4^4 = \beta^4 = 1,\ \alpha_4^2 = \beta^2,\ \alpha_4^{-1}\beta\alpha_4 = \beta^{-1} \right\rangle.$$

We pose $A := \mathbb{C}^2/\Lambda$ and

$$\begin{aligned} \alpha_4 \colon A &\longrightarrow A, & (z_1, z_2) &\longmapsto (iz_1, -iz_2) \\ \beta \colon A &\longrightarrow A, & (z_1, z_2) &\longmapsto (-z_2, z_1). \end{aligned}$$

The actions of α_4 and β are algebraic automorphisms on both $A := E_i \times E_i$ (where E_i is the elliptic curve with j-invariant equal to 1728, *i.e.,* the elliptic curve associated with the lattice $\langle 1, i\rangle$) and the Abelian surface $A' := \mathbb{C}^2/\Lambda$, where

$$\Lambda := \left\langle (1,0), (i,0), \left(\tfrac{1+i}{2}, \tfrac{1+i}{2}\right), \left(\tfrac{1+i}{2}, \tfrac{i-1}{2}\right) \right\rangle.$$

So the group generated by α_4 and β is both a subgroup of $\mathrm{Aut}(A)$ and by $\mathrm{Aut}(A')$. We denote it by $\mathbb{D}_8$ when it is considered as subgroup of $\mathrm{Aut}(A)$ and by $\mathbb{D}_8'$ when it is considered as subgroup of $\mathrm{Aut}(A')$.

We now identify the points of A (resp. A') that have a nontrivial stabilizer for $\mathbb{D}_8$ (resp. $\mathbb{D}_8'$). All of them are 2-torsion points and indeed are fixed for $\alpha_4^2 = \beta^2$. We obtain Tables 1 and 2

We observe that $\alpha_4^2 (= \beta^2)$ is the center of $G := \langle \alpha_4, \beta\rangle$, and in particular that it is a normal subgroup of $\langle \alpha_4, \beta\rangle$. So, in order to construct A/G, one can first consider $A/\langle \alpha_4^2\rangle$ and then $(A/\langle \alpha_4^2\rangle)/Q$, where Q is the quotient group $G/\langle \alpha_4^2\rangle$. The group Q is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and is generated by $\overline{\alpha_4}$ and $\overline{\beta}$, where $\overline{g}$ is the image of $g \in G$ under the quotient map $G \to G/\langle \alpha_4^2\rangle$.

The automorphism α_4^2 is $(z_1, z_2) \mapsto (-z_1, -z_2)$, and the surface A/α_4^2 is singular in the image of the sixteen 2-torsion points of A. We denote by $A[2]$ the set of these points on A. The desingularization, $\mathrm{Km}(A)$, of $A/\langle \alpha_4^2\rangle$ is obtained by blowing up the singular points once, and it is the Kummer surface of A. Let $(p,q) \in A[2]$; then we denote by $K_{(p,q)} \subset \mathrm{Km}(A)$ the sixteen rational curves arising from the desingularization of $A/\langle \alpha_4^2\rangle$.

The minimal resolution X_G of A/G is birational to the minimal resolution of $\mathrm{Km}(A)/Q$, where $Q = G/\langle \alpha_4^2\rangle$. Since the minimal resolution of A/G is a K3 surface, and birational K3 surfaces are isomorphic, we conclude that the minimal model of A/G is the minimal model of $\mathrm{Km}(A)/Q$. We recall that $\mathrm{Km}(A)$ is a K3 surface and $Q = (\mathbb{Z}/2\mathbb{Z})^2$ acts on $\mathrm{Km}(A)$ preserving the symplectic structure (indeed the quotient has the induced symplectic structure). The action of the group $(\mathbb{Z}/2\mathbb{Z})^2$ on a K3 surface is very well known (see [N3, Section 5]). Each copy of $\mathbb{Z}/2\mathbb{Z}$ in $(\mathbb{Z}/2\mathbb{Z})^2$ stabilizes exactly eight points and there are no points fixed by the full group. So there are 24 points with nontrivial stabilizer in $\mathrm{Km}(A)$ and then in the quotient $\mathrm{Km}(A)/Q$ we have twelve singular points, each of them of type A_1. This can be also checked by hand, considering the action of Q over the curves $K_{(p,q)}$. By Tables 1 and 2, G has no points with a nontrivial stabilizer outside the set of the 2-torsion points $A[2]$. So the points with a nontrivial stabilizer for Q on $\mathrm{Km}(A)$ are all contained in the curves $K_{(p,q)}$.

If the point $(p,q) \in A$ is fixed by G, then $Q \simeq (\mathbb{Z}/2\mathbb{Z})^2$ preserves the curve $K_{(p,q)}$, which is a copy of $\mathbb{P}^1$. Hence, Q has six points with nontrivial stabilizer on $K_{(p,q)}$. So in $\mathrm{Km}(A)/Q$ there are three singular points on the image of $K_{(p,q)}$. Hence on X_G there

Points in the same orbit	Stabilizer
$(0,0)$	$\mathbb{D}_8 = \langle \alpha_4, \beta \rangle$
$(\frac{1+i}{2}, \frac{1+i}{2})$	$\mathbb{D}_8 = \langle \alpha_4, \beta \rangle$
$(\frac{1}{2}, \frac{1}{2}), (\frac{i}{2}, \frac{i}{2})$	$\mathbb{Z}/4\mathbb{Z} = \langle \beta \rangle$
$(0, \frac{1+i}{2}), (\frac{1+i}{2}, 0)$	$\mathbb{Z}/4\mathbb{Z} = \langle \alpha_4 \rangle$
$(\frac{1}{2}, \frac{i}{2}), (\frac{i}{2}, \frac{1}{2})$	$\mathbb{Z}/4\mathbb{Z} = \langle \alpha_4 \circ \beta \rangle$
$(\frac{1}{2}, 0)$, $(\frac{i}{2}, 0)$, $(0, \frac{1}{2})$, $(0, \frac{i}{2})$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_4^2 \rangle = \langle \beta^2 \rangle$
$(\frac{1+i}{2}, \frac{1}{2})$, $(\frac{1+i}{2}, \frac{i}{2})$, $(\frac{1}{2}, \frac{1+i}{2})$, $(\frac{i}{2}, \frac{1+i}{2})$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_4^2 \rangle = \langle \beta^2 \rangle$

Table 1: Points of A with nontrivial stabilizer for $\mathbb{D}_8$

Points in the same orbit	Stabilizer
$(0,0)$	$\mathbb{D}_8' = \langle \alpha_4, \beta \rangle$
$(\frac{1+i}{2}, 0)$	$\mathbb{D}_8' = \langle \alpha_4, \beta \rangle$
$(\frac{i}{2}, \frac{i}{2})$	$\mathbb{D}_8' = \langle \alpha_4, \beta \rangle$
$(\frac{1}{2}, \frac{i}{2})$	$\mathbb{D}_8' = \langle \alpha_4, \beta \rangle$
$(\frac{1}{2}, 0), (\frac{i}{2}, 0), (0, \frac{i}{2}), (\frac{1+i}{2}, \frac{i}{2})$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_4^2 \rangle = \langle \beta^2 \rangle$
$(\frac{1+i}{4}, \frac{1+i}{4})$, $(\frac{1-i}{4}, \frac{i-1}{4})$, $(\frac{i-1}{4}, \frac{i-1}{4})$, $(\frac{-1-i}{4}, \frac{i+1}{4})$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_4^2 \rangle = \langle \beta^2 \rangle$
$(\frac{1+i}{4}, \frac{i-1}{4})$, $(\frac{1-i}{4}, \frac{i+1}{4})$, $(\frac{i-1}{4}, \frac{i+1}{4})$, $(\frac{-1-i}{4}, \frac{i-1}{4})$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_4^2 \rangle = \langle \beta^2 \rangle$

Table 2: Points of A' with nontrivial stabilizer for $\mathbb{D}_8'$

are three rational curves that intersect the image of $K_{(p,q)}$(arising from the desingularization of these points). We call these curves $K_{(p,q)}^{(i)}$, $i = 1,2,3$ and let $K_{(p,q)}^{(0)}$ be the strict transform of the image of $K_{(p,q)}$ under the quotient map $\mathrm{Km}(A) \to \mathrm{Km}(A)/Q$. The curves $K_{(p,q)}^{(i)}$, $i = 0,1,2,3$ generate a copy of the lattice D_4 in $NS(X_G)$.

If the point $(p,q) \in A$ has the group $\mathbb{Z}/4\mathbb{Z} \subset G$ as stabilizer, then there is another point (p',q') in its orbit. The quotient group Q switches the curves $K_{(p,q)}$ and $K_{(p',q')}$ and has four points with nontrivial stabilizer in $K_{(p,q)} \cup K_{(p',q')}$ (two on each curve). So on the quotient surface $\mathrm{Km}(A)/Q$ there is a curve $K_{(p,q)}^{(0)}$ which is the common image of $K_{(p,q)}$ and of $K_{(p',q')}$, and there are two singular points on such a curve. We denote by $K_{(p,q)}^{(1)}$ and $K_{(p,q)}^{(2)}$ the curves in X_G arising from the desingularization of these two singular points. The curves $K_{(p,q)}^{(i)}$, $i = 0,1,2$ generate a copy of A_3 in $NS(X_G)$ (here, with an abuse of notation, we denote by $K_{(p,q)}^{(0)}$ both a curve on $\mathrm{Km}(A)/Q$ and its strict transform on X_G).

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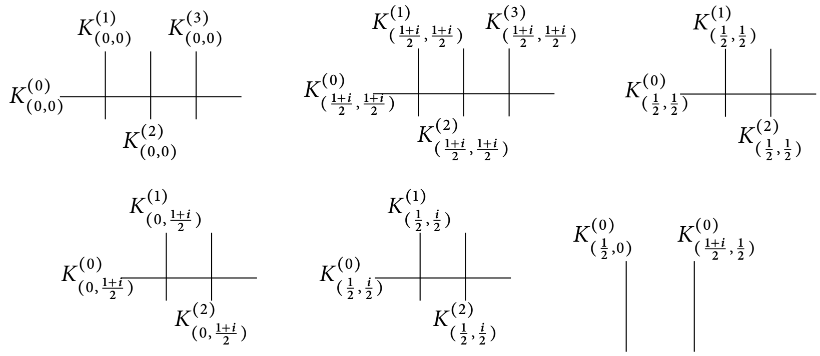

Figure 1: Curves of $F_{\mathbb{D}_8}$ on $X_{\mathbb{D}_8}$

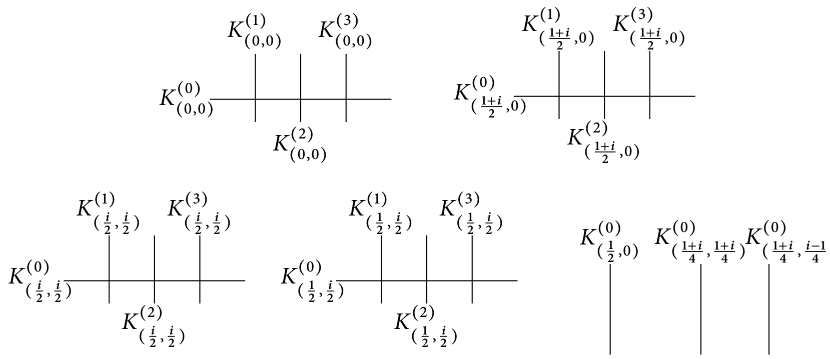

Figure 2: Curves of $F_{\mathbb{D}'_8}$ on $X_{\mathbb{D}'_8}$

If the point $(p,q) \in A$ has the group $\mathbb{Z}/2\mathbb{Z} \subset G$ as stabilizer, it is generated by α_4^2 and in the same orbit of (p,q) there are the other three points (p',q'), (p'',q''), (p''',q'''). The group Q permutes the curves

$$K_{(p,q)}, \quad K_{(p',q')}, \quad K_{(p'',q'')}, \quad K_{(p''',q''')}$$

in $\mathrm{Km}(A)$. So their image in X_G is a unique curve $K^{(0)}_{(p,q)}$.

The curves arising from the desingularization of $A/\mathbb{D}_8$ (resp. $A'/\mathbb{D}'_8$) are represented in Figure 1 (resp. Figure 2).

4.2.2 The Lattice $K_{\mathbb{D}_8}$

Let us now fix a specific action of G. In particular, let the Abelian surface be A (*e.g.*, $A \simeq E_i \times E_i$) and so $G \subset \mathrm{Aut}(A)$ is the group $\mathbb{D}_8$. In this case the lattice $F_{\mathbb{D}_8}$ is isometric to $D_4^2 \oplus A_3^3 \oplus A_1^2$; see Table 2. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^6 \times (\mathbb{Z}/4\mathbb{Z})^3$ and is

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generated by the following classes:

$$d_1 := \frac{1}{2}\Big(K^{(1)}_{(0,0)} + K^{(2)}_{(0,0)}\Big), \qquad d_2 := \frac{1}{2}\Big(K^{(1)}_{(0,0)} + K^{(3)}_{(0,0)}\Big),$$
$$d_3 := \frac{1}{2}\Big(K^{(1)}_{(\frac{1+i}{2},\frac{1+i}{2})} + K^{(2)}_{(\frac{1+i}{2},\frac{1+i}{2})}\Big), \qquad d_4 := \frac{1}{2}\Big(K^{(1)}_{(\frac{1+i}{2},\frac{1+i}{2})} + K^{(3)}_{(\frac{1+i}{2},\frac{1+i}{2})}\Big),$$
$$d_5 := \frac{1}{2}K^{(0)}_{(\frac{1}{2},0)}, \qquad d_6 := \frac{1}{2}K^{(0)}_{(\frac{1+i}{2},\frac{1}{2})},$$
$$d_7 := \frac{1}{4}\Big(K^{(1)}_{(\frac{1}{2},\frac{1}{2})} + 2K^{(0)}_{(\frac{1}{2},\frac{1}{2})} + 3K^{(2)}_{(\frac{1}{2},\frac{1}{2})}\Big), \quad d_8 := \frac{1}{4}\Big(K^{(1)}_{(0,\frac{1+i}{2})} + 2K^{(0)}_{(0,\frac{1+i}{2})} + 3K^{(2)}_{(0,\frac{1+i}{2})}\Big),$$
$$d_9 := \frac{1}{4}\Big(K^{(1)}_{(\frac{1}{2},\frac{i}{2})} + 2K^{(0)}_{(\frac{1}{2},\frac{i}{2})} + 3K^{(2)}_{(\frac{1}{2},\frac{i}{2})}\Big).$$

The set of twelve curves

$$\mathcal{S} := \Big\{K^{(j)}_{(0,0)}, K^{(j)}_{(\frac{1+i}{2},\frac{1+i}{2})}, K^{(k)}_{(\frac{1}{2},\frac{1}{2})}, K^{(k)}_{(0,\frac{1+i}{2})}, K^{(k)}_{(\frac{1}{2},\frac{i}{2})}\Big\},$$

$j = 1,2,3$, $k = 1,2$ arises from the desingularization of the quotient of a K3 surface (the surface $\mathrm{Km}(A)$) by the group $(\mathbb{Z}/2\mathbb{Z})^2$. By [N3, Section 6, case 2a), equation (6.17)], $\mathcal{S}$ contains two independent subsets of eight curves that are 2-divisible. Indeed the two classes

$$v_1 := d_1 + d_3 + 2d_7 + 2d_8, \quad v_2 := d_2 + d_4 + 2d_8 + 2d_9$$

are contained in $NS(X_G)$. Moreover, the set $\mathcal{S} \cup \{K_{(\frac{1+i}{2},0)}, K_{(\frac{1+i}{2},\frac{1}{2})}\}$ forms a set of fourteen disjoint rational curves contained in the curves of $F_{\mathbb{D}_8}$ (this set consists of the vertical curves in Figure 1). By Proposition 2.9, the minimal primitive sublattice of the Néron–Severi group that contains these fourteen curves is spanned by the curves and by three other divisible classes. So there is another divisible class contained in $NS(X_G)$, namely

$$v_3 := d_1 + d_3 + 2d_9 + d_5 + d_6.$$

Let us denote by $L_{\mathbb{D}_8}$ the lattice spanned by $F_{\mathbb{D}_8}$ and the classes v_1, v_2, v_3. Its discriminant group is $(\mathbb{Z}/4\mathbb{Z})^3$ and is generated by:

$$\delta_1 := d_4 + d_7 = \tfrac{1}{4}\Big(2K^{(1)}_{(\frac{1+i}{2},\frac{1+i}{2})} + 2K^{(2)}_{(\frac{1+i}{2},\frac{1+i}{2})} + K^{(1)}_{(\frac{1}{2},\frac{1}{2})} + 2K^{(0)}_{(\frac{1}{2},\frac{1}{2})} + 3K^{(2)}_{(\frac{1}{2},\frac{1}{2})}\Big),$$
$$\delta_2 := d_3 + d_4 + d_8 = \tfrac{1}{4}\Big(2K^{(2)}_{(\frac{1+i}{2},\frac{1+i}{2})} + 2K^{(3)}_{(\frac{1+i}{2},\frac{1+i}{2})} + K^{(1)}_{(0,\frac{1+i}{2})} + 2K^{(0)}_{(0,\frac{1+i}{2})} + 3K^{(2)}_{(0,\frac{1+i}{2})}\Big),$$
$$\delta_3 := d_3 + d_5 + d_9$$
$$= \tfrac{1}{4}\Big(2K^{(1)}_{(\frac{1+i}{2},\frac{1+i}{2})} + 2K^{(2)}_{(\frac{1+i}{2},\frac{1+i}{2})} + 2K_{(\frac{1}{2},0)} + K^{(1)}_{(\frac{1}{2},\frac{i}{2})} + 2K^{(0)}_{(\frac{1}{2},\frac{i}{2})} + 3K^{(2)}_{(\frac{1}{2},\frac{i}{2})}\Big).$$

There are two possibilities, either $K_{\mathbb{D}_8} \simeq L_{\mathbb{D}_8}$ or $K_{\mathbb{D}_8}$ is an overlattice of $L_{\mathbb{D}_8}$ of finite index; see Section 2.1. In the latter case $K_{\mathbb{D}_8}$ contains an element w that is nontrivial in the discriminant group of $L_{\mathbb{D}_8}$. So $w = \sum_{i=1}^{3} \alpha_i\delta_i$, $\alpha_i \in \mathbb{Z}$ and $(\alpha_1,\alpha_2,\alpha_3) \not\equiv (0,0,0)$ mod 4. If $(\alpha_1,\alpha_2,\alpha_3) \equiv (2,2,2)$ mod 4, let $z := w$; otherwise, let $z := 2w$. The element $z \in K_{\mathbb{D}_8}$ consists of the sum of certain disjoint rational curves divided by 2. These curves are chosen in

$$\Big\{K^{(1)}_{(\frac{1}{2},\frac{1}{2})}, K^{(2)}_{(\frac{1}{2},\frac{1}{2})}, K^{(1)}_{(0,\frac{1+i}{2})}, K^{(2)}_{(0,\frac{1+i}{2})}, K^{(1)}_{(\frac{1}{2},\frac{i}{2})}, K^{(2)}_{(\frac{1}{2},\frac{i}{2})}\Big\}$$

(which are the ones which appear in δ_i with an odd coefficient). By Proposition 2.8 a set of at most six disjoint rational curves cannot be divisible by 2, so w cannot exist. We conclude that $L_{\mathbb{D}_8} \simeq K_{\mathbb{D}_8}$, and it is generated by the generators of $F_{\mathbb{D}_8}$ and by $\{v_1, v_2, v_3\}$.

This result agrees with the one given in [W, Proposition 2.1].

4.2.3 The Lattice $K_{\mathbb{D}'_8}$

We now consider the Abelian surface A', so $G \subset \operatorname{Aut}(A')$ is $\mathbb{D}'_8$. In this case, $F_{\mathbb{D}'_8}$ is $D_4^4 \oplus A_1^3$. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^{11}$ and is generated by

$$\begin{aligned}
d'_1 &:= \frac{1}{2}\big(K^{(1)}_{(0,0)} + K^{(2)}_{(0,0)}\big), & d'_2 &:= \frac{1}{2}\big(K^{(1)}_{(0,0)} + K^{(3)}_{(0,0)}\big),\\
d'_3 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{1+i}{2},0)} + K^{(2)}_{(\frac{1+i}{2},0)}\big), & d'_4 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{1+i}{2},0)} + K^{(3)}_{(\frac{1+i}{2},0)}\big),\\
d'_5 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{i}{2},\frac{i}{2})} + K^{(2)}_{(\frac{i}{2},\frac{i}{2})}\big), & d'_6 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{i}{2},\frac{i}{2})} + K^{(3)}_{(\frac{i}{2},\frac{i}{2})}\big),\\
d'_7 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{1}{2},\frac{i}{2})} + K^{(2)}_{(\frac{1}{2},\frac{i}{2})}\big), & d'_8 &:= \frac{1}{2}\big(K^{(1)}_{(\frac{1}{2},\frac{i}{2})} + K^{(3)}_{(\frac{1}{2},\frac{i}{2})}\big),\\
d'_9 &:= \frac{1}{2}K_{(\frac{1}{2},0)}, & d'_{10} &:= \frac{1}{2}K_{(\frac{1+i}{4},\frac{1+i}{4})},\\
d'_{11} &:= \frac{1}{2}K^{(1)}_{(\frac{1+i}{2},\frac{i-1}{2})}.
\end{aligned}$$

The set of twelve curves

$$\mathcal{S} := \Big\{K^{(j)}_{(0,0)}, K^{(j)}_{(\frac{1+i}{2},0)}, K^{(j)}_{(\frac{i}{2},\frac{i}{2})}, K^{(j)}_{(\frac{1}{2},\frac{i}{2})}\Big\}, \quad j = 1, 2, 3,$$

arises from the desingularization of the quotient of a K3 surface (the surface $\operatorname{Km}(A')$) by the group $(\mathbb{Z}/2\mathbb{Z})^2$ (*i.e.*, the group $Q := G/\langle\alpha_4^2\rangle$) so, by [N3, Section 6, case 2a), equation (6.17)], there are two divisible classes whose curves are in $\mathcal{S}$. Hence,

$$v'_1 := d'_1 + d'_3 + d'_5 + d'_7, \quad v'_2 := d'_2 + d'_4 + d'_6 + d'_8 \tag{4.1}$$

are contained in $NS(X_G)$. Moreover, in the lattice $F_{\mathbb{D}'_8}$ it is easy to identify a set of fifteen disjoint rational curves (the vertical ones in Figure 2), which contains the set $\mathcal{S}$. By Proposition 2.9 the minimal primitive sublattice of the Néron–Severi group which contains these curves is spanned by the curves and by four other divisible classes. Two of these divisible classes are v'_1 and v'_2; the others are

$$v'_3 := d'_1 + d'_3 + d'_4 + d'_6 + d'_9 + d'_{10}, \quad v'_4 := d'_1 + d'_4 + d'_7 + d'_8 + d'_9 + d'_{11}.$$

These four divisible classes are also contained in $K_{\mathbb{D}'_8}$. Let us denote by $L_{\mathbb{D}'_8}$ the lattice spanned by $F_{\mathbb{D}'_8}$ and by the classes v'_1, v'_2, v'_3, v'_4. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^3$ and is generated by

$$\delta'_1 := d'_2 + d'_3 + d'_4 + d'_5, \quad \delta'_2 := d'_3 + d'_4 + d'_6 + d'_7, \quad \delta'_3 := d'_4 + d'_5 + d'_6 + d'_7 + d'_{11}.$$

If $K_{\mathbb{D}'_8}$ does not coincide with $L_{\mathbb{D}'_8}$, then there is a vector w that is nontrivial in the discriminant group of $L_{\mathbb{D}'_8}$, and is not contained in $K_{\mathbb{D}'_8}$, by Section 2.1. The curves that appear with a nontrivial coefficient in δ'_1, δ'_2, and δ'_3 are all contained in the set of

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fifteen disjoint rational curves considered above. So if a vector as w exists, it gives an overlattice of the lattice spanned by fifteen disjoint rational curves with index greater than 2^4 and contained in the Néron–Severi group of a K3 surface, but this is impossible. Indeed, if we construct an overlattice of index 2^4 of A_1^{15}, every 2-divisible set contains exactly eight disjoint rational curves by Proposition 2.8, and two divisible sets have exactly four curves in common. Let us denote by e_i the fifteen classes generating A_1^{15}. The first divisible set contains eight classes, so up to permutation of the indices we can assume that it is $\mathcal{S}_1 := \{e_1, \ldots, e_8\}$. The second one contains eight classes, four of them in common with $\mathcal{S}_1$, so we can assume that it is $\mathcal{S}_2 := \{e_1, \ldots, e_4, e_9, \ldots, e_{12}\}$. Similarly, the third can be chosen to be $\mathcal{S}_3 := \{e_1, e_2, e_5, e_6, e_9, e_{10}, e_{13}, e_{14}\}$. This forces the fourth to be $\mathcal{S}_4 := \{e_1, e_3, e_5, e_7, e_9, e_{11}, e_{13}, e_{15}\}$. But now it is not possible to find another subset of $\{e_1, \ldots, e_{15}\}$ that contains eight elements and such that its intersection with each set $\mathcal{S}_i$ contains exactly four elements.

We conclude that $L_{\mathbb{D}_8'}$ coincides with $K_{\mathbb{D}_8'}$, which is generated by the vectors in $F_{\mathbb{D}_8'}$ and by the four vectors v_1', v_2', v_3' and v_4'.

This result is different to the one given in [W]. Indeed, the lattice of Kummer type $\Pi_{\mathbb{D}_8'}$ described in [W, Proposition 2.1] contains a vector that consists of six disjoint rational curves divided by 2, which is not possible by Proposition 2.8.

4.2.4 The Lattice $K_{\mathbb{T}}$

Let us now consider the torus A'. There is an extra automorphism, which is not contained in $\mathbb{D}_8'$ and acts on A': the automorphism

$$\gamma\colon (z_1, z_2) \longrightarrow \Big(\frac{i-1}{2}(z_1 - z_2), \frac{-i-1}{2}(z_1 + z_2) \Big).$$

The automorphism γ has order three and the group $\langle \alpha_4, \beta, \gamma \rangle$ is the binary tetrahedral group $\mathbb{T}$. It is the semidirect product $\langle \gamma \rangle \ltimes \mathbb{D}_8'$. In particular $\mathbb{D}_8'$ is a normal subgroup of $\mathbb{T}$ hence $A'/\mathbb{T}$ is birational to $(A'/\mathbb{D}_8')/\langle \overline{\gamma} \rangle$, where $\overline{\gamma}$ is the image of γ under the quotient map $\mathbb{T} \to \mathbb{T}/\mathbb{D}_8'$. Hence, the K3 surface $X_{\mathbb{D}_8'}$, desingularization of $A'/\mathbb{D}_8'$, admits a symplectic automorphism, γ_X, of order 3 induced by $\overline{\gamma}$. The K3 surface $X_{\mathbb{T}}$, desingularization of $A/\mathbb{T}$, is then isomorphic to the K3 which is the desingularization of $X_{\mathbb{D}_8'}/\gamma_X$. In order to construct $F_{\mathbb{T}}$, we consider the action of γ_X on the curves of $F_{\mathbb{D}_8'}$, see Figure 2. Since

$$\gamma\Big(\Big(\frac{1+i}{2}, 0 \Big) \Big) = \Big(\frac{1}{2}, \frac{i}{2} \Big) \quad \text{and} \quad \gamma\Big(\Big(\frac{1}{2}, \frac{i}{2} \Big) \Big) = \Big(\frac{i}{2}, \frac{i}{2} \Big),$$

the three copies of D_4, whose components are

$$K^{(j)}_{(\frac{1+i}{2},0)}, \quad K^{(j)}_{(\frac{1}{2},\frac{i}{2})}, \quad \text{and} \quad K^{(j)}_{(\frac{i}{2},\frac{i}{2})}, \quad j = 0,1,2,3$$

are permuted by γ_X. Hence these three copies of D_4 are identified on $X_{\mathbb{T}}$ and correspond to a unique copy of D_4 on $X_{\mathbb{T}}$. The same happens to the three copies of A_1, which are permuted by γ_X and thus give a unique copy of A_1 on $X_{\mathbb{T}}$. Since $(0,0)$ is a fixed point for γ, the automorphism γ_X preserves the set of curves $\{K^{(j)}_{(0,0)}\}$, $j = 0,1,2,3$. Indeed γ_X preserves the curve $K^{(0)}_{(0,0)}$ and permutes the curves $K^{(j)}_{(0,0)}$, $j = 1,2,3$. So it is not the identity on $K^{(0)}_{(0,0)}$ (since it moves the intersection points

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among $K^{(0)}_{(0,0)}$ and $K^{(j)}_{(0,0)}$, $j = 1,2,3$) and thus has two fixed point on it. On the quotient these two points correspond to two singularities of type A_2. This gives six curves on $X_{\mathbb{T}}$ (one is the image of $K^{(0)}_{(0,0)}$, one is the common image of $K^{(j)}_{(0,0)}$ for $j = 1,2,3$, four come from the desingularization of the two singular points of type A_2) and their dual graph is a copy of E_6 (the image of $K^{(0)}_{(0,0)}$ intersects the image of $K^{(j)}_{(0,0)}$ and one curve of each copy of the two A_2 arising from the desingularization).

We recall that a symplectic automorphism of order three on a K3 surface has exactly six fixed points. Since γ_X fixes two points on $K^{(0)}_{(0,0)}$ and has no fixed points on the other curves of $F_{\mathbb{D}_8'}$, it necessarily fixes four points in $X_{\mathbb{D}_8'}$ outside the curves in $F_{\mathbb{D}_8'}$. Hence the desingularization $X_{\mathbb{T}}$ introduces four disjoint A_2-configurations. Thus, the lattice $F_{\mathbb{T}}$ is isometric to $E_6 \oplus D_4 \oplus A_1 \oplus A_2^4$. We fix the following notation:

$$\begin{tikzcd}[arrows=dash]
e_3 \arrow[r] & e_2 \arrow[r] & e_0 \arrow[r] \arrow[d] & e_4 \arrow[r] & e_5 \\
 & & e_1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
f_1 \arrow[r] & f_0 \arrow[r] \arrow[d] & f_2 \\
 & f_3, &
\end{tikzcd}$$

where $\{e_j\}$ forms a basis of E_6 and $\{f_j\}$ forms a basis of D_4. We denote by $a^{(1)}$ a generator of A_1 and by $a_j^{(h)}$, $j = 1,2$, $h = 1,2,3,4$, the basis of the h-th copy of A_2. A basis for the discriminant group of $F_{\mathbb{T}}$ is given by

$$\begin{aligned}
d_1 &:= \frac{1}{3}(e_2 + 2e_3 + e_4 + 2e_5) + \frac{1}{2}(f_1 + f_2), & d_2 &:= \frac{1}{3}(a_1^{(2)} + 2a_2^{(2)}) + \frac{1}{2}(f_1 + f_3),\\
d_3 &:= \frac{1}{3}(a_1^{(3)} + 2a_2^{(3)}) + \frac{1}{2}a^{(1)}, & d_4 &:= \frac{1}{3}(a_1^{(4)} + 2a_2^{(4)}),\\
d_5 &:= \frac{1}{3}(a_1^{(5)} + 2a_2^{(5)}).
\end{aligned}$$

The curves $e_2, e_3, e_4, e_5, a_1^{(j)}, a_2^{(j)}$, $j = 2,3,4,5$ are the curves arising from the resolution of the quotient $X_{\mathbb{D}_8'}/\gamma_X$. So by Section 2.2 (see also Proposition 2.8), the class

$$\Big(e_2 + 2e_3 + e_4 + 2e_5 + \sum_{j=1}^{4}(a_1^j + 2a_2^j)\Big)/3$$

is contained in $NS(X_{\mathbb{T}})$ and hence also in $K_{\mathbb{T}}$ (which is the minimal primitive sublattice of $NS(X_{\mathbb{T}})$ which contains the curves e_h, f_j, $a_r^{(s)}$). So the vector $v := 4d_1 + 4d_2 + 4d_3 + d_4 + d_5 \mod F_{\mathbb{T}}$ is contained in $K_{\mathbb{T}}$. Let us denote by $L_{\mathbb{T}}$ the lattice generated by the curves of $F_{\mathbb{T}}$ and v. Its discriminant group is generated by $\delta_1 := d_1 + d_2$, $\delta_2 := d_1 + d_3$, $\delta_3 := d_1 + d_4$. If $L_{\mathbb{T}} \neq K_{\mathbb{T}}$, then there exists a vector $w \in K_{\mathbb{T}}$, which is a nontrivial element of the discriminant group of $L_{\mathbb{T}}$, which is $(\mathbb{Z}/6\mathbb{Z})^3$. So either w or a multiple of w generates either $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ in the discriminant group of $L_{\mathbb{T}}$. Every linear combination of δ_1, δ_2 and δ_3 which generates $\mathbb{Z}/2\mathbb{Z}$ is the sum of at most four disjoint rational curves divided by 2, and so cannot be a class in $NS(X_{\mathbb{T}})$, by Proposition 2.8. Similarly, every linear combination of δ_1, δ_2, and δ_3 that generates $\mathbb{Z}/3\mathbb{Z}$ contains at most five disjoint A_2-configurations of rational curves. By Proposition 2.8 it is impossible to construct a 3-divisible class with fewer than six disjoint

A_2-configurations. We conclude that $K_{\mathbb{T}} = L_{\mathbb{T}}$ is generated by v and by the curves in $F_{\mathbb{T}}$.

This result agrees with the one given in [W].

4.2.5 The Lattice $K_{\mathbb{D}_{12}}$

Let A be the Abelian surface $A := E_{\zeta_3} \times E_{\zeta_3}$, where ζ_3 is a primtive third root of unity and E_{ζ_3} is the elliptic curve with j-invariant 0. Let us now consider the action of the group $\mathbb{D}_{12}$, which is algebraic on A and is generated by the two automorphisms $\alpha_6\colon (z_1, z_2) \mapsto (\zeta_6 z_1, \zeta_6^5 z_2)$ (where ζ_6 is a 6-th primitive root of unity), and $\beta\colon (z_1, z_2) \mapsto (-z_2, z_1)$. We observe that there are the relations $\alpha_6^3 = \beta^2$, $\alpha_6^6 = \beta^4 = 1$, $\alpha_6^{-1}\beta\alpha_6 = \beta^{-1}$, so α_6 and β generate $\mathbb{D}_{12} \subset \mathrm{Aut}(A)$. The points of A with nontrivial stabilizer for $\mathbb{D}_{12}$ are those in Table 3:

points in the same orbit	stabilizer
$(0,0)$	$\mathbb{D}_{12} = \langle \alpha_6, \beta \rangle$
$(0, \frac{1-\zeta_3}{3}), (\frac{-1+\zeta_3}{3}, 0), (\frac{1-\zeta_3}{3}, 0), (0, \frac{-1+\zeta_3}{3})$	$\mathbb{Z}/3\mathbb{Z} = \langle \alpha_6^2 \rangle$
$(\frac{1-\zeta_3}{3}, \frac{1-\zeta_3}{3}), (\frac{-1+\zeta_3}{3}, \frac{1-\zeta_3}{3}), (\frac{1-\zeta_3}{3}, \frac{-1+\zeta_3}{3}), (\frac{-1+\zeta_3}{3}, \frac{-1+\zeta_3}{3})$	$\mathbb{Z}/3\mathbb{Z} = \langle \alpha_6^2 \rangle$
$(\frac{1}{2}, \frac{1}{2}), (\frac{\zeta_3}{2}, \frac{1+\zeta_3}{2}), (\frac{1+\zeta_3}{2}, \frac{\zeta_3}{2})$	$\mathbb{Z}/4\mathbb{Z} = \langle \beta \rangle$
$(\frac{\zeta_3}{2}, \frac{\zeta_3}{2}), (\frac{1+\zeta_3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1+\zeta_3}{2})$	$\mathbb{Z}/4\mathbb{Z} = \langle \beta \rangle$
$(\frac{1+\zeta_3}{2}, \frac{1+\zeta_3}{2}), (\frac{1}{2}, \frac{\zeta_3}{2}), (\frac{\zeta_3}{2}, \frac{1}{2})$	$\mathbb{Z}/4\mathbb{Z} = \langle \beta \rangle$
$(0, \frac{1}{2}), (0, \frac{\zeta_3}{2}), (0, \frac{1+\zeta_3}{2}), (\frac{1}{2}, 0), (\frac{\zeta_3}{2}, 0), (\frac{1+\zeta_3}{2}, 0)$	$\mathbb{Z}/2\mathbb{Z} = \langle \alpha_6^3 \rangle$

Table 3: Points of A with nontrivial stabilizer for $\mathbb{D}_{12}$

It follows that $F_{\mathbb{D}_{12}}$ is isometric to $D_5 \oplus A_2^2 \oplus A_3^3 \oplus A_1$.

First, we consider the quotient by $\langle \alpha_6^2 \rangle$, which is a normal subgroup of $\mathbb{D}_{12}$. The quotient $A/\langle \alpha_6^2 \rangle$ is a surface with nine singularities of type A_2, in the image of the points p contained in the set

$$\mathcal{P} := \Big\{ (0,0), \Big(0, \frac{1-\zeta_3}{3}\Big), \Big(\frac{-1+\zeta_3}{3}, 0\Big), \Big(\frac{1-\zeta_3}{3}, 0\Big), \Big(0, \frac{-1+\zeta_3}{3}\Big), \Big(\frac{1-\zeta_3}{3}, \frac{1-\zeta_3}{3}\Big), \Big(\frac{-1+\zeta_3}{3}, \frac{1-\zeta_3}{3}\Big), \Big(\frac{1-\zeta_3}{3}, \frac{-1+\zeta_3}{3}\Big), \Big(\frac{-1+\zeta_3}{3}, \frac{-1+\zeta_3}{3}\Big) \Big\}.$$

This introduces eighteen curves on the K3 surface $\widetilde{A/\langle \alpha_6^2 \rangle}$, which is the desingularization of $A/\langle \alpha_6^2 \rangle$, namely the curves K_p^j, $p \in \mathcal{P}$, $j = 1, 2$, which desingularize the point $p \in \mathcal{P}$. The automorphism $\beta \in \mathrm{Aut}(A)$ induces an automorphism β' on $\widetilde{A/\langle \alpha_6^2 \rangle}$. Since β fixes $(0,0)$, β' preserves the set of curves $\{K_{(0,0)}^{(j)}\}$, $j = 1, 2$. The automorphism β' fixes the intersection point $K_{(0,0)}^{(1)} \cap K_{(0,0)}^{(2)}$ and switches the curves $K_{(0,0)}^{(1)}$ and $K_{(0,0)}^{(2)}$. The square $(\beta')^2$ preserves the curves $K_{(0,0)}^{(1)}$ and $K_{(0,0)}^{(2)}$ and fixes their intersection point and another point on each curve. The points in $\mathcal{P} - \{(0,0)\}$ have a trivial stabilizer with respect to the action of $\langle \beta \rangle$ on A, so the eight A_2-configurations generated

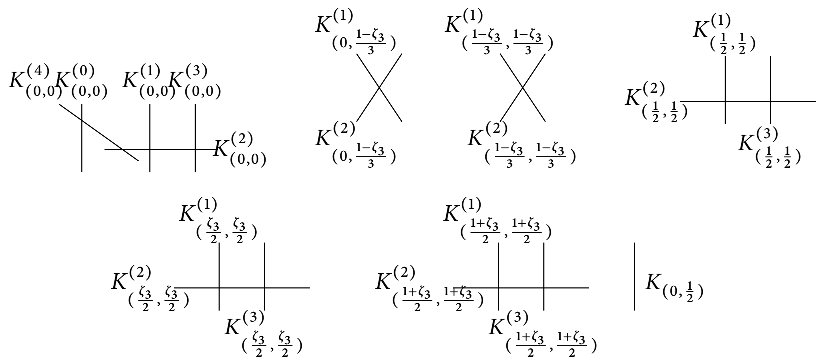

Figure 3: Curves of $F_{\mathbb{D}_{12}}$ on $X_{\mathbb{D}_{12}}$

by K_p^j, $p \in \mathcal{P} - \{(0,0)\}$, $j = 1,2$, are moved by β'. In particular, neither β' or $(\beta')^2$ have fixed points on these curves. The automorphism β fixes other nine points of A (see Table 3), which correspond to three points on $\widetilde{A/\langle\alpha_6^2\rangle} - \{K_p^j\}$ (where $p \in \mathcal{P}$, $j = 1,2$) and thus to three singularities of type A_3 on $\widetilde{A/\langle\alpha_6^2\rangle}/\beta'$. The automorphism β^2 fixes other six points on A (see Table 3), which correspond to 2 points of $\widetilde{A/\langle\alpha_6^2\rangle} - \{K_p^{(j)}\}$ (where $p \in \mathcal{P}$, $j = 1,2$) and thus to one singular point of type A_1 on $\widetilde{A/\langle\alpha_6^2\rangle}/\beta'$.

Hence in the desingularization of $(\widetilde{A/\langle\alpha_6^2\rangle})/\beta'$, which is isomorphic to $X_{\mathbb{D}_{12}}$, there are the following curves:

$$K^{(h)}_{(0,0)}, \quad h = 0,\dots,4,$$

which form a D_5; the curves

$$K^{(j)}_{(0,\frac{1-\zeta_3}{3})} \quad \text{and} \quad K^{(j)}_{(\frac{1-\zeta_3}{3},\frac{1-\zeta_3}{3})}, \quad j = 1,2,$$

which form two disjoint copies of A_2 and that are image of the eight copies of A_2 not preserved by β'; the curves

$$K^{(j)}_{(\frac{1}{2},\frac{1}{2})}, \quad K^{(j)}_{(\frac{\zeta_3}{2},\frac{\zeta_3}{2})}, \quad K^{(j)}_{(\frac{1+\zeta_3}{2},\frac{1+\zeta_3}{2})}, \quad j = 1,2,3,$$

which form three disjoint copies of A_3, and the curve $K_{(0,\frac{1}{2})}$, which is a copy of A_1.

The intersection properties of these curves are presented in Figure 3.

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The discriminant group of $F_{\mathbb{D}_{12}}$ is $(\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$, generated by

$$\begin{aligned}
d_1 &:= \frac{1}{4}\Big(2K^{(4)}_{(0,0)} + K^{(1)}_{(0,0)} + 2K^{(2)}_{(0,0)} + 3K^{(3)}_{(0,0)}\Big),\\
d_2 &:= \frac{1}{4}\Big(K^{(1)}_{(\frac{1}{2},\frac{1}{2})} + 2K^{(2)}_{(\frac{1}{2},\frac{1}{2})} + 3K^{(3)}_{(\frac{1}{2},\frac{1}{2})}\Big) + \frac{1}{3}\Big(K^{(1)}_{(0,\frac{1-\zeta_3}{3})} + 2K^{(2)}_{(0,\frac{1-\zeta_3}{3})}\Big),\\
d_3 &:= \frac{1}{4}\Big(K^{(1)}_{(\frac{\zeta_3}{2},\frac{\zeta_3}{2})} + 2K^{(2)}_{(\frac{\zeta_3}{2},\frac{\zeta_3}{2})} + 3K^{(3)}_{(\frac{\zeta_3}{2},\frac{\zeta_3}{2})}\Big) + \frac{1}{3}\Big(K^{(1)}_{(\frac{1-\zeta_3}{3},\frac{1-\zeta_3}{3})} + 2K^{(2)}_{(\frac{1-\zeta_3}{3},\frac{1-\zeta_3}{3})}\Big),\\
d_4 &:= \frac{1}{4}\Big(K^{(1)}_{(\frac{1+\zeta_3}{2},\frac{1+\zeta_3}{2})} + 2K^{(2)}_{(\frac{1+\zeta_3}{2},\frac{1+\zeta_3}{2})} + 3K^{(3)}_{(\frac{1+\zeta_3}{2},\frac{1+\zeta_3}{2})}\Big),\\
d_5 &:= \frac{1}{2}K_{(0,\frac{1}{2})}.
\end{aligned}$$

The curves

$$K^{(j)}_{(0,0)},\ j=1,2,3,4,5,\quad K^{(h)}_{(\frac{1}{2},\frac{1}{2})},\quad K^{(h)}_{(\frac{\zeta_3}{2},\frac{\zeta_3}{2})},\quad K^{(h)}_{(\frac{1+\zeta_3}{2},\frac{1+\zeta_3}{2})},\ h=1,2,3,\quad \text{and}\quad K_{(0,\frac{1}{2})}$$

arise from the desingularization of the $\mathbb{Z}/4\mathbb{Z}$ quotient $\widetilde{A/\langle\alpha_6^2\rangle} \to (\widetilde{A/\langle\alpha_6^2\rangle})/\beta'$. So the class $v_{\mathbb{D}_{12}} := d_1 + 9d_2 + 9d_3 + d_4 + d_5$ is contained in $NS(X_{\mathbb{D}_{12}})$, because $X_{\mathbb{D}_{12}}$ is the resolution of $\widetilde{A/\langle\alpha_6^2\rangle})/\beta'$.

Let us denote by $L_{\mathbb{D}_{12}}$ the lattice generated by the curves of $F_{\mathbb{D}_{12}}$ and by $v_{\mathbb{D}_{12}}$. The discriminant group of $L_{\mathbb{D}_{12}}$ is $(\mathbb{Z}/12\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$ and is generated by the vectors $\delta_1 := d_1 + 9d_2$, $\delta_2 := d_1 + 9d_3$, $\delta_3 := 2d_1 + d_5$. Either $K_{\mathbb{D}_{12}}$ coincides with $L_{\mathbb{D}_{12}}$ or it is an overlattice of finite index of $L_{\mathbb{D}_{12}}$. In the latter case there would be a nontrivial vector w in the discriminant group of $L_{\mathbb{D}_{12}}$, which is contained in $K_{\mathbb{D}_{12}}$. Either w or a multiple of w generates either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ in the discriminant group. It is easy to check that there is no vector w as required, since it should correspond either to the sum of n, $n \le 7$, disjoint rational curves divided by 2 or to the sum of m disjoint A_2-configurations divided by 3, with $m \le 2$. By Proposition 2.8, these two possibilities are not acceptable, so $K_{\mathbb{D}_{12}}$ coincides with $L_{\mathbb{D}_{12}}$.

This result contradicts the one given in [W]. In our construction the lattice of Kummer type is generated by the classes of the curves arising from the desingularization of A/G and by a class 4-divisible (*i.e.*, the vector $v_{\mathbb{D}_{12}}$). In [W, Proposition 2.1] the lattice of Kummer type (Π_{12}, with the notation used in [W]), is claimed to be generated by the classes of the curves arising from the desingularization of A/G and by a class 2-divisible (and not 4-divisible). The discriminant group of the lattice $\Pi_{\mathbb{D}_{12}}$ described in [W] is $(\mathbb{Z}/12\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3$. This group has five generators. Since the rank of $\Pi_{\mathbb{D}_{12}}$ is 19 and the rank of Λ_{K3} is 22, this is impossible because of Proposition 2.3.

4.3 The Kummer Type Lattices

Here we collect the results obtained above and the known ones in order to give a description of all the lattices of Kummer type. In particular, we show that for all the lattices K_G of Kummer type, the roots of K_G coincide with the roots of F_G, which will be very useful later.

Proposition 4.3 (See [N1] for $G = \mathbb{Z}/2\mathbb{Z}$; [Be] for $G = \mathbb{Z}/n\mathbb{Z}$, $n = 3, 4, 6$; Section 4.2 and [W] for $G = \mathbb{D}_8, \mathbb{D}'_8, \mathbb{D}_{12}, \mathbb{T}$).) *Let A be an Abelian surface with an action of a finite group G that does not contain translations. Let X_G be the minimal model of A/G. If X_G is a K3 surface, then G is one of the following seven groups: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{D}_8$, $\mathbb{D}_{12}$, $\mathbb{T}$. We recall that there are two different actions of the quaternion group, denoted by $\mathbb{D}_8$ and $\mathbb{D}'_8$.*

Let us assume that X_G is a K3 surface (so G is one of the seven groups listed above). Let K_i be the curves on X_G arising by the resolution of the singularities of A/G. Then the lattice F_G spanned by the curves K_i is one of the following root lattices:

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$
F_G	A_1^{16}	A_2^9	$A_3^4 \oplus A_1^6$	$A_5 \oplus A_2^4 \oplus A_1^5$

G	$\mathbb{D}_8$	$\mathbb{D}'_8$	$\mathbb{D}_{12}$	$\mathbb{T}$
F_G	$D_4^2 \oplus A_3^3 \oplus A_1^2$	$D_4^4 \oplus A_1^3$	$D_5 \oplus A_3^3 \oplus A_2^2 \oplus A_1$	$E_6 \oplus D_4 \oplus A_2^4 \oplus A_1$

.

Let K_G be the minimal primitive sublattice of $NS(X_G)$ containing the curves K_i, then K_G is an overlattice of finite index r_G of F_G with the following properties: The roots of

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$
r_G	2^5	3^3	2^4	6
$\mathrm{rank}(K_G)$	16	18	18	18
$K_G^\vee/K_G$	$(\mathbb{Z}/2\mathbb{Z})^6$	$(\mathbb{Z}/3\mathbb{Z})^3$	$(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/6\mathbb{Z})^3 \times \mathbb{Z}/2\mathbb{Z}$

G	$\mathbb{D}_8$	$\mathbb{D}'_8$	$\mathbb{D}_{12}$	$\mathbb{T}$
r_G	2^3	2^4	4	3
$\mathrm{rank}(K_G)$	19	19	19	19
$K_G^\vee/K_G$	$(\mathbb{Z}/4\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/12\mathbb{Z})^2$	$(\mathbb{Z}/6\mathbb{Z})^3$

the lattice K_G coincide with the roots of the lattice F_G for all G.

By construction, K_G is a negative definite lattice primitively embedded in $NS(X_G)$, and thus $\rho(X_G) \geq 1 + \mathrm{rank}(K_G)$.

Proof The groups G that act on A in such a way that the resolution of A/G is a K3 surface are classified by [F, Lemma 3.3]. The properties of F_G and K_G are proved in [N1, Section 1] for $G = \mathbb{Z}/2\mathbb{Z}$; [Be, Section 1 and Theorem 2.5] for $G = \mathbb{Z}/n\mathbb{Z}$, $n = 3, 4, 6$; in Section 4.2 and [W, Proposition 2.1] for $G = \mathbb{D}_8, \mathbb{D}'_8, \mathbb{D}_{12}, \mathbb{T}$. The unique observation that needs to be proved is that the root system of F_G coincides with the one of K_G. This was explicitly proved in [Be, Proposition 1.3] for $G = \mathbb{Z}/n\mathbb{Z}$, $n = 3, 4, 6$. In Section 4.2 we described a basis for F_G and K_G if G is noncyclic, and in [N1] a basis for $K_{\mathbb{Z}/2\mathbb{Z}}$ is given. One can explicitly write down a Gram matrix for the lattice K_G. Since K_G is a negative definite lattice, the number of vectors with a given self intersection is finite and can be computed. In particular, one computes

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the number of vectors of self-intersection -2 in K_G (for example, using the command `ShortestVectors(-`K_G`)` in `Magma`) and one compares it with the number of vectors of self-intersection -2 in F_G. They coincide for every group G in the list, and this concludes the proof. ∎

4.4 The Main Results

The aim of this section is to present and to prove our main result (Theorem 4.4). One can deduce whether a K3 surface is the quotient of an Abelian surface by checking if a certain lattice is primitively embedded in its Néron–Severi group. This essentially implies that one can construct the moduli space of a K3 surface that is a desingularization of the quotient of an Abelian surface by a finite group as a moduli space of a lattice polarized K3 surfaces.

The other result of this section (Theorem 4.7) is that one can deduce whether a K3 surface is rationally G-covered by an Abelian surface by checking if a certain configuration of rational curves is present on the K3 surface.

Combinating these two results we deduce a synthetic description of the lattices of Kummer type as overlattices with certain properties of the lattices F_G (see Corollary 4.8).

Theorem 4.4 *Let G be one of the groups $\mathbb{Z}/n\mathbb{Z}$, $n = 2, 3, 4, 6$, $\mathbb{D}_8$, $\mathbb{D}_8'$, $\mathbb{D}_{12}$, and $\mathbb{T}$, and let K_G be the lattice of Kummer type defined above. A K3 surface is the minimal model of A/G for a certain Abelian surface A if and only if K_G is primitively embedded in $NS(X_G)$.*

Proof One of the implications is trivial. If X_G is the desingularization of A/G, then $NS(X_G)$ contains the classes of the curves arising from the desingularization of A/G, so it contains the lattice F_G. By definition K_G is the minimal primitive sublattice of $NS(X_G)$ containing F_G, and so K_G is primitively embedded in $NS(X_G)$.

Let X_G be a K3 surface such that K_G is primitively embedded in $NS(X_G)$. We first prove our result in the case where $\rho(X_G) = 1 + \mathrm{rank}(K_G)$, *i.e.*, where it is the minimal possible. Let us denote by h the generator of the 1-dimensional subspace of $NS(X_G)$ that is orthogonal to K_G, so $NS(X_G)$ is an overlattice of finite index of $\mathbb{Z}h \oplus K_G$. Up to the action of the Weyl group, we can assume that h is a pseudoample divisor on X_G. Since K_G is an overlattice of finite index of F_G, F_G is a root lattice, and the roots of F_G coincide with the roots of K_G, the assumptions of Proposition 3.2 (with $L := K_G$ and $R := F_G$) are satisfied. Hence we can assume that the classes generating F_G are supported on smooth irreducible rational curves. This fact suffices to reconstruct the surface A that is the minimal model of the G-cover of X_G. This is well known in the case where $G = \mathbb{Z}/2\mathbb{Z}$, see [N1]. The cases $G = \mathbb{Z}/3\mathbb{Z}$ and $G = \mathbb{Z}/4\mathbb{Z}$ are described in [Be, Sections (4.1) and (4.2)]. As an example, we describe how one can reconstruct A in the cases $G = \mathbb{Z}/6\mathbb{Z}$, $G = \mathbb{D}_8'$, and $G = \mathbb{T}$.

Let us assume $G = \mathbb{Z}/6\mathbb{Z}$. Then $F_G \simeq A_5 \oplus A_2^4 \oplus A_1^5$ and K_G is obtained by adding to F_G the class

$$v := \frac{1}{6}\Big(\sum_{j=1}^{5} jK_1^{(j)}\Big) + \frac{1}{3}\sum_{i=2}^{5}(K_i^{(1)} + 2K_i^{(2)}) + \frac{1}{2}\Big(\sum_{i=6}^{10} K_i^{(1)}\Big),$$

where $K_1^{(j)}$, $j=1,\dots,5$ is a basis of A_5, $K_i^{(j)}$, $j=1,2$, $i=2,3,4,5$ is a basis of the $(i-1)$-th copy of A_2, and $K_i^{(1)}$, $i=6,\dots,10$ is a generator of the $(i-5)$-the copy of A_1. Let us now consider $3v$. It exhibits the set of curves

$$\{K_1^{(1)},K_1^{(3)},K_1^{(5)},K_6^{(1)},K_7^{(1)},K_8^{(1)},K_9^{(1)},K_{10}^{(1)}\}$$

as a set of eight disjoint rational curves divisible by 2 on a K3 surface. Then there exists a 2:1 cover of X_G, $\widetilde{Y}\to X_G$, branched along these curves and such that the minimal model Y of $\widetilde{Y}$ is a K3 surface. The minimal model Y is obtained contracting the eight (-1)-curves which are the 2:1 cover of the branch curves. Let us consider the rational 2:1 maps $\pi\colon Y\dashrightarrow X_G$. Then $\pi^{(-1)}(K_i^{(j)})$ splits into two rational curves for $j=1,2$, $i=2,3,4,5$, this gives eight A_2-configurations on Y; $\pi^{-1}(K_1^{(2)})$ is a rational curve which is a 2:1 cover of $K_1^{(2)}$ branched in two points; $\pi^{(-1)}(K_1^{(3)})$ is a rational curve which is a 2:1 cover of $K_1^{(3)}$ branched in two points and we observe that after the contraction $\widetilde{Y}\to Y$, $\pi^{-1}(K_1^{(2)})$ and $\pi^{-1}(K_1^{(3)})$ form a copy of A_2. So we have nine copies of A_2 on Y. By Proposition 2.9, there exists an Abelian surface A that is a 3:1 rational cover of Y. The minimal model of this cover is an Abelian surface A, which is indeed a (rational) G-cover of X_G.

In case $\rho(X)>1+\operatorname{rank}(K_G)$, the proof follows by a standard deformation argument that we summarize here: If $\rho(X)>1+\operatorname{rank}(K_G)$, then $\rho(X)=20$. There exists a 1-dimensional family $\{X_t\}_{t\in\mathbb{C}}$ that deforms X such that the generic member X_t has Picard number 19 and K_G is primitively embedded in $NS(X_t)$. Since generically $\rho(X_t)=1+\operatorname{rank}(K_G)$, for a generic t there exists an Abelian surface A_t. As described in [MTW, Section 5.2], the members of the 1-dimensional family of Abelian surfaces A_t admits a symplectic G action and the desingularization of A_t/G is X_t. Generically $\rho(A_t)=3$ but there are special members, $A_{\overline{t}}$ in the family $\{A_t\}$ such that $\rho(A_{\overline{t}})=4$. These Abelian surfaces are (rational) G-covers of K3 surfaces $X_{\overline{t}}$, which has Picard number 20. In particular there exist an Abelian surface A, special member of the family $\{A_t\}$, which is a (rational) G-cover of X.

Let us now consider the non Abelian case. We remark that in this case $\rho(X)$ is necessarily equal to $1+\rho(K_G)$, because the latter is 20.

Let $G=\mathbb{D}_8'$ and $K_{\mathbb{D}_8'}$ as described in Section 4.2 and let X be a K3 surface such that $K_{\mathbb{D}_8'}$ is primitively embedded in $NS(X)$. The classes v_1' and v_2' given in (4.1) allows one to construct a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of X. Let us denote by Y the minimal model of the $(\mathbb{Z}/2\mathbb{Z})^2$-cover of X branched with multiplicity 2 along each the twelve curves in the set

$$\mathcal{S}:=\Big\{K^{(j)}_{(0,0)},K^{(j)}_{(\frac{1+i}{2},0)},K^{(j)}_{(\frac{i}{2},\frac{i}{2})},K^{(j)}_{(\frac{1}{2},\frac{i}{2})}\Big\},\quad j=1,2,3.$$

Let $\pi_Y\colon Y\dashrightarrow X$ be the rational map induced by the $(\mathbb{Z}/2\mathbb{Z})^2$-cover and let $\langle\mu,\nu\rangle$ the cover group. We observe that $\pi_Y^{(-1)}(K^{(0)}_{(0,0)})$ consists of a unique irreducible rational curve and coincides with the inverse image of the D_4-configuration $K^{(j)}_{(0,0)}$, $j=0,1,2,3$. We denote this curve on Y by $K_{(0,0)}$. Similarly the inverse images of the D_4-configuration $K^{(j)}_{(\frac{1+i}{2},0)}$, (resp. $K^{(j)}_{(\frac{i}{2},\frac{i}{2})}$, $K^{(j)}_{(\frac{1}{2},\frac{i}{2})}$) for $j=0,1,2,3$, consists of a unique irreducible rational curve denoted by $K_{(\frac{1+i}{2},0)}$, (resp. $K_{(\frac{i}{2},\frac{i}{2})}$, $K_{(\frac{1}{2},\frac{i}{2})}$). Since the curve

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$K_{(\frac{1}{2},0)}$ (resp. $K_{(\frac{1+i}{4},\frac{1+i}{4})}$, $K_{(\frac{1+i}{4},\frac{i-1}{4})}$) is not in the branch locus of the $(\mathbb{Z}/2\mathbb{Z})^2$-cover of X and does not meet the branch locus, its inverse image on Y consists of four disjoint rational curves, denoted by $K^{(j)}_{(\frac{1}{2},0)}$ (resp. $K^{(j)}_{(\frac{1+i}{4},\frac{1+i}{4})}$, $K^{(j)}_{(\frac{1+i}{4},\frac{i-1}{4})}$) for $j=1,2,3,4$. Thus, there are sixteen disjoint rational curves on Y . Hence, by Proposition 2.9, Y is a Kummer surface of an Abelian surface B, and there exists the rational map $\pi_B\colon B \dashrightarrow Y$, whose cover involution will be denoted by ι_B. Hence, there is a 8:1 map, $\pi_Y\circ\pi_B\colon B \dashrightarrow X$. By construction, the automorphisms μ and ν of Y preserve the branch locus of the map $\pi_B\colon B \dashrightarrow Y$, and thus they induce two automorphisms μ_B and ν_B on B. Let us denote by H_B the group generated by ι_B, μ_B and ν_B. By construction, $B \dashrightarrow X$ is the map induced by the desingularization of the quotient B/H_B. In particular, the group H_B has order 8.

Let $\gamma\colon X \to S$ be the contraction of all the curves in $F_{\mathbb{D}'_8}$. The singular surface S has four singularities of type D_4 and 3 singularities of type A_1. It is immediate to check by our construction that $B \dashrightarrow X\xrightarrow{\gamma}S$ coincides with the quotient $B \to B/H_B$ and so $S = B/H_B$. The quotient singularities of type D_4 correspond to points whose stabilizer is the quaternion group, so the quaternion group $\mathbb{D}'_8$ has to be a subgroup of the group H_B, but the order of H_B is 8, as the order of the quaternion group, so H_B is the quaternion group. This implies that X is the desingularization of the quotient $B/\mathbb{D}'_8$.

The case $G = \mathbb{D}_8$ is analogous. In the case where $G = \mathbb{D}_{12}$ one first considers a 4:1 cover of the K3 surface X_G. The minimal model of such a cover, say Y, contains nine disjoint A_2-configurations, hence there exists an Abelian surface B that is a 3:1 cover of Y. Then one proves that X is the desingularization of B/H_B, where H_B is a group generated by certain automorphisms and, considering the singularities, one proves that H_B must be $\mathbb{D}_{12}$ (since it has order 12 and has to contain $\mathbb{D}_{12}$).

Let us now consider the case $G = \mathbb{T}$. Let X be a K3 surface such that $K_{\mathbb{T}}$ is primitively embedded in $NS(X)$. So there are nineteen curves that span the lattice $E_6\oplus D_4\oplus A_1\oplus A_2^4$ and there is a 3-divisible class that involves six disjoint A_2-configurations. So there exists a 3:1 cover of X whose minimal model is a K3 surface Y. We denote the 3:1 rational map by $\pi\colon Y \dashrightarrow X$. The inverse image on Y of the curves in $F_{\mathbb{T}}$ consists of nineteen rational curves that span the lattice $D_4^4\oplus A_1^3$ (where a copy of D_4 is mapped by π to the E_6 contained in $F_{\mathbb{T}}$, three other copies of D_4 are mapped to the unique copy of D_4 in $F_{\mathbb{T}}$, the three copies of A_1 are mapped by π to the unique copy of A_1 in $F_{\mathbb{T}}$). We observe that $D_4^4\oplus A_1^3 \simeq F_{\mathbb{D}'_8}$. In order to reconstruct the Abelian surface that is the cover of X, it suffices to prove that not only $F_{\mathbb{D}'_8}$, but exactly $K_{\mathbb{D}'_8}$ is primitively embedded in $NS(Y)$. Once one proves this, one finds an Abelian surface B such that Y is the minimal resolution of $B/\mathbb{D}'_8$ (we already proved this result), and one deduces that X is the minimal resolution of $B/\mathbb{T}$ as in the previous cases. In Section 4.2 we constructed the lattice $K_{\mathbb{D}'_8}$ introducing four divisible vectors. Two of them (v'_1 and v'_2) are strictly related to the geometry of the quotient that we are considering. The property of these two vectors, which is essential in order to reconstruct the Abelian surface B with a $\mathbb{D}'_8$-action, is that the curves appearing in these two divisible classes are all contained in the D_4-configurations; *i.e.*, the curves that generate the three copies of A_1 in $F_{\mathbb{D}'_8}$ do not appear in these divisible classes. Since there are fifteen disjoint rational curves contained in the set of the nineteen curves that span

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$D_4^4 \oplus A_1^3$, we know that there are also four independent divisible classes in $NS(Y)$ by Proposition 2.9. Now we have to show that at least two of them can be chosen to have no components in the direct summands A_1^3 of $D_4^4 \oplus A_1^3$. Suppose the opposite; this means that there is a choice of three divisible vectors n_1, n_2, and n_3 such that all the elements in $\langle n_1, n_2, n_3\rangle$ have components among the generators of A_1^3. Just to fix the notation we gives to the curves in $F_{\mathbb{D}_8'}$ the same names as in Section 4.2. We choose the first class n_1 in such a way that it has some components in A_1^3. We recall that the divisible classes are the sum of eight disjoint rational curves divided by 2 and that they are linear combinations of the elements of the discriminant group. We observe that a divisible class has components among the generators of A_1^3 if and only if at least one of the vectors (of the discriminant group) d_9, d_{10}, and d_{11} appears with a nontrivial coefficient in its expression. Since the generators of the discriminant group d_i with $i \neq 9, 10, 11$ are the sum of two rational curves divided by 2 an even number of vectors d_9, d_{10} and d_{11} with a nontrivial coefficient appears in the expression of n_1. So we can assume that $n_1 = d_9 + d_{10} + m_1$, where $m_1 \in \langle d_j\rangle$, $j = 1, \ldots, 8$. Now we construct a second divisible class n_2, assuming that it has some components among the generators of A_1^3. If $n_2 = d_9 + d_{10} + m_2$, where $m_2 \in \langle d_j\rangle$, $j = 1, \ldots, 8$, then $n_1 + n_2 \in \langle d_j\rangle$, $j = 1, \ldots, 8$, *i.e.*, it has no components among the generators of A_1^3. So we can assume that $n_2 := d_9 + d_{11} + m_2$, $m_2 \in \langle d_j\rangle$, $j = 1, \ldots, 8$. We observe that $n_1 + n_2 = d_{10} + d_{11} + m_3$, $m_3 \in \langle d_j\rangle$, $j = 1, \ldots, 8$. But now there is no way to choose n_3 is such a way that all the elements in $\langle n_1, n_2, n_3\rangle$ have components among the curves generating A_1^3. Indeed, every pair of elements in $\{d_9, d_{10}, d_{11}\}$ appears with a nontrivial coefficient in n_1 or in n_2 or in $n_1 + n_2$. This proves that if there is configuration of nineteen rational curves on a K3 surface Y that span the lattice $F_{\mathbb{D}_8'}$, then the lattice $K_{\mathbb{D}_8'}$ is primitively embedded in $NS(Y)$, and so Y is the minimal resolution of the quotient of an Abelian surface B by the group $\mathbb{D}_8'$. This concludes the proof in the unique remaining case $G = \mathbb{T}$. ∎

Remark 4.5 In [Be] the proof of the previous result is given in case G is a cyclic group of order greater than 2. The proof given in the case where $\rho(X_G)$ is the minimal possible coincides with our proof. In case $\rho(X_G)$ is greater (and indeed 20), in [Be] it is observed that one can use a deformation argument as we did, but an alternative proof is given. Unfortunately, it is based on [Be, Lemma 3.2], which contains a mistake. Indeed, using the notation of [Be, Lemma 3.2], it is true that there exists an orthogonal embedding η of $\{A_{k_1}, \ldots, A_{k_n}\}$ in a system of roots, Q, of type $\mathbb{A}$ such that (up to the action of the Weyl group), $\eta(A_{k_i})$ is contained in a chosen basis of Q for every $i = 1, \ldots, n$, but the same result is not necessarily true if the system of roots Q is of type $\mathbb{D}$. A simple counterexample is given by the orthogonal embedding of $\{A_1, A_1, A_1, A_1\}$ in D_4 given by $\{\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_3 + \epsilon_4, \epsilon_3 - \epsilon_4\}$ that cannot be contained in a basis of D_4 (up to the action of the Weyl group of D_4).

The advantage of the result in Theorem 4.4 is that one relates a purely geometric property with a purely lattice theoretic property. This is what is needed in order to describe the lattice polarized moduli space of the K3 surfaces with a certain geometric property, so we immediately obtain the following corollary.

Corollary 4.6 *Let $\mathcal{L}_G$ be the set of lattices L_G satisfying the following:*

(i) *L_G has rank* $1 + \mathrm{rank}(K_G)$*;*
(ii) *L_G is hyperbolic;*
(iii) *L_G admits a primitive embedding in Λ_{K3};*
(iv) *there exists a primitive embedding of K_G in L_G.*

A K3 surface is the desingularization of the quotient of an Abelian surface by G if and only if it is an L_G-polarized K3 surface for an $L_G \in \mathcal{L}_G$.

In particular, the coarse moduli space of the K3 surfaces that are desingularization of the quotient A/G for an Abelian surface A has infinitely many components of dimension $19 - \mathrm{rank}(K_G)$.

Proof As a direct consequence of Theorem 4.4, one has that a K3 surface is (rationally) G-covered by an Abelian surface if and only if it is L_G-polarized. It remains only to show that the components of the moduli space of the K3 surfaces that are (rationally) G-covered by an Abelian surface A are infinitely many. This depends on the possible choices for the lattice L_G. Generically L_G is the Nèron–Severi group of an L_G-polarized K3 surface and its orthogonal group in Λ_{K3} is the transcendental group.

Let us consider first the cyclic groups G, in particular, the case where $G \neq \mathbb{Z}/2\mathbb{Z}$ (since the case $\mathbb{Z}/2\mathbb{Z}$ is very well known). The transcendental lattice of a generic K3 surface X that is (rationally) G covered by an Abelian surface A is determined by that of A. With the notation introduced in [MTW, Section 5.2], an Abelian surface A admits a symplectic action of G if and only if the transcendental lattice T_A is primitively embedded in $T(G_\sigma) \simeq H^2(A,\mathbb{Z})^G$. If $G \neq \mathbb{Z}/2\mathbb{Z}$, then $T(G_\sigma)$ has signature $(3,1)$. In order to identify a possible choice for T_A, it suffices to define T_A as the orthogonal complement to a vector, say v_a with a positive self intersection in $T(G_\sigma)$. So it suffices to show that there are infinitely many choices for v, which determine infinitely many lattices T_A. By [MTW, Section 4] for every cyclic group G, the lattice $T(G_\sigma)$ splits in the direct sum of a copy of U and another lattice, say R_G. Let us assume that v_a has nontrivial components only in U and that these components are of the form $(1,a)$. Then $v_a^2 = 2a$ and T_A is isometric to

$$\langle -2a\rangle \oplus R_G \simeq v_a^{\perp_{T(G_\sigma)}}.$$

Since we have infinitely many possible choices for a, we have infinitely many choices for T_A and thus for T_X.

If G is not cyclic, we did not explicitly compute the lattice $H^2(A,\mathbb{Z})^G$ where A is the Abelian surface that (rationally) G-covers X. Hence, in this case we compute the possible lattice L_G directly. By definition, L_G is an overlattice of a finite index of $\mathbb{Z}v_b \oplus K_G$, where v_b is a vector with a positive self intersection $2b$ in $K_G^{\perp_{\Lambda_{K3}}}$. Since K_G is negative definite of rank 19, $K_G^{\perp_{\Lambda_{K3}}}$ is positive definite. Hence, the number of vectors in $K_G^{\perp_{\Lambda_{K3}}}$ with a given length is finite. Since the number of vectors in $K_G^{\perp_{\Lambda_{K3}}}$ is clearly infinite, there are infinitely many choices for the length of v_b and thus for the overlattice of finite index of $\mathbb{Z}v_b \oplus K_G$. ∎

We observe that the conditions (i), (ii), and (iv) in Corollary 4.6 imply that L_G is an overlattices of finite index l_G of $\mathbb{Z}h \oplus K_G$, where h is a vector with a positive self intersection h^2. Condition (iii) implies that h^2 is even and imposes several restriction to l_G. The concrete possibilities for the lattices in $\mathcal{L}_G$ are classically known for $G =$

$\mathbb{Z}/2\mathbb{Z}$ (see, for example, [GS2, Theorem 2.7] for a recent reference) and for $G = \mathbb{Z}/3\mathbb{Z}$ (see [Ba2]).

In [N1], it is proved that it is not necessary to check the existence of a primitive embedding of $K_{\mathbb{Z}/2\mathbb{Z}}$ in the Néron–Severi group of a K3 surface to conclude that it is a Kummer surface. It suffices to know that it contains sixteen disjoint smooth irreducible rational curves. We underline that from the point of view of the description of the moduli space this result is not very useful, because we have no a way to translate the condition "certain -2 classes correspond to irreducible curves" in the context of the lattice polarized K3 surfaces. On the other hand this result is very nice from a geometric point of view, since it can also be stated in the following way. If a K3 surface admits a model with sixteen nodes, then it is a Kummer surface (for example, this can be used to conclude that a quartic with sixteen nodes is a Kummer surface). A similar result was generalized to the group $G = \mathbb{Z}/3\mathbb{Z}$ by Barth [Ba1]. Here we generalize this result to all the other admissible groups.

Theorem 4.7 *Let G be one of the groups $\mathbb{Z}/n\mathbb{Z}$, $n = 2,3,4,6$, $\mathbb{D}_8$, $\mathbb{D}'_8$, $\mathbb{D}_{12}$, and $\mathbb{T}$, and let F_G be the lattice defined above. Then a K3 surface is the minimal model of A/G for some Abelian surface A if and only if F_G is embedded in $NS(X_G)$ and there exists a basis of F_G that represents irreducible smooth curves on X_G.*

Proof This result is known if $G = \mathbb{Z}/2\mathbb{Z}$ (see [N1]) and if $G = \mathbb{Z}/3\mathbb{Z}$ (see [Ba1]). In the proof of the Theorem 4.4 we proved the statement in the case where $G = \mathbb{D}'_8$. Here we give a complete proof in the case $G = \mathbb{Z}/4\mathbb{Z}$. The other cases are very similar. The lattice $F_{\mathbb{Z}/4\mathbb{Z}}$ has rank 18 and length 10. Since the length of a lattice of rank 18 primitively embedded in Λ_{K3} is at most 4, we know that $F_{\mathbb{Z}/4\mathbb{Z}}$ is not primitively embedded in Λ_{K3} and so there is an overlattice of finite index of $F_{\mathbb{Z}/4\mathbb{Z}}$, called $R_{\mathbb{Z}/4\mathbb{Z}}$, which is primitively embedded in Λ_{K3}. In order to construct an overlattice $R_{\mathbb{Z}/4\mathbb{Z}}$ of $F_{\mathbb{Z}/4\mathbb{Z}}$ we have to add to $F_{\mathbb{Z}/4\mathbb{Z}}$ certain elements that are nontrivial in the discriminant group of $F_{\mathbb{Z}/4\mathbb{Z}}$ and that have an even self intersection. Moreover, we have to recall that if the sum of m disjoint rational curves is divided by 2, then m is either 16 or 8.

Let us consider the lattice $F_{\mathbb{Z}/4\mathbb{Z}} = A_3^4 \oplus A_1^6$. We denote the basis of the j-th copy of A_3 by $a_i^{(j)}$, $i = 1,2,3$, $j = 1,2,3,4$ and the generator of the $(j-4)$-th copy of A_1 by $a^{(j)}$, $j = 5,6,7,8,9,10$. The discriminant of $F_{\mathbb{Z}/4\mathbb{Z}}$ is generated by

$$d_j := \frac{1}{4}\big(a_1^{(j)} + 2a_2^{(j)} + 3a_3^{(j)} \big), \quad j = 1,2,3,4, \quad d_j := \frac{a^{(j)}}{2}, \quad j = 5,6,7,8,9,10.$$

Since $l(F_{\mathbb{Z}/4\mathbb{Z}}) - l(R_{\mathbb{Z}/4\mathbb{Z}})$ has to be at least six, we have to add at least three divisible vectors to $F_{\mathbb{Z}/4\mathbb{Z}}$ in order to obtain $R_{\mathbb{Z}/4\mathbb{Z}}$. First, suppose we add three vectors, v_1, v_2, v_3 such that $\langle v_1, v_2, v_3 \rangle = (\mathbb{Z}/2\mathbb{Z})^3$ in the discriminant group (*i.e.*, no vectors among v_1, v_2 ,v_3 has order 4 in the discriminant group of $F_{\mathbb{Z}/4\mathbb{Z}}$). Every vector that generates $\mathbb{Z}/2\mathbb{Z}$ in the discriminant group of $F_{\mathbb{Z}/4\mathbb{Z}}$ is a linear combination of $2d_j$ for $j = 1,2,3,4$ and d_k for $k = 5,\dots,10$. The curves that appear with a nontrivial coefficient in each of these linear combinations are among the fourteen disjoint rational curves $\{a_1^{(j)}, a_3^{(j)}, a^{(k)}\}$ for $j = 1,2,3,4$ and $k = 5,\dots,10$. We recall that it is possible to add three independent divisible 2-classes starting from fourteen disjoint rational curves, but it is not possible to add four independent divisible classes

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using only fourteen rational curves. So we can add exactly the three vectors v_1, v_2, and v_3. Up to permutations of the indices the unique possibility for the three vectors v_1, v_2, and v_3 is $v_1 := 2(d_1 + d_2 + d_3 + d_4)$, $v_2 := 2d_1 + 2d_2 + d_5 + d_6 + d_7 + d_8$, $v_3 := 2d_1 + 2d_3 + d_7 + d_8 + d_9 + d_{10}$. The lattice $R_{\mathbb{Z}/4\mathbb{Z}}$ obtained adding the vectors v_1, v_2 and v_3 to $F_{\mathbb{Z}/4\mathbb{Z}}$ is an overlattice of index 2^3. One can directly compute its discriminant group, and one finds that the discriminant group of this lattice is $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^4$. But the length of this lattice is 6, which is not admissible.

We conclude that there is at least one vector, say v_1, in $F_{\mathbb{Z}/4\mathbb{Z}}/R_{\mathbb{Z}/4\mathbb{Z}}$ that generates a copy of $\mathbb{Z}/4\mathbb{Z}$ in the discriminant group of $F_{\mathbb{Z}/4\mathbb{Z}}$. We recall that $(v_1)^2$ has to be an even number, that $(d_j)^2 = -3/4$ if $j = 1, 2, 3, 4$, and that $(d_k)^2 = -1/2$ if $k = 5, \ldots, 10$. Moreover, $2v_1 \mod F_{\mathbb{Z}/4\mathbb{Z}}$ has to be the sum of eight disjoint rational curves divided by 2 (since the sum of n rational curves cannot be divided by 2 if $n \leq 14$ and $n \neq 8$). So there are only the following two possibilities modulo $F_{\mathbb{Z}/4\mathbb{Z}}$ (up to a permutation of the indices)

(a) $v_1 := d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 + d_{10}$ or
(b) $v_1 := d_1 + d_2 + d_3 + d_4 + d_5 + d_6$.

In case (a) one can construct a 4:1 cover of X whose branch divisor is v_1. So we have a map $Y \to X$ that is 4:1. By construction, the minimal model of Y has a trivial canonical bundle and its Euler characteristic is 0, so this surface is an Abelian surface, and we conclude the proof. We remark that it suffices to observe that the divisor v_1 in case (a) is the one described by Bertin in [Be, p. 270] where it was proved that the minimal model of a 4:1 cover of a K3 surface whose branch locus has a certain property has to be an Abelian surface.

In case (b) the 4:1 cover associated with the vector v_1 produces a K3 surface, and not an Abelian surface. Thus we have to analyze not only the vector v_1, but also the vectors v_2 and v_3 in order to show that $R_{\mathbb{Z}/4\mathbb{Z}}$ coincides with $K_{\mathbb{Z}/4\mathbb{Z}}$. We now consider the vectors v_2 and v_3. Up to replacing, possibly, v_2 (resp. v_3) with $2v_2$ (resp. $2v_3$), we have that v_2 (resp. v_3) generates a copy of $\mathbb{Z}/2\mathbb{Z}$ and consists of the sum of eight disjoint rational curves divided by 2; four of these curves have to be chosen among $\{a_1^{(j)}, a_3^{(j)}\}$, $j = 1, 2, 3, 4$, since these are the eight disjoint rational curves of the divisible vector $2v_1$. Up to a permutation of the indices, we can assume that $v_2 = 2d_1 + 2d_2 + d_5 + d_6 + d_7 + d_8$ and $v_3 = 2d_1 + 2d_3 + d_7 + d_8 + d_9 + d_{10}$. Now we consider the vector $v_1 + v_3$ (which is surely contained in $R_{\mathbb{Z}/4\mathbb{Z}}$). It is $3d_1 + d_2 + 3d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 + d_{10}$. Modulo $F_{\mathbb{Z}/4\mathbb{Z}}$ and a change of the indices of the generators of A_3, this coincides with the vector v_1 in case (a). So the minimal model of 4:1 cover of X whose branch divisor is $v_1 + v_3$ is an Abelian surface and we conclude the proof as before.

The other cases are similar (but easier). One checks that the length of F_G is greater than $22 - \operatorname{rank}(F_G)$; one deduces that one has to add some divisible classes in order to construct the lattice R_G, which is the minimal primitive sublattice of Λ_{K3} containing F_G. One identifies these classes (recalling the condition that they are linear combinations of elements of the discriminant group of F_G and the conditions imposed by Proposition 2.9). Then one compares the lattice R_G with K_G or one explicitly constructs a certain cover of X in order to show either that $R_G = K_G$ (which implies that X is the desingularization of A/G by Theorem 4.4) or directly that there exists an Abelian surface A such that X is the resolution of A/G. ∎

Corollary 4.8 *Let G be one of the groups $\mathbb{Z}/n\mathbb{Z}$, $n = 2,3,4,6$, $\mathbb{D}_8$, $\mathbb{D}_8'$, $\mathbb{D}_{12}$, and $\mathbb{T}$, and let F_G be the lattice defined above. Let H_G be the minimal primitive sublattice of Λ_{K3} that contains F_G and such that the root lattice of F_G coincides with the one of H_G. Then $H_G \simeq K_G$.*

Proof By hypothesis, H_G is a negative definite lattice primitively embedded in Λ_{K3} and $\operatorname{rank}(H_G) = \operatorname{rank}(F_G)$. Let D be a vector in Λ_{K3} that is orthogonal to H_G and has a positive self intersection. By the Torelli theorem, there exists a K3 surface, X, whose transcendental lattice is the orthogonal complement to $\mathbb{Z}D \oplus H_G$ in Λ_{K3}. The Néron–Severi group of X is an overlattice of finite index of $\mathbb{Z}D \oplus H_G$ such that H_G is primitively embedded in it, and, without loss of generality, we can assume that D is pseudoample. Under our assumptions on H_G, we can apply Proposition 3.2 to $h = D$, $L = H_G$ and $R = F_G$. So the lattice F_G is spanned by irreducible rational curves on X. By Theorem 4.7, it follows that X is the desingularization of the quotient A/G for a certain Abelian surface A. In this case the minimal primitive sublattice that contains the curves of the lattice F_G is K_G, but by the hypothesis the minimal primitive sublattice of $NS(X) \subset \Lambda_{K3}$ which contains F_G is H_G, so K_G coincides with H_G. ∎

Remark 4.9 The hypothesis that the roots of H_G coincide with the ones of F_G in Corollary 4.8 is essential. Indeed, let us consider the case $G = \mathbb{Z}/2\mathbb{Z}$. The lattice F_G is A_1^{16}; let us denote by K_i, $i = 1, \dots, 16$ the generators of this lattice. Let us consider the vectors

$$
\begin{aligned}
v_j &:= \Big(\sum_{i=1}^{4} K_{4j+i}\Big)/2, \qquad j = 0,1,2,3,\\
w_1 &:= (K_1 + K_2 + K_5 + K_6 + K_9 + K_{10} + K_{13} + K_{14})/2,\\
w_2 &:= (K_1 + K_3 + K_5 + K_7 + K_9 + K_{11} + K_{13} + K_{15}).
\end{aligned}
$$

Let us denote by $H_{\mathbb{Z}/2\mathbb{Z}}$ the lattice obtained adding the vectors v_i, $i = 0,1,2,3$ and w_h, $h = 1,2$ to F_G. It is an overlattice (of index 2^6) of F_G that admits a primitive embedding in Λ_{K3}, but it is not isometric to the Kummer lattice (which, in fact, is an overlattice of index 2^5 of F_G). In this case, v_1 is a root of H_G that is not contained in F_G.

4.5 K3 Surfaces (Rationally) $\mathbb{Z}/3\mathbb{Z}$-covered by Abelian Surfaces

In [GS2] it was observed that every Kummer surface $\operatorname{Km}(A)$ (*i.e.*, every K3 surface that is the desingularization of $A/(\mathbb{Z}/2\mathbb{Z})$) admits the group $(\mathbb{Z}/2\mathbb{Z})^4$ as group of symplectic automorphisms. Moreover, $\operatorname{Km}(A)$ is also the quotient of a K3 surface by the symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$. This result is based on the observation that if a K3 surface is a Kummer surface $\operatorname{Km}(A)$, then the translations by the two torsion points of A induce symplectic automorphisms on $\operatorname{Km}(A)$.

A similar result can be obtained if the K3 surface X_G is the (desingularization of) quotient of an Abelian surface by an action of the group $\mathbb{Z}/3\mathbb{Z}$.

Proposition 4.10 *Let X be the desingularization of the quotient of an Abelian surface A by the group $\mathbb{Z}/3\mathbb{Z}$. Then X admits a symplectic action of the group $(\mathbb{Z}/3\mathbb{Z})^2$. Moreover, there exists a K3 surface Y that admits a symplectic action of $(\mathbb{Z}/3\mathbb{Z})^2$ such that X is the desingularization of $Y/(\mathbb{Z}/3\mathbb{Z})^2$.*

Proof Let A be an Abelian surface admitting an automorphism α_A of order 3 such that X is the desingularization of A/α_A. Let $A[3]$ be the group of three torsion points of A and let $\langle P, Q\rangle \subset A[3]$ be the set of points fixed by α_A. Let us denote by t_P and t_Q the translation on A by the points P and Q, respectively. Then $(\mathbb{Z}/3\mathbb{Z})^2 \simeq \langle t_P, t_Q\rangle \subset \operatorname{Aut}(A)$, and the automorphisms t_P and t_Q commute with α_A. So t_P and t_Q induce two automorphisms of order 3 on A/α_A that lift to two automorphisms, τ_P and τ_Q, on X. The period of X (*i.e.*, the generator of $H^{2,0}(X)$) is induced by the generator of $H^{2,0}(A)$, which is preserved by the translations. So τ_P and τ_Q are symplectic automorphisms of X. This gives a symplectic action of $(\mathbb{Z}/3\mathbb{Z})^2$ on X.

On the other hand, X contains nine disjoint A_2-configurations of rational curves (which generate the lattice $F_{\mathbb{Z}/3\mathbb{Z}}$) and the minimal primitive sublattice $K_{\mathbb{Z}/3\mathbb{Z}}$ that contains all these curves also contains several divisible classes. In particular, let us denote by $a_i^{(j)}$, $i = 1, 2$, $j = 1, \dots, 9$ basis of the j-th copy of A_2. Up to a choice of the indices, $K_{\mathbb{Z}/3\mathbb{Z}}$ also contains the classes (mod $F_{\mathbb{Z}/3\mathbb{Z}}$)

$$v_1 := \frac{1}{3}\Big(\sum_{i=1}^{6} a_1^{(j)} - a_2^{(j)}\Big),\ v_2 = \frac{1}{3}\Big(\sum_{j=1}^{2}(a_1^{(j)} - a_2^{(j)}) - \sum_{h=3}^{4}(a_1^{(h)} - a_2^{(h)}) + \sum_{k=7}^{8}(a_1^{(k)} - a_2^{(k)})\Big),$$

as shown in [Be, p. 269] with a slightly different notation. But the presence of these divisible classes allows one to reconstruct a $(\mathbb{Z}/3\mathbb{Z})^2$ cover of X (one first constructs the 3:1 cover associated with the class v_1 as in Section 2.1, and then one considers the pull back of the class v_2, which allows one to construct another 3:1 cover). Using this process, one obtains a non minimal surface, whose minimal model Y is a K3 surface that is a (rational) $(\mathbb{Z}/3\mathbb{Z})^2$-cover of X, hence X is the desingularization of the quotient of the K3 surface Y by the group $(\mathbb{Z}/3\mathbb{Z})^2$. ■

Corollary 4.11 *The 1-dimensional families of K3 surfaces that are desingularizations of the quotients $A/\mathbb{Z}/3\mathbb{Z}$ for certain Abelian surfaces A are contained in the intersection between the 3-dimensional families of the K3 surfaces that are (desingularization of) quotients of K3 surfaces by a symplectic action of $(\mathbb{Z}/3\mathbb{Z})^2$ and the 3-dimensional families of K3 surfaces that admit a symplectic action of $(\mathbb{Z}/3\mathbb{Z})^2$.*

Remark 4.12 The existence of the surface Y in Proposition 4.10 directly follows by the primitive embedding of lattice $M_{(\mathbb{Z}/3\mathbb{Z})^2}$ in the lattice $K_{\mathbb{Z}/3\mathbb{Z}}$, after proving Theorem 5.2. Similarly one obtains that if X is the minimal model of the quotient $A/(\mathbb{Z}/4\mathbb{Z})$ for a certain Abelian surface, then it is also the minimal model of the quotient $Y/(\mathbb{Z}/4\mathbb{Z})$ for a certain K3 surface Y, since $M_{\mathbb{Z}/4\mathbb{Z}} \subset K_{\mathbb{Z}/4\mathbb{Z}}$.

In Proposition 4.10 we proved that a K3 surface X that is (rationally) $\mathbb{Z}/3\mathbb{Z}$-covered by an Abelian surface, necessarily admits certain symplectic automorphisms induced by translation on the Abelian surface. Here we observe that there exists another automorphism on A that induces a symplectic automorphism on X.

Proposition 4.13 *Let X be a K3 surface such that $K_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $NS(X)$; then X admits a symplectic involution ι_X such that $K_{\mathbb{Z}/6\mathbb{Z}}$ is primitively embedded in $NS(W)$ where W is the K3 surface minimal model of X/ι_X.*

Proof Every Abelian surface admits an involution $\iota_A: A \to A$ that sends every point to its inverse with respect to the group law of A. Under the hypothesis on X there exists an Abelian surface A with an automorphism $\alpha_A \in \mathrm{Aut}(A)$ of order 3 such that X is the desingularization of A/α_A. The automorphisms ι_A and α_A commute and generate an automorphism $\alpha_A \circ \iota_A$ of order 6 that preserves the non-vanishing holomorphic 2-form of A. The involution ι_A induces an involution ι_X on X. The singular surface $A/(\alpha_A \circ \iota)$ is birational to X/ι_X. Since the minimal model of $A/(\alpha_A \circ \iota)$ is a K3 surface, the minimal model of X/ι_X is also a K3 surface, and these surfaces are isomorphic. We call this surface W, and we observe that it is constructed as minimal model of the quotient of an Abelian surface by the action of $\mathbb{Z}/6\mathbb{Z} = \langle \alpha_A \circ \iota \rangle$, so $K_{\mathbb{Z}/6\mathbb{Z}}$ is primitively embedded in $NS(W)$. ∎

A generalization of the previous result can be obtained by replacing $(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ with $(\mathbb{D}'_8, \mathbb{T})$.

Corollary 4.14 *Let S be a K3 surface such that $K_{\mathbb{D}'_8}$ is primitively embedded in $NS(S)$; then S admits an automorphism of order 3, γ_S, such that $K_{\mathbb{T}}$ is primitively embedded in $NS(\widetilde{S/\gamma_S})$, where $\widetilde{S}/\gamma_S$ is the minimal resolution of S/γ_S.*

Putting together the Propositions 4.10 and 4.13, one obtains the following corollary.

Corollary 4.15 *Let X be a K3 surface that is (rationally) $(\mathbb{Z}/3\mathbb{Z})$-covered by an Abelian surface. The group $\mathfrak{A}_{3,3}$ acts symplectically on X.*

Proof It suffices to prove that the involution ι_A and the translations t_P and t_Q introduced in proofs of Propositions 4.13 and 4.10 generate $\mathfrak{A}_{3,3}$. This can be easily checked; for example, one can specialize the Abelian surface A to the product of two elliptic curves with j-invariant equal to 0. The order 3 automorphism α_A (defined in proof of Proposition 4.10) fixes the points $(0,0)$, $P := (\frac{1}{3}(1-\zeta_3), 0)$, and $Q := (0, \frac{1}{3}(1-\zeta_3))$. This identifies the translation t_P and t_Q, and it is immediate to verify that $\langle t_P, \iota \rangle \simeq \langle t_Q, \iota \rangle$ is the dihedral group of order 6 and then $\langle t_P, t_Q, \iota \rangle$ is $\mathfrak{A}_{(3,3)}$. ∎

5 K3 Surface Quotients of K3 Surfaces

The aim of this section is to extend some of the results proved for the K3 surfaces that are (rationally) covered by an Abelian surface, to the K3 surfaces that are (rationally) covered by a K3 surface. We will denote by Y_G a K3 surface that admits a symplectic action of the group G and by S_G the minimal resolution of the quotient Y_G/G. It is well known that S_G is a K3 surface (see [N3]).

Proposition 5.1 *Let Y_G be a K3 surface and let $G \in \mathrm{Aut}(Y_G)$ be a finite group. Let S_G be the minimal model of Y_G/G. Then S_G is a K3 surface if and only if G acts symplectically on Y_G. If G is Abelian, then it is one of the following fourteen groups $\mathbb{Z}/n\mathbb{Z}$, $n = 2, \dots, 8$, $(\mathbb{Z}/m\mathbb{Z})^2$, $m = 2, 3, 4$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$, $t = 4, 6$, $(\mathbb{Z}/2\mathbb{Z})^j$, $j = 3, 4$.*

Let M_i be the curves on S_G arising from the resolution of the singularities of Y_G/G. Then the lattices E_G spanned by the curves M_i is one of the following root lattices:

G	$\mathbb{Z}/2\mathbb{Z}$	$/Z/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$
E_G	A_1^8	A_2^6	$A_3^4 \oplus A_1^2$	A_4^4	$A_5^2 \oplus A_2^2 \oplus A_1^2$	A_6^3	$A_7^2 \oplus A_3 \oplus A_1$

G	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^4$	$\mathbb{Z}/2 \times \mathbb{Z}/4$	$\mathbb{Z}/2 \times \mathbb{Z}/6$	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/4\mathbb{Z})^2$
E_G	A_1^{12}	A_1^{14}	A_1^{15}	$A_3^4 \oplus A_1^4$	$A_5^3 \oplus A_1^3$	A_2^8	A_3^6

Let M_G be the minimal primitive sublattice of $NS(S_G)$ that contains the curves M_i; then M_G is an overlattice of finite index r_G of E_G and its properties are as follows:

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$
r_G	2	3	4	5	6	7	8
$\mathrm{rank}(M_G)$	8	12	14	16	16	18	18
$M_G^\vee/M_G$	$(\mathbb{Z}/2\mathbb{Z})^6$	$(\mathbb{Z}/3\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})^2$	$(\mathbb{Z}/5\mathbb{Z})^2$	$(\mathbb{Z}/6\mathbb{Z})^2$	$(\mathbb{Z}/7\mathbb{Z})$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

G	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^4$	$\mathbb{Z}/2 \times \mathbb{Z}/4$	$\mathbb{Z}/2 \times \mathbb{Z}/6$	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/4\mathbb{Z})^2$
r_G	2^2	2^3	2^4	8	12	3^2	4^2
$\mathrm{rank}(M_G)$	12	14	15	16	18	16	18
$M_G^\vee/M_G$	$(\mathbb{Z}/2\mathbb{Z})^8$	$(\mathbb{Z}/2\mathbb{Z})^8$	$(\mathbb{Z}/2\mathbb{Z})^7$	$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^4$	$(\mathbb{Z}/4\mathbb{Z})^2$

The roots of the lattice M_G coincide with the roots of the lattice E_G for all the abelian groups G.

By construction, M_G is a negative definite lattice primitively embedded in $NS(S_G)$, and thus $\rho(S_G) \geq 1 + \mathrm{rank}(M_G)$.

Proof The classification of the Abelian groups acting symplectically on a K3 surface is given in [N3, Theorem 4.5], where it was also proved that S_G is a K3 surface if and only if G acts symplectically on Y_G. The lattices E_G and M_G are described in [N3, § 6 and 7]. The fact that the root lattices of M_G and of E_G coincide can be checked by a Magma computation as in proof of Proposition 4.3. ■

We obtain an analogue of Theorem 4.4.

Theorem 5.2 *Let G be one of the Abelian groups acting symplectically on a K3 surface. A K3 surface S_G is the desingularization of the quotient Y_G/G for a certain K3 surface Y_G if and only if M_G is primitively embedded in $NS(S_G)$.*

Proof The proof is similar to (but easier than) the one of Theorem 4.4. Since the Abelian groups G acting symplectically on a K3 surface are either cyclic or free products of cyclic groups, there is a correspondence between the divisible classes of M_G

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and covers of S_G, given by Section 2.2. So it is immediate to reconstruct the covering surface and its minimal model Y_G from the following data: S_G, the lattice M_G, the knowledge that certain (-2) classes in M_G represent smooth irreducible rational curves on S_G. The latter condition is guaranteed by Lemma 3.1 and the fact that the roots of E_G coincide with those of M_G; see Proposition 5.1. ■

It is not possible to generalize Theorem 4.7 or Corollary 4.8 to all the Abelian groups acting symplectically on a K3 surface. Indeed, for example, there exist K3 surfaces that contain a set of eight disjoint rational curves, but this set is not divisible by 2, hence these K3 surfaces are not necessarily desingularization of quotient of another K3 surface by $\mathbb{Z}/2\mathbb{Z}$: an example is given by the K3 surface that is the minimal model of the 2:1 cover of $\mathbb{P}^2$ branched along a sextic with eight nodes. Indeed, the cover of $\mathbb{P}^2$ is singular and has eight singularities of type A_1. So on the K3 surface there are eight disjoint rational curves arising from the desingularization of these singularities. But these curves are not a divisible set: this can be checked considering that the fixed locus of the cover involution is a curve of genus 2, and this determines, by [N2], the Néron–Severi group of the K3 surface. It is known that Theorem 4.7 can be extended to the K3 surfaces that contain at least fourteen disjoint rational curves, see [GS2].

Remark 5.3 Theorem 5.2 was proved for $G = \mathbb{Z}/2\mathbb{Z}$ in [GS1, Proposition 2.3] using a different method. The approach used in [GS1] was strictly based on a careful description of the action induced by a symplectic involution on Λ_{K3}. This allows one to give stronger results, but a similar description of the action induced by a group of symplectic automorphisms on Λ_{K3} is not known for groups G different from $\mathbb{Z}/2\mathbb{Z}$.

Theorem 5.2 allows one to describe the moduli space of the K3 surfaces that are covered by other K3 surfaces in terms of lattice polarized K3 surfaces:

Corollary 5.4 *Let G be a finite abelian group acting symplectically on a K3 surface. Let $\mathcal{W}_G$ be the set of lattices W_G satisfying*

(i) *W_G has rank $1 + \operatorname{rank}(M_G)$;*
(ii) *W_G is hyperbolic;*
(iii) *W_G admits a primitive embedding in Λ_{K3};*
(iv) *M_G is primitively embedded in W_G.*

Then a K3 surface is the desingularization of the quotient of a K3 surface by G if and only if it is a W_G-polarized K3 surface for a $W_G \in \mathcal{W}_G$.

In particular, the coarse moduli space of the K3 surfaces that are desingularization of the quotient Y/G for a K3 surface Y has infinitely many components of dimension $19 - \operatorname{rank}(M_G)$.

In the case where $G = \mathbb{Z}/2\mathbb{Z}$ all the admissible lattices that appear in $\mathcal{W}_{\mathbb{Z}/2\mathbb{Z}}$ are described in [GS1, Proposition 2.1 and Corollary 2.1]. Here we obtain the analogous result for $G = \mathbb{Z}/3\mathbb{Z}$. First we fix the following notation. The lattice $E_{\mathbb{Z}/3\mathbb{Z}}$ is isometric to A_2^6. We denote by $M_i^{(j)}$, $i = 1, 2$ the two curves that generate the j-th copy of A_2 in E_G and by $d_j := (M_1^{(j)} + 2M_2^{(j)})/3$. We can assume that $M_{\mathbb{Z}/3\mathbb{Z}}$ is generated by the generators of $E_{\mathbb{Z}/3\mathbb{Z}}$ and by the class $\sum_{j=1}^{6} d_j$.

Proposition 5.5 *Let $Y_{\mathbb{Z}/3\mathbb{Z}}$ be a K3 surface that admits a symplectic action of $\mathbb{Z}/3\mathbb{Z}$. Let $S_{\mathbb{Z}/3\mathbb{Z}}$ be the K3 surface desingularization of $(Y_{\mathbb{Z}/3\mathbb{Z}})/(\mathbb{Z}/3\mathbb{Z})$. Let us assume that $\rho(S_{\mathbb{Z}/3\mathbb{Z}}) = 13$. There is a primitive embedding of $M_{\mathbb{Z}/3\mathbb{Z}}$ in $NS(S_{\mathbb{Z}/3\mathbb{Z}})$. Let us denote by H a generator of the 1-dimensional subspace of $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ orthogonal to $M_{\mathbb{Z}/3\mathbb{Z}}$ in $NS(S_{\mathbb{Z}/3\mathbb{Z}})$. So $H^2 = 2d$ for a positive integer d, and without loss of generality, we can assume that H is pseudoample. Then there are the following possibilities, and all of them appear:*

- $d \not\equiv 0 \mod 3$*: in this case* $NS(S_{\mathbb{Z}/3\mathbb{Z}}) \simeq \mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$;
- $d \equiv 0 \mod 3$*: in this case there are two possibilities, either* $NS(S_{\mathbb{Z}/3\mathbb{Z}}) = \mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ *or* $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ *is an overlattice of index 3 of* $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$. *In the latter case* $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ *is generated by the generators of* $M_{\mathbb{Z}/3\mathbb{Z}}$ *and by a class* v. *Up to isometries the class* v *(mod* $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$*) is uniquely determined by* $d \mod 9$ *and it is the following:*
  - *if* $d \equiv 0 \mod 9$, *then* $v := H/3 + \sum_{j=1}^{3} d_j$;
  - *if* $d \equiv 3 \mod 9$, *then* $v := H/3 + \sum_{j=1}^{2}(d_j) + 2\sum_{h=3}^{4}(d_h)$;
  - *if* $d \equiv 6 \mod 9$, *then* $v := H/3 + d_1 + 2d_2$.

Proof The proof is based on the lattice theory and is analogous to that of [GS1, Propositions 2.1, 2.2 and Corollary 2.1].

Let $S_{\mathbb{Z}/3\mathbb{Z}}$ be a K3 surface that is a desingularization of $Y_{\mathbb{Z}/3\mathbb{Z}}/(\mathbb{Z}/3\mathbb{Z})$ for a certain K3 surface $Y_{\mathbb{Z}/3\mathbb{Z}}$. Then $M_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ and its orthogonal complement is a positive definite sublattice of rank 1.

So $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ is an overlattice of finite index, s, of $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ where $H^2 = 2d > 0$. The discriminant group of the lattice $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ is $\mathbb{Z}/2d\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4$, so the lattice $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ has length 5 if $d \equiv 0 \mod 3$, and 4 otherwise. A lattice of length at most 5 and of rank 13 admits a primitive embedding in Λ_{K3}. Thus, for each value of d there are K3 surfaces $S_{\mathbb{Z}/3\mathbb{Z}}$ with $NS(S_{\mathbb{Z}/3\mathbb{Z}}) \simeq \mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$, and so for any value of d there is a K3 surface obtained as quotient of $Y_{\mathbb{Z}/3\mathbb{Z}}$ by $\mathbb{Z}/3\mathbb{Z}$ and such that $NS(S_{\mathbb{Z}/3\mathbb{Z}}) \simeq \mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}}$.

Let us now assume that the index s of the inclusion $\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}} \hookrightarrow NS(S_{\mathbb{Z}/3\mathbb{Z}})$ is not 1. Then there is a nontrivial vector v in $(\mathbb{Z}H \oplus M_{\mathbb{Z}/3\mathbb{Z}})/NS(S_{\mathbb{Z}/3\mathbb{Z}})$. Since $M_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $NS(S_{\mathbb{Z}/3\mathbb{Z}})$, the vector v is of the form $v := \frac{1}{s}(H+m)$, where $m \in M_{\mathbb{Z}/3\mathbb{Z}}$ and m/s is a nontrivial element in the discriminant group of $M_{\mathbb{Z}/3\mathbb{Z}}$. This forces s to be 3. The condition $vH = 2d/3 \in \mathbb{Z}$ forces d to be a multiple of 3.

In order to identify v we describe the discriminant group of $M_{\mathbb{Z}/3\mathbb{Z}}$. Let us recall that $M_{\mathbb{Z}/3\mathbb{Z}}$ is an overlattice of index 3 of $E_{\mathbb{Z}/3\mathbb{Z}} \simeq A_2^6$. Since the lattice $M_{\mathbb{Z}/3\mathbb{Z}}$ is obtained by $E_{\mathbb{Z}/3\mathbb{Z}}$ adding the vector $\sum_{j=1}^{6} d_j$, the vectors in the discriminant group of $M_{\mathbb{Z}/3\mathbb{Z}}$ are the vectors $\sum_{j=1}^{6} \alpha_j d_j$ with $\alpha_i \in \mathbb{Z}/3\mathbb{Z}$ such that $\sum_{i=1}^{6} \alpha_i \equiv 0 \mod 3$. So the vector v is of the form $H/3 + w$, where $w = \sum_{j=1}^{6} \alpha_j d_j$ with $\alpha_i \in \mathbb{Z}/3\mathbb{Z}$ such that $\sum_{i=1}^{6} \alpha_i \equiv 0 \mod 3$. The self intersection of v is $2d/9 + \sum_{i=1}^{6} \alpha_i^2(-2/3)$. We observe that α_i^2 is either 0, if α_i is 0, or 1. The number $k := \sum_{i=1}^{6} \alpha_i^2$ is the number of $\alpha_i \in \mathbb{Z}/3\mathbb{Z}$ which are different from 0. The condition $v^2 \in 2\mathbb{Z}$ is then equivalent to $2d - 6k \equiv 0 \mod 18$ and so to $d - 3k \equiv 0 \mod 9$. Since we already know that $d \equiv 0 \mod 3$, we have that d is equivalent to one of the values 0, 3, or 6 mod 9. If $d \equiv 0 \mod 9$, then $3k \equiv 0 \mod 9$, so $k \equiv 0 \mod 3$. If $k = 0$, then the divisor $H/3$ is contained in $NS(X_{\mathbb{Z}/3\mathbb{Z}})$, which is impossible, since by definition H is a generator of the sublattice of $NS(X_{\mathbb{Z}/3\mathbb{Z}})$, orthogonal to $M_{\mathbb{Z}/3\mathbb{Z}}$. If $k = 3$, then, up to a permutation of the indices, the unique choice for v is $v := H/3 + d_1 + d_2 + d_3$. We observe that in this case the vector $H/3 + 2(d_1 + d_2 + d_3) + d_4 + d_5 + d_6$ is contained in $NS(X_{\mathbb{Z}/3\mathbb{Z}})$, because it is $v + \sum_{i=1}^{6} d_i$. If $k = 6$ a priori, we have two possible choices for v: either $v := H/3+d_1+d_2+d_3+d_4+d_5+d_6$ or $H/3+2(d_1+d_2+d_3)+d_4+d_5+d_6$. The first is not admissible, since it implies $H/3 \in NS(X_{\mathbb{Z}/3\mathbb{Z}})$. The second is equivalent to the choice $v := H/3 + d_1 + d_2 + d_3$. So if $d \equiv 0 \mod 9$, then $NS(S_{\mathbb{Z}/3\mathbb{Z}})$ is generated by the generators of $M_{\mathbb{Z}/3\mathbb{Z}}$ and by $v = H/3 + d_1 + d_2 + d_3$. Similarly, if $d \equiv 3 \mod 9$, then either $k = 1$ or $k = 4$. Since $\sum_{i=1}^{6} \alpha_i \equiv 0 \mod 3$, $k = 1$ is not admissible, so (up to a permutation of the indices) we can assume that $v := H/3 + d_1 + d_2 + 2d_3 + 2d_4$. If $d \equiv 6 \mod 9$, then either $k = 2$ or $k = 5$. If $k = 2$, we can assume that $v := H/3 + d_1 + 2d_2$. In this case we observe that the vector $H/3 + 2d_1 + d_3 + d_4 + d_5 + d_6$ is contained in $NS(X_{\mathbb{Z}/3\mathbb{Z}})$, because it is the sum of v and $\sum_{i=1}^{6} d_i$. But the vector $H/3 + 2d_1 + d_3 + d_4 + d_5 + d_6$ is the unique admissible choice for v (up to a permutation of the indices) with $k = 5$. So if $d \equiv 6 \mod 9$, we can assume that $v := H/3 + d_1 + 2d_2$. ■

Remark 5.6 There is a clear geometric meaning of H and d. Indeed, for every value of d there is a projective model of $S_{\mathbb{Z}/3\mathbb{Z}}$, given by $\phi_{|H|}: S_{\mathbb{Z}/3\mathbb{Z}} \to \mathbb{P}(H^0(X, H)^\vee)$. The image $\phi_{|H|}(S_{\mathbb{Z}/3\mathbb{Z}})$ is a surface with six singularities of type A_2, and it is in fact the quotient surface $Y_{\mathbb{Z}/3\mathbb{Z}}/\mathbb{Z}/3\mathbb{Z}$. The self intersection of H determines the dimension of the ambient space of $\phi_{|H|}(S_{\mathbb{Z}/3\mathbb{Z}})$, which is $\mathbb{P}^{d+1}$. This is the smallest projective space in which one can describe the quotient $Y_{\mathbb{Z}/3\mathbb{Z}}/\mathbb{Z}/3\mathbb{Z}$.

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References

[Ba1] W. Barth, *K3 surfaces with nine cusps*. Geom. Dedicata **72**(1998), no. 2, 171–178. http://dx.doi.org/10.1023/A:1005074306462

[Ba2] ———, *On the classification of K3 surfaces with nine cusps*. In: Complex analysis and algebraic geometry, de Gruyter, Berlin, 2000, pp. 41–59.

[Be] J. Bertin, *Réseaux de Kummer et surfaces K3*. Invent. Math. **93**(1988), no. 2, 267–284. http://dx.doi.org/10.1007/BF01394333

[BHPV] W. Barth, K. Hulek, C. Peters, and A. van de Ven, *Compact complex surfaces*. Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004. http://dx.doi.org/10.1007/978-3-642-57739-0

[F] A. Fujiki, *Finite automorphism groups of complex tori of dimension two*. Publ. Res. Inst. Math. Sci. **24**(1988), no. 1, 1–97. http://dx.doi.org/10.2977/prims/1195175326

[GS1] A. Garbagnati and A. Sarti, *Projective models of K3 surfaces with an even set*. Adv. Geom. **8**(2008), no. 3, 413–440. http://dx.doi.org/10.1515/ADVGEOM.2008.027

[GS2] ———, *Kummer surfaces and K3 surfaces with* $(\mathbb{Z}/2\mathbb{Z})^4$ *symplectic action*. Rocky Mountain J. Math., to appear. arxiv:1305.3514

[M] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*. Invent. Math. **94**(1988), no. 1, 183–221. http://dx.doi.org/10.1007/BF01394352

[MTW] G. Mongardi, K. Tari, and M. Wandel, *Prime order automorphisms of abelian surfaces: a lattice-theoretic point of view*. arxiv:1506.05679

[N1] V. V. Nikulin, *Kummer surfaces*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **39**(1975), 278–293; (English) Math. USSR. Izv. **9**(1975/1976), 261–275.

[N2] ———, *Factor groups of groups of the automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*. Soviet Math. Dokl. **20**(1979), 1156–1158.

[N3] ———, *Finite automorphism groups of Kähler K3 surfaces*. Trans. Moscow Math. Soc. **38**(1980), 71–135.

[N4] ———, *Integral symmetric bilinear forms and some applications*. Math. USSR Izvestija **14**(1980), 103–167.

[W] K. Wendland, *Consistency of orbifold conformal field theories on K3*. Adv. Theor. Math. Phys. **5**(2001), no. 3, 429–456.

[X] G. Xiao, *Galois covers between K3 surfaces*. Ann. Inst. Fourier (Grenoble) **46**(1996), no. 1, 73–88. http://dx.doi.org/10.5802/aif.1507

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