Can. J. Math. Vol. 46 (6), 1994 pp. 1150–1174

QUASI-MULTIPLIERS AND EMBEDDINGS OF HILBERT C^*-BIMODULES

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ABSTRACT. This paper considers Hilbert C^*-bimodules, a slight generalization of imprimitivity bimodules which were introduced by Rieffel [20]. Brown, Green, and Rieffel [7] showed that every imprimitivity bimodule X can be embedded into a certain C^*-algebra L, called the linking algebra of X. We consider arbitrary embeddings of Hilbert C^*-bimodules into C^*-algebras; *i.e.* we describe the relative position of two arbitrary hereditary C^*-algebras of a C^*-algebra, in an analogy with Dixmier's description [10] of the relative position of two subspaces of a Hilbert space.

The main result of this paper (Theorem 4.3) is taken from the doctoral dissertation of the third author [22], although the proof here follows a different approach. In Section 1 we set out the definitions and basic properties (mostly folklore) of Hilbert C^*-bimodules. In Section 2 we show how every quasi-multiplier gives rise to an embedding of a bimodule. In Section 3 we show that $C^*(A^\bullet)$, the enveloping C^*-algebra of the C^*-algebra A with its product perturbed by a positive quasi-multiplier s: $a \bullet b = asb$, is isomorphic to the closure of $s^{1/2}As^{1/2}$ (Proposition 3.1). Section 4 contains the main theorem (4.3), and in Section 5 we explain the analogy with the relative position of two subspaces of a Hilbert spaces and present some complements.

1. **Definitions and basic properties.** Many of the definitions and results used in this section go back to Paschke [16] and Rieffel [20]. However we shall use the terminology of Kasparov. Let us recall the notion of a Hilbert C^*-module as given in Kasparov [14, §2].

DEFINITION 1.1. Let A be a C^*-algebra and X a complex vector space and right A-module with a sesqui-linear map $(\cdot|\cdot)_A : X \times X \longrightarrow A$ which is conjugate linear in the first variable and linear in the second variable such that, for all $\xi, \eta \in X, a \in A$

(i) $(\xi|\xi)_A \geq 0$
(ii) $(\xi|\xi)_A = 0$ implies $\xi = 0$
(iii) $(\xi|\eta)_A^* = (\eta|\xi)_A$
(iv) $(\xi|\eta a)_A = (\xi|\eta)_A a$
(v) with the norm $\|\xi\| = \|(\xi|\xi)_A\|^{\frac{1}{2}}$, X is complete.

Then X is a *right Hilbert A-module*.

REMARK 1.2. In [14] Kasparov only considered right modules, so the object just defined was simply called a Hilbert A-module. We intend to consider both left and right modules, so we shall always indicate which it is we are considering.

The first author was partially supported by the Danish Research Council.

The second author was supported by l'Université d'Orléans and the Natural Sciences and Engineering Research Council of Canada.

Received by the editors August 2, 1992.

DEFINITION 1.3. Let A be a C^*-algebra and X a complex vector space and left A-module with a sesqui-linear form ${}_A(\cdot|\cdot): X \times X \longrightarrow A$ which is linear in the first variable and conjugate linear in the second satisfying (i), (ii), (iii), and (v) of Definition 1 and

(iv)′ ${}_A(a\xi|\eta) = a_A(\xi|\eta)$.

Then X is a *left Hilbert A-module.*

There is an operation which converts right Hilbert A-modules into left Hilbert A-modules and vice-versa.

DEFINITION 1.4. Let X be a right Hilbert A-module. Let $X^* = \{\xi^* \mid \xi \in X\}$. We will make X^* into a complex vector space as follows:

(i) $\xi^* + \eta^* = (\xi + \eta)^*$

(ii) $\lambda \cdot \xi^* = (\bar{\lambda}\xi)^*$, for $\xi^*, \eta^* \in X^*$ and $\lambda \in \mathbb{C}$.

We give X^* a left A-action and an A valued inner product:

(iii) $a \cdot \xi^* = (\xi a^*)^*$

(iv) ${}_A(\xi^*|\eta^*) = (\xi|\eta)_A$.

It is now easy to verify that X^* is a left Hilbert A-module. To convert a left Hilbert A-module into a right Hilbert A-module we use (i) and (ii) to give X^* a complex structure but define the action and inner product in the analgous way

(iii)′ $\xi^* \cdot a = (a^*\xi)^*$

(iv)′ $(\xi^*|\eta^*)_A = {}_A(\xi|\eta)$.

It is routine to verify that $X^{**} = X$.

Starting with a right Hilbert A-module X there are two C^*-algebras acting as algebras of A-module endomorphisms of X [14, Definition 3 and Definition 4].

DEFINITION 1.5. Let X be a right Hilbert A-module and T a bounded linear operator on the Banach space X such that

(i) $T(\xi a) = T(\xi)a \;\forall a \in A$, and

(ii) there exists a bounded operator S on X such that $\forall \xi, \eta \in X \; (T\xi|\eta)_A = (\xi|S\eta)_A$.

Then T is an *adjointable* operator on X. It turns out that there can be at most one S satisfying (ii) and this unique S is called the *adjoint* of T and denoted T^*. The set of all adjointable operators is denoted $\mathcal{L}(X)$.

If we give $\mathcal{L}(X)$ the operator norm and the involution $T \longmapsto T^*$, $\mathcal{L}(X)$ is a C^*-algebra. Given $\xi, \eta \in X$ we may define an operator $\theta_{\xi,\eta}$ by $\theta_{\xi,\eta}(\mu) = \xi(\eta|\mu)_A$. Proposition 2.9 of [20], an analogue of the Cauchy-Schwartz inequality implies that $\|(\eta|\mu)_A\| \leq \|\eta\| \, \|\mu\|$, and hence $\vartheta_{\xi,\eta}$ is bounded.

DEFINITION 1.6. Let $\mathcal{K}(X)$ be the closure of the linear span of $\{\theta_{\xi,\eta}|\xi, \eta \in X\}$.

In [14, Lemma 3, Lemma 4, and Theorem 1] Kasparov shows that $\mathcal{K}(X)$ is a closed two sided ideal in $\mathcal{L}(X)$ and $\mathcal{L}(X) = M\big(\mathcal{K}(X)\big)$. We will need two facts about $\mathcal{K}(X)$ (Proposition 1.7 and Proposition 1.10), which we now prove as we are unable to provide a reference. But first a bit of notation: if A_1 and A_2 are linear subspaces of a C^*-algebra, let A_1A_2 be the linear span of the set of products a_1a_2 with $a_i \in A_i$. Similarly we extend this to the case of three subspaces A_1, A_2, and A_3; $A_1A_2A_3$ is the linear span of the set of products $a_1a_2a_3$ with $a_i \in A_i$.

PROPOSITION 1.7. *Let X be a right Hilbert B-module. Then*
(i) $\overline{XB} = X$.
(ii) $\|\theta_{\xi,\xi}\| = \|(\xi|\xi)_B\| \ \forall \xi \in X$.

PROOF. We have $\xi = \lim_{n\to\infty} \xi(\xi|\xi)_B[(\xi|\xi)_B + n^{-1}]^{-1}$. This proves (i).
To prove (ii) first notice that for $\xi \in X$

$$\|\xi\|^2 = \|(\xi|\xi)_B\| \leq \sup\big\{\|(\xi|\eta)_B\| \ \big| \ \|\eta\| \leq 1\big\} \cdot \|\xi\| \leq \|\xi\|^2.$$

Hence $\|\xi\| = \sup\big\{\|(\xi|\eta)_B\| \ \big| \ \|\eta\| \leq 1\big\}$. Next

$$\begin{aligned}\|\theta_{\xi,\eta}\|^2 &= \sup\big\{\|\xi(\eta|\mu)_B\|^2 \ \big| \ \|\mu\| \leq 1\big\}\\ &= \sup\big\{\|(\mu|\eta)_B(\xi|\xi)_B(\eta|\mu)_B\| \ \big| \ \|\mu\| \leq 1\big\}\\ &= \sup\Big\{\Big\|\big(\eta(\xi|\xi)_B^{\frac12}|\mu\big)_B\Big\|^2 \ \Big| \ \|\mu\| \leq 1\Big\}\\ &= \|\eta(\xi|\xi)_B^{\frac12}\|^2.\end{aligned}$$

Hence $\|\theta_{\xi,\xi}\|^2 = \|\xi(\xi|\xi)_B^{\frac12}\|^2 = \|(\xi|\xi)_B\|^2$. ■

We can now present the main object of study.

DEFINITION 1.8. Let A and B be C^*-algebras and X a complex vector space and A-B-bimodule. Suppose that we have sesqui-linear forms ${}_A(\cdot|\cdot)$ and $(\cdot|\cdot)_B$ so that X is both a left Hilbert A-module and a right Hilbert B-module and that the forms are related by the equation

$${}_A(\xi|\eta)\mu = \xi(\eta|\mu)_B$$

for all $\xi, \eta, \mu \in X$. Then X is a *Hilbert A-B-bimodule*.

REMARK 1.9. It may appear from the definition that X has two norms on it, one from each inner product. We shall show that $\|{}_A(\xi|\xi)\| = \|(\xi|\xi)_B\|$ for all $\xi \in X$; so the two ways of norming X agree. To do this let us introduce some notation. If X is a Hilbert A-module (either right or left) let I_A be the closed linear span of $\{{}_A(\xi|\eta) \mid \xi, \eta \in X\}$ (here assuming that X is a left A-module). Note that I_A is always a closed two sided ideal of A and that a Hilbert A-B-bimodule is by restriction a Hilbert I_A-I_B-bimodule.

Recall that I_A has an approximate identity $\{u_\alpha\}$ where each u_α is a finite sum $\sum_{i=1}^n {}_A(\eta_i^\alpha|\eta_i^\alpha)$ with η_i^α in X. In fact as indicated in Brown [4, Theorem 2.1] this follows from Dixmier's argument [11, 1.7.2]: given $\alpha = \{\xi_1, \ldots, \xi_n\} \subseteq X$. Let $\eta_i^\alpha = \big(n^{-1} + \sum_{i=1}^n {}_A(\xi_i|\xi_i)\big)^{-1/2}\xi_i$ and $u_\alpha = \sum_{i=1}^n {}_A(\eta_i^\alpha|\eta_i^\alpha)$. $\{u_\alpha\}$ is an approximate identity for I_A where α ranges over the finite subsets of X.

An immediate consequence of this is that if $a \in I_A$ and $a\xi = 0$ for all ξ in X, then $a = 0$. Secondly, if X is a Hilbert A-B-bimodule and $a \in A$, then for all $\xi, \eta, \mu \in X$ $\xi(a\eta|\mu)_B = {}_A(\xi|a\eta)\mu = {}_A(\xi|\eta)a^*\mu = \xi(\eta|a^*\mu)_B$. Hence $(a\eta|\mu)_B = (\eta|a^*\mu)_B$, for all $a \in A$ and all $\eta, \mu \in X$.

The notion of a Hilbert A-B-bimodule is a generalization of the notion of an A-B-imprimitivity bimodule as introduced by Rieffel [20, Definition 6.10]. Every A-B-imprimitivity bimodule is a Hilbert A-B-bimodule but a Hilbert A-B-bimodule is an A-B-imprimitivity bimodule only when $A = I_A$ and $I_B = B$.

PROPOSITION 1.10. *Let X be a Hilbert A-B-module. Then* $\mathcal{K}(X) \cong I_A$ *and* $\mathcal{K}(X^*) \cong I_B$.

PROOF. Recall that for $\xi, \eta \in X$, for all $\mu \in X$ $\theta_{\xi,\eta}(\mu) = \xi(\eta|\mu)_B = {}_A(\xi|\eta)\mu$. Hence left multiplication by ${}_A(\xi|\eta)$ is the operator $\theta_{\xi,\eta}$ on X. Let us denote this map by λ_a: *i.e.* $\lambda_a(\xi) = a\xi$ for $a \in I_A$ $\xi \in X$. We need to show that λ is an isomorphism. For this it suffices to show that λ is isometric as $\lambda\big(\sum_i {}_A(\xi_i|\eta_i)\big) = \sum \theta_{\xi_i,\eta_i}$ and so will take a dense set into a dense set. Now it is easy to check that λ is bounded; *i.e.*, $\|\lambda_a\| \leq \|a\|$. To show that λ is isometric we shall show that it has trivial kernel.

Now suppose $a \in I_A$ and $\lambda_a = 0$. Let u_α be the approximate identity constructed above. Then $au_\alpha = \sum_i {}_A(\lambda_a \eta_i^\alpha | \eta_i^\alpha) = 0$ implies $a = 0$ since u_α is an approximate identity. Hence λ is an isomophism. Thus $I_A \simeq \mathcal{K}(X)$. As X^* is a Hilbert B-A-bimodule, we have $I_B \simeq \mathcal{K}(X^*)$.

COROLLARY 1.11. *If X is a Hilbert A-B-bimodule, then* $\|{}_A(\xi|\xi)\| = \|(\xi|\xi)_B\|$ *for all* $\xi \in X$.

PROOF.

$$\begin{aligned}\|{}_A(\xi|\xi)\| &= \|\theta_{\xi,\xi}\| \quad \text{(by Proposition 1.10)}\\ &= \|(\xi,\xi)_B\| \quad \text{(by Proposition 1.7).}\end{aligned}$$

■

Let us conclude this section by recalling some facts about hereditary C^*-algebras from Brown [4, §1].

DEFINITION 1.12. Let A be a C^*-algebra and B a C^*-subalgebra. B is a *hereditary* subalgebra if whenever $b \in B$, $a \in A$, and $0 \leq a \leq b$, we have that $a \in B$. Equivalently B is hereditary if $BAB \subseteq B$ (one implication is clear, the other can be obtained, for example, from Pedersen [17, Proposition 1.4.5]).

A hereditary subalgebra B of A is called *full* if B is not contained in any proper closed two sided ideal of A, *i.e.* $\overline{ABA} = A$.

If p is a projection in the multiplier algebra of A then $B = pAp$ is a hereditary subalgebra of A, and p is called *full* if pAp is full; *i.e.* $\overline{ApA} = A$ (see Brown [4, Lemma 1.1]).

If X is any subset of a C^*-algebra A then we define her(X) to be the intersection of all hereditary C^*-subalgebras containing X. If B is a C^*-subalgebra of A then her$(B) = \overline{BAB}$.

PROPOSITION 1.13. *Let A be a C^*-algebra and $\{B_\alpha\}$ a family of hereditary C^*-subalgebras. Suppose* her$\big(\bigcup_\alpha B_\alpha\big) = A$. *Then for every non-trivial closed two sided ideal I of A, $I \cap B_\alpha \neq \{0\}$ for at least one α.*

PROOF. Let B be the C^*-algebra generated by $\bigcup_\alpha B_\alpha$. Then her$(B) = \overline{BAB} = A$. Let I be a closed two sided ideal in A. Suppose $I \cap B_\alpha = \{0\}$, $\forall \alpha$. Since B_α is hereditary $B_\alpha I B_\alpha \subseteq I \cap B_\alpha = \{0\}$. Thus $IB_\alpha = \{0\}$, $\forall \alpha$. Hence $IB = \{0\}$. Therefore $I = \overline{BABIBAB} = \{0\}$.

2. **Embeddings of Hilbert C^*-bimodules and quasi-multipliers.** Let us begin by recalling the notion of a linking algebra from Brown, Green, and Rieffel [7]. Suppose A and B are strongly Morita equivalent C^*-algebras; *i.e.* there is a Hilbert A-B-bimodule X, such that $I_A = A$ and $I_B = B$. Form the right Hilbert B-module $X \oplus B$ (as in Kasparov [14, Definition 2]) and let $L = \mathcal{K}(X \oplus B)$. In $M(L)$ there are two projections $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Now $pLp = \mathcal{K}(X) = A$, $qLq = \mathcal{K}(B) = B$, $pLq = \mathcal{K}(B;X) = X$ (via $\theta_{\xi,b} \longmapsto \xi b^*$), and $qLp = \mathcal{K}(X,B) = X^*$ (via $\theta_{b,\xi} \longmapsto b\xi^*$). So we may write $L = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$. L is called the *linking algebra* of X.

Conversely starting with a C^*-algebra C and projections $p, q \in M(C)$ with $p + q = 1$ we may let $A = pCp$, $X = pCq$, and $B = qCq$. In this way we obtain hereditary C^*-subalgebras A and B, and a Hilbert A-B-bimodule X with inner products ${}_A(pc_1q|pc_2q) = pc_1qc_2^*p$ and $(pc_1q|pc_2q)_B = qc_1^*pc_2q$. If in addition p and q are full then $\overline{X^*X} = B$ and $\overline{XX^*} = A$ and C is the linking algebra of the A-B-imprimitivity bimodule X. Even when p and q are not full we shall say that C is the linking algebra of the Hilbert A-B-bimodule X.

We can obtain a more general situation as follows. Suppose A and B are hereditary subalgebras of a C^*-algebra C and $C = \mathrm{her}(A \cup B)$. Let $X = \overline{ACB}$, then X is a Hilbert A-B-bimodule and we say (A, X, B) is *embedded* into C.The first problem we wish to consider is: given a Hilbert A-B-bimodule X, is there an embedding of (A, X, B) into some C^*-algebra C? Secondly, given two embeddings, one into C_1 and the other into C_2 say, when are they equivalent in that there is a $*$-isomorphism of C_1 to C_2 takes one to another?

A small adjustment to the argument of Brown, Green, and Rieffel shows that at least one embedding, the linking algebra, always exists. To each embedding we associate a quasi-multiplier and with these, one can describe the embeddings.

DEFINITION 2.1. Let X be a Hilbert A-B-bimodule. An *embedding* $f = (f_A, f_X, f_B)$ of (A, X, B) into a C^*-algebra C is a triple (f_A, f_X, f_B) of isometries of Banach spaces such that

(i) $f_A : A \longrightarrow C$ and $f_B : B \longrightarrow C$ are $*$-homomorphisms such that $f_A(A)$ and $f_B(B)$ are hereditary subalgebras of C whose union hereditarily generates C; *i.e.*, $[f_A(A) \cup f_B(B)]C[f_A(A) \cup f_B(B)]$ is dense in C.

(ii) $f_X(X) = \overline{f_A(A)Cf_B(B)}$

(iii) $f : (A, X, B) \longrightarrow \big(f_A(A), f_X(X), f_B(B)\big)$ is an isomorphism of Hilbert C^*-bimodules *i.e.*

(α) $f_A(a)f_X(\xi)f_B(b) = f_X(a\xi b)$

(β) $f_X(\xi)^* f_X(\eta) = f_B\big((\xi|\eta)_B\big)$

(γ) $f_X(\xi)f_X(\eta)^* = f_A\big({}_A(\xi|\eta)\big)$

Given two embeddings of (A, X, B), $f^1 : (A, X, B) \longrightarrow C_1$, and $f^2 : (A, X, B) \longrightarrow C_2$ we say that f^1 is *equivalent* to f^2 if there is an isomorphism $\theta : C_1 \longrightarrow C_2$ such that

(a) $\theta \circ f_A^1 = f_A^2$,

https://doi.org/10.4153/CJM-1994-065-5 Published online by Cambridge University Press

(b) $\theta \circ f_X^1 = f_X^2$, and
(c) $\theta \circ f_B^1 = f_B^2$.

There is always at least one embedding of a Hilbert A-B-bimodule. As this generalizes the construction of Brown, Green, and Rieffel [7] we shall call it the *linking algebra* of the bimodule.

DEFINITION 2.2. Let X be a Hilbert A-B-bimodule. Let

$$L = \left\{ \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \;\middle|\; a \in A, b \in B, \xi, \eta \in X \right\}.$$

L has the linear structure coming from A, B, X, and X^*. L has the same product as in [7] viz

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + {}_A(\xi_1|\eta_2) & a_1\xi_2 + \xi_1 b_2 \\ \eta_1^* a_2 + b_1\eta_2^* & (\eta_1|\xi_2)_B + b_1 b_2 \end{pmatrix}.$$

Also we give L an involution

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}.$$

L now becomes an involutive algebra. We give L a norm as follows. As before $X \oplus B$ is a right Hilbert B-module and $A \oplus X^*$ is a right Hilbert A-module. We get two representations of L. $\pi_A : L \longrightarrow \mathcal{L}(A \oplus X^*)$ and $\pi_B : L \longrightarrow \mathcal{L}(X \oplus B)$.

$$\pi_A \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} a_1 \\ \eta_1^* \end{pmatrix} = \begin{pmatrix} aa_1 + {}_A(\xi|\eta_1) \\ \eta^* a_1 + b\eta_1^* \end{pmatrix}$$

$$\pi_B \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} \xi_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a\xi_1 + \xi b_1 \\ (\eta|\xi_1)_B + bb_1 \end{pmatrix}.$$

In the next proposition we shall show that L is a C^*-algebra. We call L the *linking algebra* of X.

PROPOSITION 2.3. *For $c \in L$ let $\|c\| = \max\{\|\pi_A(c)\|, \|\pi_B(c)\|\}$. Then*
(i) for $c = \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \in L$,

$$\max\{\|a\|, \|\xi\|, \|\eta\|, \|b\|\} \leq \|c\| \leq 4\max\{\|a\|, \|\xi\|, \|\eta\|, \|b\|\}.$$

(ii) $f_A(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $f_X(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}$, $f_B(b) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ *define an embedding of X into the C^*-algebra L.*

PROOF. (i) It is easy to check that $\|a\|, \|\xi\| \leq \left\| \pi_A \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \right\|$. We shall show that

$$\left\| \pi_A \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \right\| \leq 4\max\{\|a\|, \|\xi\|, \|\eta\|, \|b\|\}.$$

Combining this with the analogous statement for π_B we get both of the inequalities of (i).

For $\begin{pmatrix} \tilde{a} \\ \tilde{\eta}^* \end{pmatrix} \in A \oplus X^*$, $\left\| \begin{pmatrix} \tilde{a} \\ \tilde{\eta}^* \end{pmatrix} \right\| = \|\tilde{a}^*\tilde{a} + {}_A(\tilde{\eta}|\tilde{\eta})\|^{\frac{1}{2}}$, so

$$\begin{aligned}\left\|\pi_A \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{\eta}^* \end{pmatrix}\right\| &= \|[a\tilde{a} + {}_A(\xi|\tilde{\eta})]^*[a\tilde{a} + {}_A(\xi|\tilde{\eta})] + {}_A(\eta^*\tilde{a} + b\tilde{\eta}^*|\eta^*\tilde{a} + b\tilde{\eta}^*)\|^{\frac{1}{2}} \\ &\leq \|a\|\,\|\tilde{a}\| + \|\xi\|\,\|\tilde{\eta}\| + \|\tilde{a}\|\,\|\eta\| + \|\tilde{\eta}\|\,\|b\| \\ &\leq 4\max\{\|a\|, \|\xi\|, \|\eta\|, \|b\|\}\|\tilde{a}^*\tilde{a} + {}_A(\tilde{\eta}|\tilde{\eta})\|^{\frac{1}{2}}.\end{aligned}$$

This completes the proof of (i).

(ii) As each of $\|\pi_A(\cdot)\|$ and $\|\pi_B(\cdot)\|$ are C^*-semi-norms on L, $\|\cdot\|$ is a C^*-semi-norm on L. But by (i) it is a norm, and L is complete. Thus L is a C^*-algebra with this norm. Moreover f_A and f_B are injective. Since they are $*$-homomorphisms they are isometric. Thus

$$\begin{aligned}\|f_X(\xi)\| &= \|f_X(\xi)^*f_X(\xi)\|^{\frac{1}{2}} = \left\|f_B\big((\xi|\xi)_B\big)\right\|^{\frac{1}{2}} \\ &= \|(\xi|\xi)_B\|^{\frac{1}{2}} = \|\xi\|.\end{aligned}$$

So all of the maps are isometric. It is easy to check that condition (iii) of Definition 2.1 is satisfied. By Proposition 1.7(i)

$$f_A(A)Lf_B(B) = \operatorname{span}\left\{\begin{pmatrix} 0 & a\xi b \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in A, \xi \in X, b \in B\right\}$$

is dense in $f_X(X)$, so condition (ii) of Definition 2.1 holds.

Now let $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; p and q are projections in $M(L)$ and $f_A(A) = pLp$ and $f_B(B) = qLq$ so these algebras are hereditary. If $\{e_\alpha\}$ is an approximate identity for L, then $\{pe_\alpha p + qe_\alpha q\} \subseteq f_A(A) + f_B(B)$ is also an approximate identity for L. Hence $[f_A(A) \cup f_B(B)]L[f_A(A) \cup f_B(B)]$ is dense in L. This proves that condition (i) of Definition 2.1 holds and that $f = (f_A, f_X, f_B)$ is an embedding. ∎

Let us recall from Pedersen [17, 3.12.1] the notion of a quasi-multiplier. Let A be a C^*-algebra and A'' its second dual as a Banach space. By definition $QM(A) = \{t \in A'' \mid atb \in A,\ \forall a, b \in A\}$. From Proposition 2.3 (i) we see that $L'' = \begin{pmatrix} A'' & X'' \\ X''^* & B'' \end{pmatrix}$. So we may identify X'' with $pL''q$.

LEMMA 2.4. *(i) Suppose $t \in X''$ and $atb \in X\ \forall a \in A,\ b \in B$. Then $\forall \xi, \eta \in X$, $\eta^*t \in LM(B)$, $t^*\eta \in RM(B)$, $t\eta^* \in RM(A)$, $\eta t^* \in LM(A)$ and $\eta^*t\xi^* \in X^*$.*

(ii) $\left\{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \,\middle|\, t \in X'' \text{ and } atb \in X\ \forall a \in A, b \in B\right\} = \{psq \mid s \in QM(L)\}$.

PROOF. (i) By Proposition 1.7 (i) AXB is dense in X. If $\eta \in AXB$ say $\eta = a\xi b$ and $\tilde{b} \in B$ then $(\eta^*t)\tilde{b} = b^*\xi^*a^*t\tilde{b} = b^*(\xi|a^*t\tilde{b})_B \in B$. By taking limits we see that $X^*t \subseteq LM(B)$. Similarly $t^*X \subseteq RM(B)$, $tX^* \in RM(A)$ and $Xt^* \in LM(A)$. If $\xi_1 = a_1\eta_1 b_1$ $\xi_2 = a_2\eta_2 b_2$ then $\xi_1^*t\xi_2^* = b_1^*\eta_1^*a_1^*tb_2^*\eta_2^*a_2^* = b_1^*(\eta_1|a_1^*tb_2^*)_B\eta_2^*a_2^* \in X^*$.

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(ii) It is clear that $pQM(L)q \subseteq \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \;\middle|\; t \in X'', atb \in X \, \forall a \in A, b \in B \right\}$. Suppose $t \in X''$ and $atb \in X \, \forall a \in A, b \in B$. Then by (i) and straight forward computation $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \in QM(L)$.

DEFINITION 2.5. Let X be a Hilbert A-B-bimodule and L its linking algebra. $QM(X) = \{t \in X'' \mid atb \in X \, \forall a \in A \, \forall b \in B\}$ is the set of *quasi-multipliers* of X. By Lemma 2.4 $QM(X) = pQM(L)q$.

REMARK 2.6. Now let us demonstrate how a quasi-multiplier of a Hilbert A-B-module X gives an embedding of (A, X, B).

Suppose $t \in QM(X)$ and $\|t\| \leq 1$. Let $s = \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}^* \in QM(L)$, where L is the linking algebra of X. Let $L_t = \overline{s^{1/2} L s^{1/2}}$. L_t is a closed self-adjoint subspace of L''; but if $c_1, c_2 \in L$, $(s^{\frac{1}{2}} c_1 s^{\frac{1}{2}}) \cdot (s^{\frac{1}{2}} c_2 s^{\frac{1}{2}}) = s^{\frac{1}{2}} (c_1 s c_2) s^{\frac{1}{2}} \in s^{\frac{1}{2}} L s^{\frac{1}{2}}$. So L_t is a C^*-algebra. Next we shall construct an embedding $f_t \colon (A, X, B) \longrightarrow L_t$ as follows:

$$f_A^t(a) = s^{\frac{1}{2}} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}},$$

$$f_X^t(\xi) = s^{\frac{1}{2}} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}},$$

$$f_B^t(b) = s^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} s^{\frac{1}{2}}.$$

PROPOSITION 2.7. *Suppose $t \in QM(X)$, with $\|t\| \leq 1$, and $s = \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix}$. Then $f^t = (f_A^t, f_X^t, f_B^t)$ is an embedding of (A, X, B) into $L_t = \overline{s^{1/2} L s^{1/2}}$*

PROOF. To prove that each of f_A^t, f_X^t, f_B^t is isometric we shall show that f_A^t and f_B^t are faithful $*$-homomorphisms of A and B into L_t respectively and f_X^t is a morphism of Hilbert C^*-bimodules *i.e.*

$$f_X^t(\xi)^* f_X^t(\eta) = f_B^t\big((\xi|\eta)_B\big) \quad \text{and}$$
$$f_X^t(\xi) f_X^t(\eta)^* = f_A^t\big({}_A(\xi|\eta)\big).$$

The easily checked equations

$$\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} s \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} s \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} {}_A(\xi|\eta) & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} s \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b_1 b_2 \end{pmatrix} \quad \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} s \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a\xi \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}^* s \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (\xi|\eta)_B \end{pmatrix} \quad \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} s \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & \xi b \\ 0 & 0 \end{pmatrix}$$

show that f_A^t and f_B^t are $*$-homomorphisms and that f_X^t is a morphism of Hilbert C^*-bimodules. To show that f_A^t (and similarly f_B^t) is faithful observe that if

$$0 = \left(s^{\frac{1}{2}} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}} \right)^* \left(s^{\frac{1}{2}} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}} \right) = s^{\frac{1}{2}} \begin{pmatrix} a^* a & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}}$$

then

$$0 = s\begin{pmatrix} a^*a & 0 \\ 0 & 0 \end{pmatrix} s = \begin{pmatrix} a^*a & a^*at \\ t^*a^*a & t^*a^*at \end{pmatrix};$$

so $a = 0$. Thus f_A^t, f_X^t, f_B^t are isometric. All that is left to check is that $f_A^t(A)$ and $f_B^t(B)$ are hereditary subalgebras of L_t, and together hereditarily generate L_t.

To prove that $f_A^t(A)$ and $f_B^t(B)$ are hereditary note that

$$f_A^t(A) = s^{\frac{1}{2}}pLps^{\frac{1}{2}}$$

and

$$f_B^t(B) = s^{\frac{1}{2}}qLqs^{\frac{1}{2}};$$

i.e., these sets are closed. Now

$$\begin{aligned} s^{\frac{1}{2}}pLps^{\frac{1}{2}} \cdot \overline{s^{\frac{1}{2}}Ls^{\frac{1}{2}}} \cdot s^{\frac{1}{2}}pLps^{\frac{1}{2}} &\subseteq s^{\frac{1}{2}}pLps^{\frac{1}{2}} \cdot \overline{s^{\frac{1}{2}}LLs^{\frac{1}{2}}} \cdot s^{\frac{1}{2}}pLps^{\frac{1}{2}} \\ &\subseteq s^{\frac{1}{2}}p\overline{(LpsL)\cdot(LspL)}ps^{\frac{1}{2}} \\ &\subseteq s^{\frac{1}{2}}pLps^{\frac{1}{2}} = f_A(A). \end{aligned}$$

So $f_A^t(A)$ is hereditary; similarly $f_B^t(B)$ is hereditary.

In order to show that $f_A^t(A) \cup f_B^t(B)$ hereditarily generates L_t it is enough to show that the hereditary subalgebra generated by $f_A^t(A) \cup f_B^t(B)$ contains $s^{\frac{1}{2}}pLqs^{\frac{1}{2}}$. For then by taking adjoints it will also contain $s^{\frac{1}{2}}qLps^{\frac{1}{2}}$, and it already contains $s^{\frac{1}{2}}pLps^{\frac{1}{2}}$ and $s^{\frac{1}{2}}qLqs^{\frac{1}{2}}$; since the union of these four sets is total in L_t we shall be done.

Now

$$\begin{aligned} f_A^t(A)f_X^t(X)f_B^t(B) &= s^{1/2}pLps^{1/2} \cdot s^{1/2}pLqs^{1/2} \cdot s^{1/2}qLqs^{1/2} \\ &= s^{1/2}pLp \cdot pLq \cdot qLq \cdot s^{1/2}, \end{aligned}$$

and the closure of the latter contains $f_X^t(X) = s^{1/2}pLqs^{1/2}$. Thus $f_A^t(A) \cup f_B^t(B)$ hereditarily generates L_t.

Finally we must check condition (ii) of Definition 2.1. Now

$$\begin{aligned} f_X^t(X) &= s^{\frac{1}{2}}pLqs^{\frac{1}{2}} = s^{\frac{1}{2}}\overline{(pLpLqLq)}s^{\frac{1}{2}} \\ &= s^{\frac{1}{2}}\overline{(pLpspLqsqLq)}s^{\frac{1}{2}} \\ &= \overline{s^{\frac{1}{2}}pLps^{\frac{1}{2}} \cdot s^{\frac{1}{2}}pLqs^{\frac{1}{2}} \cdot s^{\frac{1}{2}}qLqs^{\frac{1}{2}}} \\ &= \overline{f_A^t(A) \cdot f_X^t(X) \cdot f_B^t(B)} \subseteq \overline{f_A^t(A)L_tf_B^t(B)} \\ &= \overline{s^{\frac{1}{2}}pLps^{\frac{1}{2}} \cdot s^{\frac{1}{2}}Ls^{\frac{1}{2}} \cdot s^{\frac{1}{2}}qLqs^{\frac{1}{2}}} \\ &\subseteq \overline{s^{\frac{1}{2}}p(LpsL) \cdot (LsqL)qs^{\frac{1}{2}}} \subseteq s^{\frac{1}{2}}pLqs^{\frac{1}{2}} \\ &\subseteq f_X^t(X). \end{aligned}$$

This concludes the proof that we have an embedding.

3. **The C^*-algebra generated by a quasi-multiplier.** Given $s \in QM(A)_+$ we may define a new product on A by $a_1 \bullet a_2 = a_1 s a_2$. In this section we consider the C^*-algebra so generated.

PROPOSITION 3.1. *Let A be a C^*-algebra and s a positive quasi-muliplier of A. Let $A^\bullet$ equal A as a complex vector space; but give $A^\bullet$ a new (but equivalent) norm and a new product: for $a, b \in A^\bullet$*

(i) $a \bullet b = asb$

(ii) $\|a\|_s = \|s\| \, \|a\|$.

Give $A^\bullet$ the involution coming from A. Then $A^\bullet$ is an involutive Banach algebra and its enveloping C^*-algebra is isomorphic to $\overline{s^{1/2}As^{1/2}} \subseteq A''$ via the canonical extension of the map

$$a \longmapsto s^{\frac{1}{2}} a s^{\frac{1}{2}} : A^\bullet \longrightarrow s^{\frac{1}{2}} A s^{\frac{1}{2}}.$$

PROOF. (We are grateful to Man-Duen Choi who simplified our original proof of this proposition.)

As $\|\cdot\|_s$ is equivalent to $\|\cdot\|$ it is clear that $A^\bullet$ is a Banach space. For $a, b \in A^\bullet \|a \bullet b\|_s = \|s\| \, \|asb\| \leq \|a\|_s \|b\|_s$. Also $\|a^*\|_s = \|a\|_s$. So $A^\bullet$ is an involutive Banach algebra.

Recall that we construct $C^*(A^\bullet)$ the enveloping C^*-algebra of $A^\bullet$ as follows. For $a \in A^\bullet$ let $\|a\|_* = \sup\big\{ \|\pi(a)\| \;\big|\; \pi \text{ is a } *\text{-homomorphism of } A^\bullet \text{ into a } C^*\text{-algebra} \big\}$. This supremum is finite as $\|\pi(a)\| \leq \|a\|_s$, $\forall a \in A$ (see Dixmier [11, Proposition 1.3.7]). $\|\cdot\|_*$ is not necessarily a norm on $A^\bullet$, however we may mod out by the elements of length zero and complete to form $C^*(A^\bullet)$.

We then see that the map $a \longmapsto s^{1/2} a s^{1/2}$ extends to surjection $C^*(A^\bullet) \longrightarrow \overline{s^{1/2}As^{1/2}}$. To prove that this map is an isomorphism we must show that for every $*$-homomorphism $\pi: A^\bullet \longrightarrow B$, for some C^*-algebra B, $\|\pi(a)\| \leq \|s^{\frac{1}{2}} a s^{\frac{1}{2}}\|$, $\forall a \in A^\bullet$.

It suffices to check this for $a = a^*$. Now

$$\begin{aligned} \|\pi(a)\|^n &= \|\pi(a^n)\| = \|\pi(asa \cdots asa)\| \leq \|asa \cdots asa\|_s \\ &= \|s\| \, \|asa \cdots asa\| \\ &= \|s\| \, \|as^{\frac{1}{2}}(s^{\frac{1}{2}}as^{\frac{1}{2}})^{n-2}s^{\frac{1}{2}}a\| \leq \|s\| \, \|as^{\frac{1}{2}}\|^2 \|s^{\frac{1}{2}}as^{\frac{1}{2}}\|^{n-2}. \end{aligned}$$

Thus

$$\|\pi(a)\| \leq \|s\|^{\frac{1}{n}} \|as^{\frac{1}{2}}\|^{\frac{2}{n}} \|s^{\frac{1}{2}}as^{\frac{1}{2}}\|^{\frac{n-2}{n}}, \quad \forall n.$$

If $\|s^{\frac{1}{2}}as^{\frac{1}{2}}\| = 0$ then $\|\pi(a)\| = 0$ and we are done. If $\|s^{\frac{1}{2}}as^{\frac{1}{2}}\| \neq 0$ then $\|as^{\frac{1}{2}}\| \neq 0$ and so

$$\|\pi(a)\| \leq \lim_{n \to \infty} \|s\|^{\frac{1}{n}} \|as^{\frac{1}{2}}\|^{\frac{2}{n}} \|s^{\frac{1}{2}}as^{\frac{1}{2}}\|^{\frac{n-2}{n}} = \|s^{\frac{1}{2}}as^{\frac{1}{2}}\|.$$

PROPOSITION 3.2. *Let q be the range projection of the quasi-multiplier s. The enveloping von Neumann algebra of $(s^{1/2}As^{1/2})^-$ is $qA''q$, where A'' is the enveloping von Neumann algebra of A.*

PROOF. By elementary spectral theory $qA''q$ is the weak closure in A'' of $s^{1/2}As^{1/2}$. One must show that every representation π of $(s^{1/2}As^{1/2})^-$ extends to a normal representation of $qA''q$.

So let π be a non-degenerate representation of $(s^{1/2}As^{1/2})^-$ on a Hilbert space H. As in the proof of the previous proposition let $\pi^\bullet: A \longrightarrow B(H)$ be given by $\pi^\bullet(a) = \pi(s^{1/2}as^{1/2})$. So the extension we seek, let's call it $\tilde{\pi}$, will have to satisfy $\tilde{\pi}(s)^{1/2}\tilde{\pi}(qaq)\tilde{\pi}(s)^{1/2} = \pi(s^{1/2}as^{1/2}) = \pi^\bullet(a)$, for a in A. We shall use this equation to define $\tilde{\pi}$.

Note that $\pi^\bullet(a_1)\pi^\bullet(a_2) = \pi^\bullet(a_1sa_2)$, for all $a_1, a_2 \in A$. $\pi^\bullet$ is a completely positive normal map and thus extends to a completely positive normal map $\pi^\bullet: A'' \longrightarrow B(H)$ (see for example [13, 10.1.13]). By normality, this extension of $\pi^\bullet$ will share the same property. Let $S = \pi^\bullet(1)$. Note that $\pi^\bullet(q)\pi^\bullet(a) = \pi^\bullet(qsa) = \pi^\bullet(sa) = \pi^\bullet(1)\pi^\bullet(a)$, so by the non-degeneracy of π, $\pi^\bullet(q) = \pi^\bullet(1)$. Also $\pi^\bullet(q^\perp a) = \pi^\bullet(aq^\perp) = 0$ for all a in A''. Again by the non-degeneracy of π we have that S is one-to-one and has dense range.

Now by [8, Lemma 2.2] there is a completely positive map $\tilde{\pi}: A'' \longrightarrow B(H)$ such that $S^{1/2}\tilde{\pi}(a)S^{1/2} = \pi^\bullet(a)$, for all a in A''. By the one-to-oneness of S there is only one such map $\tilde{\pi}$ satisfying the equation. Moreover $\tilde{\pi}$ is automatically normal: for if $\xi, \eta \in H$ then $a \longmapsto \big(\tilde{\pi}(a)S^{1/2}\xi | S^{1/2}\eta\big) = \big(\pi^\bullet(a)\xi|\eta\big)$ is normal and since vectors of the form $S^{1/2}\xi$ are dense in H, we have that $\tilde{\pi}$ is normal. As with $\pi^\bullet$, $\tilde{\pi}$ is supported on the corner $qA''q$. What remains to show is that $\tilde{\pi}$ is a homomorphism extending π.

Since $S^{1/2}\tilde{\pi}(1)S^{1/2} = \pi^\bullet(1) = S$, we have that $\tilde{\pi}(1) = 1$. Also,

$$S^{1/2}\tilde{\pi}(s)S^{1/2} = \pi^\bullet(s) = S^2;$$

thus $\tilde{\pi}(s) = S$. Recalling the property $\pi^\bullet(a_1)\pi^\bullet(a_2) = \pi^\bullet(a_1sa_2)$, for all $a_1, a_2 \in A''$, we have $\tilde{\pi}(a_1sa_2) = \tilde{\pi}(a_1)\tilde{\pi}(s)\tilde{\pi}(a_2)$. Hence $\tilde{\pi}(as) = \tilde{\pi}(a)\tilde{\pi}(s)$. Thus

$$\tilde{\pi}(a_1s^na_2) = \tilde{\pi}(a_1s^{n-1})\tilde{\pi}(s)\tilde{\pi}(a_2) = \cdots = \tilde{\pi}(a_1)\tilde{\pi}(s)^n\tilde{\pi}(a_2).$$

So by writing $s^{1/n}$ as a limit of polynomials we have that

$$\tilde{\pi}(a_1s^{1/n}a_2) = \tilde{\pi}(a_1)\tilde{\pi}(s)^{1/n}\tilde{\pi}(a_2).$$

Now as n tends to infinity, $s^{1/n}a$ tends to a for a in $qA''q$ and $\tilde{\pi}(s)^{1/n}$ tends to 1. Hence $\tilde{\pi}(a_1a_2) = \tilde{\pi}(a_1)\tilde{\pi}(a_2)$ and $\tilde{\pi}$ is a homomorphism as desired. For a in A we have $\pi(s^{1/2}as^{1/2}) = \pi^\bullet(a) = S^{1/2}\tilde{\pi}(a)S^{1/2} = \tilde{\pi}(s^{1/2}as^{1/2})$; thus $\tilde{\pi}$ extends π.

COROLLARY 3.3. *Let $s \in QM(A)_+$ be a positive quasi-multiplier and q its support projection. Then the map $A^\bullet \longrightarrow s^{1/2}As^{1/2}$ is surjective, i.e. $s^{1/2}As^{1/2}$ is closed, if and only if the spectrum of s omits an interval* $(0, \epsilon)$ *for some* $\epsilon > 0$.

PROOF. Consider the map $a \longmapsto s^{1/2}as^{1/2}: A \longrightarrow \overline{s^{1/2}As^{1/2}}$. It has closed range if and only if the second adjoint map in the category of Banach spaces: $A'' \longrightarrow \overline{s^{1/2}As^{1/2}}'' = qA''q$ has closed range. However it is well known (*cf.* [3, Lemma III.2.9]) that $s^{1/2}A''s^{1/2}$ is closed if and only if there is $\epsilon > 0$ such that $(0, \epsilon)$ avoids the spectrum of s.

REMARK 3.4. It was shown in [6, 2.44.b] that if the spectrum of s omits $(0, \epsilon)$ the the kernel projection of s is open.

PROPOSITION 3.5. *Let $s \in QM(A)_+$ be a positive quasi-multiplier and r the kernel projection of s (i.e. the complement of the range projection). Then the map $a \mapsto s^{1/2}as^{1/2}: A^{\bullet} \longrightarrow s^{1/2}As^{1/2}$ is one-to-one if and only if there are no non-zero open sub-projections of r.*

PROOF. If r_1 were an open sub-projection of r then $r_1A''r_1 \cap A$ would be a hereditary subalgebra of A in the kernel of $a \mapsto s^{1/2}as^{1/2}$. So one direction is clear.

Suppose now that $0 \neq a \in A$ and $s^{1/2}as^{1/2} = 0$. Recall that the range projection of an element of A is open. So if $a^*sa = 0$ then $s^{1/2}a = 0$ and thus the range projection of a would be a non-zero open sub-projection of r. If $a^*sa \neq 0$ then the fact that $s^{1/2}(a^*sa)s^{1/2} = 0$ implies that $(a^*sa)^{1/2}s^{1/2} = 0$ and thus the range projection of $(a^*sa)^{1/2}$ is a non-zero open sub-projection of r.

4. **The classification of embeddings.** Suppose X is a Hilbert A-B-bimodule with linking algebra L. We have shown that for t in $QM(X)$ with $\|t\| \leq 1$ there is an embedding f^t of (A, X, B) into L_t. When $t = 0$ we get the original embedding of (A, X, B) into its linking algebra. Our main result is that all embeddings occur this way: given an embedding we can find a unique quasi-multiplier, t, such that the original embedding is equivalent to $f^t: (A, X, B) \longrightarrow L_t$.

LEMMA 4.1. *(i) Let A be a C^*-algebra and s in $QM(A)$; then $\|s\| = \sup\{\|asb\| \mid a, b \in A, \|a\| \leq 1, \|b\| \leq 1\}$.*

(ii) Suppose A is a C^*-algebra and $\{s_\alpha\}$ is a bounded net in A such that for all $a, b \in A$, $\{as_\alpha b\}$ is a norm convergent net in A. Then there is a unique s in $QM(A)$ such that $\lim_\alpha as_\alpha b = asb$ (in norm) for all a, b in A.

(iii) Let X be a Hilbert A-B-bimodule and $t \in QM(X)$, then $\|t\| = \sup\{\|atb\| \mid a \in A, b \in B; \|a\|, \|b\| \leq 1\}$.

(iv) Suppose X is a Hilbert A-B-bimodule and $\{t_\alpha\} \subseteq X$ is a bounded net such that $\forall a \in A$, $b \in B$ $\{at_\alpha b\}$ is a norm convergent net. Then there is a unique t in $QM(X)$ such that $\{at_\alpha b\}$ converges in norm to atb $\forall a \in A$, $\forall b \in B$.

PROOF. (i) Clearly $\|s\| \geq \sup\{\|asb\| \mid \|a\|, \|b\| \leq 1\}$. Let $\{e_\alpha\} \subseteq A$ be an approximate identity. Then $\{e_\alpha\}$ converges to 1 $s(A'', A')$ (see Sakai [21, Definition 1.8.6]). So $e_\alpha s e_\alpha \longrightarrow s$, $s(A'', A')$ ([21, Proposition 1.8.12]). Thus $\|s\| \leq \sup_\alpha \|e_\alpha s e_\alpha\| \leq \sup\{\|asb\| \mid \|a\|, \|b\| \leq 1\}$.

(ii) Let s be a $\sigma(A'', A')$ cluster point of $\{s_\alpha\}$. Then for a, b in A, $\{as_\alpha b\}$ is norm convergent, but has a $\sigma(A'', A')$ convergent subnet converging to asb. Hence $\{as_\alpha b\}$ converges to asb in norm, so $asb \in A$ and thus $s \in QM(A)$. Uniqueness follows from (i).

The proofs of (iii) and (iv) are analogous to (i) and (ii).

PROPOSITION 4.2 (*cf.* AKEMANN AND PEDERSEN [1, PROPOSITION 4.2]). *Let X be a Hilbert A-B-bimodule and $f: (A, X, B) \longrightarrow C$ an embedding. Then there is a unique t in $QM(X)$ such that $f_X(atb) = f_A(a)f_B(b)$ $\forall a \in A$, $b \in B$. Moreover for such a t we have $\|t\| \leq 1$.*

PROOF. By property (ii) of Definition 2.1 $f_A(A)f_B(B) \subseteq f_X(X)$. So choose $\{e_\alpha\} \subseteq A$, $\{f_\beta\} \subseteq B$ approximate identities. Then $\{f_X^{-1}\big(f_A(e_\alpha)f_B(f_\beta)\big)\}$ is a bounded net in X such that for a in A and b in B

$$\{af_X^{-1}\big(f_A(e_\alpha)f_B(f_\beta)\big)b\} = \{f_X^{-1}\big(f_A(ae_\alpha)f_B(f_\beta b)\big)\}$$

is norm convergent to $f_X^{-1}\big(f_A(a)f_B(b)\big)$. So by Lemma 4.1 (iv) there is a unique t in $QM(X)$ such that $\{af_X^{-1}\big(f_A(e_\alpha)f_B(f_\beta)\big)b\}$ converges in norm to atb. Hence $\{f_A(ae_\alpha)f_B(f_\beta b)\}$ converges in norm to $f_X(atb)$. But it also converges to $f_A(a)f_B(b)$. Hence $f_X(atb) = f_A(a)f_B(b)$ for all $a \in A$, $b \in B$. The uniqueness of t follows from Lemma 4.1 (iv). Since $\|atb\| = \|f_X(atb)\| = \|f_A(a)f_B(b)\| \leq \|a\|\,\|b\|$, we see that $\|t\| \leq 1$. ∎

We are now in a position to formulate the main theorem. We have seen in Proposition 2.7 that given t in $QM(X)$ with $\|t\| \leq 1$ we may construct an embedding f^t such that $f_A^t(a)f_B^t(b) = f_X^t(atb)$. Conversely given any embedding we have just shown that there is a unique quasi-multiplier t in $QM(X)$ with $\|t\| \leq 1$ such that $f_X(atb) = f_A(a)f_B(b)$. Our main theorem states that the construction of Proposition 2.7 exhausts all possible embeddings and these two constructions are inverses of each other.

THEOREM 4.3. *Let $f\colon (A,X,B) \longrightarrow C$ be an embedding of the Hilbert A-B-bimodule X into C. Let t in $QM(X)$ be the corresponding quasi-multiplier (constructed in Proposition 4.2). Then there is an isomorphism $\varphi\colon L_t \longrightarrow C$ such that we have a commutative diagram*

$$\begin{array}{ccc} (A,X,B) & \xrightarrow{f^t} & L_t \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} \\ & & C \end{array}$$

PROOF. Let $f = (f_A, f_X, f_B)\colon (A,X,B) \longrightarrow C$ be an embedding and let t in $QM(X)$ be the quasi-multiplier associated to this embedding as in Proposition 4.2. This means that $\|t\| \leq 1$ and $f_A(a)f_B(b) = f_X(atb)$ $\forall a \in A$, $b \in B$. This equation and the fact that AXB is dense in X can also be used to show that

$$(*)\qquad \begin{aligned} f_A(a)f_X(\xi)^* &= f_A(at\xi^*) \\ f_B(b)f_X(\xi) &= f_B(bt^*\xi) \\ f_X(\xi)f_X(\eta) &= f_X(\xi t^*\eta). \end{aligned}$$

Let us denote the embedding $f^t\colon (A,X,B) \longrightarrow L_t = \overline{s^{1/2}Ls^{1/2}}$ by (f_A^t, f_X^t, f_B^t) (recall that $s = \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix}$). If we are to have that $\varphi \circ f_t = f$ then there is only one form φ can take: $\varphi\left(s^{\frac{1}{2}} \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} s^{\frac{1}{2}}\right) = f_A(a) + f_X(\xi) + f_X(\eta)^* + f_B(b)$. The main problem is to show that such a φ exists.

To accomplish this we define $\varphi^\bullet\colon L^\bullet \longrightarrow C$ by $\varphi^\bullet\left(\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}\right) = f_A(a) + f_X(\xi) + f_X(\eta)^* + f_B(b)$. Recall that $L^\bullet$ is the involutive Banach algebra obtained by giving L the new product $\ell_1 \bullet \ell_2 = \ell_1 s \ell_2$ and the new norm $\|\ell\|_s = \|s\| \, \|\ell\|$ (as in Proposition 3.1). Now $\varphi^\bullet$ is clearly well-defined, linear, and $*$-preserving. To show that $\varphi^\bullet$ is a homomorphism; *i.e.*

$$\varphi^\bullet \begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \varphi^\bullet \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \varphi^\bullet \left(\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} \right),$$

we must check the equality of the sixteen terms on the left hand side with the sixteen terms on the right hand side. Using the relations $(*)$ with the relations coming from the fact that f is a morphism of Hilbert C^*-bimodules it is a routine computation to verify that $\varphi^\bullet$ is multiplicative.

Let $\mathrm{Rep}(L^\bullet)$ be the set of $*$-homomorphisms of $L^\bullet$ into some C^*-algebra. Such $*$-homomorphisms are automatically norm decreasing (see Dixmier [11, 1.3.7]). So $\|x\|_* = \sup\{\|\pi(x)\| \mid \pi \in \mathrm{Rep}(L^\bullet)\} \leq \|x\|_s$. Let $I = \{x \mid \|x\|_* = 0\}$. Then $\|\cdot\|_*$ is a norm on the pre-C^*-algebra $L^\bullet/I$ and its completion is $C^*(L^\bullet)$, the enveloping C^*-algebra of $L^\bullet$. In Proposition 3.1 we showed that $\|\ell\|_* = \|s^{\frac{1}{2}} \ell s^{\frac{1}{2}}\|$ for $\ell \in L^\bullet$. So if $[\ell]$ denotes the class of ℓ in $L^\bullet/I$ then $s^{\frac{1}{2}} \ell s^{\frac{1}{2}} \longmapsto [\ell]$ is an isometric $*$-isomorphism of pre-C^*-algebras from $s^{\frac{1}{2}} L s^{\frac{1}{2}}$ to $L^\bullet/I$. As $\varphi^\bullet \in \mathrm{Rep}(L^\bullet)$, we have $\varphi^\bullet(I) = \{0\}$; so $\varphi^\bullet$ descends to a $*$-homomorphism of $L^\bullet/I$ to C. Now $s^{1/2}\ell s^{1/2} \longmapsto [\ell] \longmapsto \varphi^\bullet(\ell)$ is clearly a well defined $*$-homomorphism: $s^{1/2} L s^{1/2} \longrightarrow C$, which sends $s^{1/2} \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} s^{1/2}$ to $\varphi^\bullet \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} = f_A(a) + f_X(\xi) + f_X(\eta)^* + f_B(b)$. This is exactly the map φ we have been seeking; *i.e.* $\varphi\left(s^{1/2} \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} s^{1/2}\right) = \varphi^\bullet \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}$. As noted above $\varphi \circ f^t = f$. So now we must show that φ is one-to-one and onto.

By Proposition 1.13 $\ker(\varphi)$, if non-zero, must intersect either

$$s^{\frac{1}{2}} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}} \quad \text{or} \quad s^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} s^{\frac{1}{2}}.$$

But φ restricted to $s^{\frac{1}{2}} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}}$ is f_A which is one-to-one. So $\ker(\varphi)$ does not meet $s^{\frac{1}{2}} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} s^{\frac{1}{2}}$, and the same argument applies to $s^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} s^{\frac{1}{2}}$.

To show that φ is onto we must show that $\mathrm{im}(\varphi)$, the image of φ, is dense. As $\mathrm{im}(\varphi)$ contains $f_A(A)$ and $f_B(B)$ it will be enough to show that $\mathrm{im}(\varphi)$ is hereditary (see Definition 2.1 (i)). To show that $\mathrm{im}(\varphi)$ is hereditary it will suffice to show that for the subalgebra of C, $C_0 := \varphi\left(s^{\frac{1}{2}} \begin{pmatrix} A & AXB \\ BX^*A & B \end{pmatrix} s^{\frac{1}{2}}\right) = f_A(A) + f_X(AXB) + f_X(BX^*A) + f_B(B)$, $C_0 C C_0 \subseteq C_0$, as C_0 is dense in $\mathrm{im}(\varphi)$.

There are sixteen terms in $C_0 C C_0$. In each case C appears in one of the four forms:

$$f_A(A) C f_A(A)$$
$$f_B(B) C f_B(B)$$
$$f_A(A) C f_B(B)$$
$$f_B(B) C f_A(A).$$

The first two are contained in $f_A(A)$ and $f_B(B)$ respectively as $f_A(A)$ and $f_B(B)$ are assumed to be hereditary. The second two are contained in $f_X(X)$ and $f_X(X^*)$ respectively by Definition 2.1(ii). Hence $\mathrm{im}(\varphi)$ is hereditary and thus φ is an isomorphism.

5. **Concluding Remarks: the relative position of two hereditary subalgebras.** Let X be a Hilbert A-B-bimodule. In Section 2 we showed that to every quasi-multiplier t of X with $\|t\| \leq 1$ one could associate as C^*-algebra, L_t, and an embedding of (A, X, B) into L_t. In Section 4 we showed that, up to isomorphism, every embedding was of this form for an appropriate quasi-multiplier. We now want to consider what happens to this pairing under various equivalence relations. Let us begin by making some definitions.

- ▷ A triple (A, C, B) of C^*-algebras is a *hereditary triple* if A and B are hereditary subalgebras of C and $C = \mathrm{her}(A \cup B)$.
- ▷ Two triples (A_1, C_1, B_1) and (A_2, C_2, B_2) are *isomorphic* if there is an isomorphism $\vartheta\colon C_1 \longmapsto C_2$ such that $\vartheta(A_1) = A_2$ and $\vartheta(B_1) = B_2$.
- ▷ Given a hereditary triple (A, C, B), $X_C = (ACB)^-$ is the Hilbert A-B-bimodule *associated to* C.
- ▷ For X_1 a Hilbert A_1-B_1-bimodule and X_2 a Hilbert A_2-B_2-bimodule, (A_1, X_1, B_1) is *isomorphic* to (A_2, X_2, B_2) if there is a triple of isomorphisms $(\vartheta_A, \vartheta_X, \vartheta_B)$, $\vartheta_A\colon A_1 \longmapsto A_2$, $\vartheta_B\colon B_1 \longmapsto B_2$, and $\vartheta_X\colon X_1 \longmapsto X_2$, such that $\vartheta_A(a)\vartheta_X(\xi)\vartheta_B(b) = \vartheta_X(a\xi b)$, and ϑ_X preserves the inner products.
- ▷ One can see that $(A_1, X_1, B_1) \simeq (A_2, X_2, B_2)$ if and only if there is an isomorphism of linking algebras $\vartheta\colon L_1 = \begin{pmatrix} A_1 & X_1 \\ X_1^* & B_1 \end{pmatrix} \longmapsto L_2 = \begin{pmatrix} A_2 & X_2 \\ X_2^* & B_2 \end{pmatrix}$ such that $\vartheta''\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
- ▷ $\mathrm{Aut}(A, X, B)$ is the set of automophisms of (A, X, B).

We want to consider embeddings of a pair of C^*-algebras (A, B) into a third C which produce a hereditary triple. By an *embedding* we mean a pair (f_A, f_B) of monomorphisms from A and B respectively into C such that $\big(f_A(A), C, f_B(B)\big)$ is a hereditary triple. An embedding of (A, B) always produces a Hilbert A-B-bimodule: $X_f = \big(f_A(A)Cf_B(B)\big)^-$ which is a Hilbert $f_A(A)$-$f_B(B)$-bimobule, and then we obtain a A-B-bimodule by pulling back the actions and inner products via f.

Now fix a pair of C^*-algebras A and B and a Hilbert A-B-bimodule X. Let ${}_A\mathfrak{X}_B$ be the set of hereditary triples (A_1, C_1, B_1) such that $\big(A_1, (A_1C_1B_1)^-, B_1\big) \simeq (A, X, B)$.

THEOREM 5.1. *Two elements, (A_1, C_1, B_1) and (A_2, C_2, B_2), of ${}_A\mathfrak{X}_B$ are isomorphic if and only is there is ϑ in* $\mathrm{Aut}(A, X, B)$ *such that $\vartheta''(t_1) = t_2$ where t_1 and t_2 are the quasi-multipliers corresponding to the two embeddings of* (A, X, B).

PROOF. We have

$$\begin{array}{ccccc} & & (A_1, X_1, B_1) & \subseteq & C_1 \\ & {\scriptstyle f_1}\nearrow & & & \\ (A, X, B) & & & & \big\downarrow {\scriptstyle \vartheta_0} \\ & {\scriptstyle f_2}\searrow & & & \\ & & (A_2, X_2, B_2) & \subseteq & C_2 \end{array}$$

so $f_2(at_2b) = f_2(a)f_2(b) = \vartheta_0\big(f_1(a)f_2(b)\big) = \vartheta_0\big(f_1(at_1b)\big)$. So pulling ϑ_0 back (via f_1 and f_2) to an automorphism ϑ of (A,X,B) we have $\vartheta(at_1b) = \vartheta(a)t_2\vartheta(b)$, hence $\vartheta''(t_1) = t_2$.

Conversely given ϑ in $\mathrm{Aut}(A,X,B)$ with $\vartheta''(t_1) = t_2$ we get an isomorphism of $L^{\bullet_1}$ onto $L^{\bullet_2}$ and hence of L_{t_1} onto L_{t_2}. By Theorem 4.3 $C_1 \simeq L_{t_1}$ and $C_2 \simeq L_{t_2}$, hence $(A_1,C_1,B_1) \simeq (A_2,C_2,B_2)$. ∎

Consider now a second equivalence relation on embeddings of Hilbert A-B-bimodules. We shall say that two embeddings $f^1\colon (A,X,B) \longrightarrow C_1$ and $f^2\colon (A,X,B) \longrightarrow C_2$ are *weakly equivalent* if they satisfy the two conditions (a) and (c) of Definition 2.1 that is

(a) $\theta \circ f_A^1 = f_A^2$, and

(c) $\theta \circ f_B^1 = f_B^2$.

This is exactly the equivalence relation obtained if we consider embeddings of a pair (A,B) instead of a triple (A,X,B) and fix the isomorphism type of X. Reasoning just like that in Theorem 5.1 leads to a simular result with $\mathrm{Aut}(A,X,B)$ replaced by $\mathrm{Aut}(X) = \{\theta \in \mathrm{Aut}(A,X,B) \mid \theta|_A = id_A \text{ and } \theta|_B = id_B\}$. It is easy to see that $\mathrm{Aut}(X)$is unaffected if we replace A and B by I_A and I_B, thus obtaining an imprimitivity bimodule. Then Section 3 of [7] implies that $\mathrm{Aut}(X)$ can be identified with $\mathfrak{Z}U(I_A)$ the set of unitary elements in the centre of $M(I_A)$, or with $\mathfrak{Z}U(I_B)$. Note that there is an isomorphism between $\mathfrak{Z}U(I_A)$ and $\mathfrak{Z}U(I_B)$ such that if u corresponds to v then $ux = xv$ for all x in X.

THEOREM 5.2. *If X is a Hilbert A-B-bimodule, then weak equivalence classes of embeddings of (A,X,B) are in one-to-one correspondence with equivalence classes of elements of $\{t \in QM(X) \mid \|t\| \le 1\}$; where t_1 is equivalent to t_2 if and only if there is u in $\mathfrak{Z}U(I_A)$ such that $t_2 = ut_1$ if and only if there is v in $\mathfrak{Z}U(I_B)$ such that $t_2 = t_1v$.* ∎

To complete this discussion, we compare Theorem 5.1 with the classification of the relative positions of a pair (M,N) of closed subspaces of a Hilbert space H. This classification was first given by Dixmier [10] and Krein, Krasnosel'skiĭ, and Mil'man [15]; see also [9], [12], [18], and [19]. The triples (M_1,N_1,H_1) and (M_2,N_2,H_2) *determine the same relative position* if there is a unitary $U\colon H_1 \longmapsto H_2$ such that $UM_1 = M_2$ and $UN_1 = N_2$. It is harmless, though not completely standard, to impose the extra requirement that $H_i = (M_i + N_i)^-$. We can fix the dimensions of M_i and N_i, which is analogous to fixing the isomorphism type of (A_i,X_i,B_i). Then possible relative positions are in one-to-one corespondence with the equivalence classes of contractions T in $B(N,M)$, where T_1 is *equivalent* to T_2 if there are linear isometries $U\colon M_1 \longmapsto M_2$ and $V\colon N_1 \longmapsto N_2$ such that $T_2 = UT_1V^*$. If p and q are the projections in $B(H)$ corresponding to M and N, then the contraction T which determines the relative position of M and N is pq regarded as an operator from N to M. This is in very close analogy with our theory, since the operator t produced by our Proposition 4.2 is $f_X''^{-1}(p_Cq_C)$, where p_C and q_C are the open projections in C'' corresponding to the hereditary subalgebras $f_A(A)$ and $f_B(B)$. We can obtain explicit formulas for p_C and q_C by using $L_t = (s^{1/2}Ls^{1/2})^-$, where $s = \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix}$. Let

$p_t = s^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s^{1/2}$ and $q_t = s^{1/2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} s^{1/2}$; then p_t and q_t are the open projections in L''_t corresponding to $(s^{1/2}As^{1/2})^-$ and $(s^{1/2}Bs^{1/2})^-$. Then

$$p_tq_t = s^{1/2}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} s^{1/2} = \frac{1}{2}\begin{pmatrix} tt^* & t + t\sqrt{1-t^*t} \\ t^* - t^*\sqrt{1-tt^*} & t^*t \end{pmatrix}$$

using that

$$s^{1/2} = \frac{1}{2}\begin{pmatrix} \sqrt{1+|t^*|} + \sqrt{1-|t^*|} & u\big[\sqrt{1+|t|} - \sqrt{1-|t|}\big] \\ u^*\big[\sqrt{1+|t^*|} - \sqrt{1-|t^*|}\big] & \sqrt{1+|t|} + \sqrt{1-|t|} \end{pmatrix},$$

where $t = u|t|$, and

$$p_t = \frac{1}{2}\left(s + \begin{pmatrix} \sqrt{1-tt^*} & 0 \\ 0 & -\sqrt{1-t^*t} \end{pmatrix}\right)$$

and

$$q_t = \frac{1}{2}\left(s - \begin{pmatrix} \sqrt{1-tt^*} & 0 \\ 0 & -\sqrt{1-t^*t} \end{pmatrix}\right).$$

From this we have

$$\begin{aligned} p_tq_tp_t &= \frac{1}{2}\begin{pmatrix} tt^*(1+\sqrt{1-tt^*}) & tt^*t \\ t^*tt^* & t^*t(1-\sqrt{1-t^*t}) \end{pmatrix} \\ &= \begin{pmatrix} |t^*|^2 & 0 \\ 0 & |t|^2 \end{pmatrix} p_t \\ &= p_t\begin{pmatrix} |t^*|^2 & 0 \\ 0 & |t|^2 \end{pmatrix}. \end{aligned}$$

To verify these equations we have used the following

$$\begin{gathered} \big[\sqrt{1+|t^*|} + \sqrt{1-|t^*|}\big]^2 = 2\big[1 + \sqrt{1-|t^*|^2}\big] \\ \big[\sqrt{1+|t^*|} + \sqrt{1-|t^*|}\big]\big[\sqrt{1+|t^*|} - \sqrt{1-|t^*|}\big] = 2|t^*| \\ \big[\sqrt{1+|t^*|} - \sqrt{1-|t^*|}\big]^2 = 2\big[1 - \sqrt{1-|t^*|^2}\big] \\ u^*\big[1 - \sqrt{1-|t^*|^2}\big]u = \big[1 - \sqrt{1-|t|^2}\big]. \end{gathered}$$

There is however a difference, which is that in the Hilbert space setting one can solve the equivalence relation on contraction operators, thus producing a more explicit classification. (The reader familar with the literature may have already noticed that our description of the relative position of two subspaces is not a usual one). As a final remark on the analogy, we point out that any pair (U, V) of unitaries in $\big(M(A), M(B)\big)$ yields an automorphism $\mathrm{Ad}_{\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}}$ of (A, X, B). Although such automorphisms typically do not exhaust $\mathrm{Aut}(A, X, B)$, they do form a natural subgroup ([7]). Moreover the description of the relative position of two subspaces given above actually follows from Theorem 5.1 in the case where A and B are elementary C^*-algebras, and X is an imprimitivity bimodule.

If X is a Hilbert A-B-bimodule, we define $LM(X)$ the *left multipliers of* X, as $\{t \in X'' \mid tb \in X, \forall b \in B\}$, and (*cf.* 2.4) we have $LM(X) = pLM(L)q$. $RM(L)$ is defined similarly, and $M(X) = LM(X) \cap RM(X)$.

PROPOSITION 5.3. *If $t \in QM(X)$ and $\|t\| \leq 1$, then $f_A^t(A)$ is a corner of L_t if and only if $t \in LM(X)$; $f_B^t(B)$ is a corner of L_t if and only if $t \in RM(X)$.*

PROOF. If $f_A^t(A)$ is a corner, there is a projection r in $M(L_t)$ such that $f_A^t(A) = rL_tr$. Thus $rf_B^t(b) \in rf_B^t(B) \subseteq \left(rL_trf_B^t(B)\right)^- = \left(f_A^t(A)f_B^t(B)\right)^- \subseteq f_X^t(X)$. Then as in the proof of Proposition 4.2, we see that there is t' in $LM(X)$ such that $f_X^t(t'b) = rf_B^t(b)$ for all b in B. It follows from this that $f_X^t(at'b) = f_A^t(a) \cdot r \cdot f_B^t(b) = f_A^t(a)f_B^t(b)$ for all a in A and all b in B. Hence $t' = t$ by the uniqueness in Proposition 4.2.

Conversely, suppose that t is in $LM(X)$, and let $\{e_\alpha\}$ be an approximate identity of A. If $\{f_A^t(e_\alpha)\}$ is convergent in the strict topology of $M(L_t)$, then the limit, r, will be a projection such that $rL_tr = f_A^t(A)$. Since $e_\alpha^* = e_\alpha$, it is enough to show left strict convergence: *i.e.* $\{f_A^t(e_\alpha)l\}$ is norm convergent for l in L_t; and by the construction of L_t, it is enough to show that $\{e_\alpha sx\}$ is convergent in L for x in A, B, X, or X^*. If x is in A or X, then $\{e_\alpha sx\} = \{e_\alpha x\}$, which converges to x. If x is in X^* or B, then $\{e_\alpha sx\} = \{e_\alpha tx\}$. Now, tx is in L, since $t \in LM(X)$, and hence $tx \in A+X = pL$. Therefore $\{e_\alpha tx\}$ converges to tx.

The case of right multipliers follows from taking adjoints. ∎

It it interesting to know about the kernel of the canonical map $x \longmapsto s^{1/2}xs^{1/2}$ from $L^\bullet$ to L_t. Clearly x is in this kernel if and only if $s^{1/2}xs^{1/2} = 0$, and it is not hard to see that $s^{1/2}xs^{1/2} = 0$ if and only if $sxs = 0$. In fact, $sxs = 0$ implies that $g(s)xg(s) = 0$ for any polynomial g such that $g(0) = 0$, and there is a sequence $\{g_n\}$ of such polynomials such that $g_n(s) \longrightarrow s^{1/2}$. Note that the calculation of sxs is just an elementary matrix multiplication.

For the definition and basic properties of open projections and hereditary subalgebras the reader may refer to Pedersen [18, §1.5 and §3.12]. Note that if A and B are hereditary subalgebras of a C^*-algebra C and p and q in C'' are the corresponding open projections then

$$C = \operatorname{her}(A \cup B) \Leftrightarrow p \vee q = 1 \text{ and}$$
$$A \cap B = \{0\} \Leftrightarrow p \wedge q \text{ contains no open subprojections.}$$

THEOREM 5.4. *(i) Let $a \in A$ and $b \in B$, then*

$$f_A^t(a) = f_B^t(b) \Leftrightarrow s\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}s = s\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}s \Leftrightarrow \left\{\begin{matrix} a = tbt^* \\ t^*a = bt^* \\ at = tb \\ t^*at = b \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} b = t^*at \\ tt^*a = a = att^* \end{matrix}\right\}.$$

(ii) Let p_0 be the spectral projection of tt^* for 1, and let $A_1 = \{a \in A \mid f_A^t(a) \in f_B^t(B)\}$. A_1 is a hereditary C^*-subalgebra of A, and if p_1 is its open projection, then $p_1 \leq p_0$, and p_1 is the largest open projection in A'' which is majorized by p_0.

(iii) Let q_0 be the spectral projection of t^*t for 1, and let $B_1 = \{b \in B \mid f_B^t(b) \in f_A^t(A)\}$. B_1 is a hereditary C^*-subalgebra of B, and if q_1 is its open projection, then $q_1 \leq q_0$, and q_1 is the largest open projection in B'' which is majorized by q_0.

(iv) $f_B^t(B) \subseteq f_A^t(A) \Leftrightarrow q_0 = q \Leftrightarrow t^*t = q$, *i.e. t is an 'isometry'*

$f_A^t(A) \subseteq f_B^t(B) \Leftrightarrow p_0 = p \Leftrightarrow tt^* = p$, *i.e. t is a 'co-isometry'*

$f_A^t(A) = f_B^t(B) \Leftrightarrow \begin{Bmatrix} p_0 = p \\ q_0 = q \end{Bmatrix} \Leftrightarrow t^*t = q$ *and* $tt^* = p$, *i.e. t is a 'unitary'.*

(v) Let $t = u|t|$ *be the polar decomposition of t. Then* $uq_0 = p_0u$, *and the projection*

$$r = \frac{1}{2}\begin{pmatrix} p_0 & -p_0uq_0 \\ -q_0u^*p_0 & q_0 \end{pmatrix}$$

is the kernel projection of s.

(vi) The following are equivalent:

(a) The map $a \longmapsto s^{1/2}as^{1/2}\colon L^\bullet \longrightarrow L_t$ *is one-to-one.*

(b) p_0 *contains no non-zero open subprojections.*

(c) q_0 *contains no non-zero open subprojections.*

(d) $f_A^t(A) \cap f_B^t(B) = \{0\}$.

PROOF. (i) is a straightforward computation.

(ii) It is clear that A_1 is hereditary as $f_B^t(B)$ is hereditary. To prove the claim concerning p_1 we need only show that for $a \in A$

$$(*) \qquad tt^*a = a = att^* \Longrightarrow t^*at \in B.$$

Suppose $(*)$ holds and $p' \in A''$ is open and $p' \leq p_0$. Let $\{a_\alpha\}$ be an increasing net in A_+ converging to p'. Then $p_0a_\alpha = a_\alpha = a_\alpha p_0$. By $(*)$ we also have $t^*a_\alpha t \in B$ so $a_\alpha \in A_1$. Hence $a_\alpha p_1 = a_\alpha = a_\alpha p_1$, so $p' \leq p_1$. Now let us prove $(*)$. For an a such that $tt^*a = a = att^*$, we have $at \in LM(X) \subset LM(L)$ and $(at)(at)^* = aa^* \in A \subset L$. Then an argument similar to that in Proposition 4.4 of [1] shows that at is in L; *i.e.* $at \in X$. (Look at $(at)f_\beta(at)^*$, where $\{f_\beta\}$ is an approximate identity of B, and use Dini's Theorem.) Since $X^*X \subset B$, it follows that $t^*a_2^*a_1t \in B$, and since $\{a \in A \mid tt^*a = a = att^*\}$ is a C^*-algebra, this implies that $t^*at \in B$. This establishes $(*)$.

(iii) follows from the same reasoning, and (iv) follows from (ii) and (iii).

Let us prove (v). That $up_0 = p_0u$ is elementary, by direct computation $r = r^* = r^2$ and $sr = 0$. So $r \leq \ker(s)$ the kernel projection of s. To see that r covers the kernel let $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ be in the kernel of s. Then

$$s\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \Rightarrow \begin{Bmatrix} \xi + t\eta = 0 \\ \eta + t^*\xi = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \xi = p_0\xi \\ \eta = q_0\eta \end{Bmatrix} \Rightarrow \begin{Bmatrix} p_0\xi - p_0uq_0\eta = 2\xi \\ q_0\eta - q_0u^*p_0 = 2\eta \end{Bmatrix} \Rightarrow r\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Finally let us prove (vi). By (ii), (b) and (c) are equivalent, and by (iii), (b) and (d) are equivalent. By Proposition 3.5, (a) is equivalent to the assertion that r contains no proper open subprojections. Since $r \leq \begin{pmatrix} p_0 & 0 \\ 0 & q_0 \end{pmatrix}$ a non-zero open subprojection of r would produce a non-zero open subprojection of p_0. Hence (b) implies (a). Obviously (a) implies (d). ∎

We shall need to recall the notion of the *angle* between two subspaces or equivalently the angle between two projections (see [9, §3]). Let M and N be two closed subspaces of a Hilbert space H and p and q the projections onto M and N respectively. It follows from the open mapping theorem that $M+N$ is closed if and only if

$$\inf \frac{\|\xi+\eta\|}{\|\xi\|+\|\eta\|} > 0,$$

or equivalently

$$\sup \frac{|(\xi,\eta)|}{\|\xi\|\,\|\eta\|} < 1,$$

where the infimum and the supremum are taken over pairs ξ and η where $0 \neq \xi \in M \ominus (M\cap N)$ and $0 \neq \eta \in N \ominus (M \cap N)$. It thus makes sense to define the angle ϑ between M and N to be such that

$$\cos(\vartheta) = \sup \frac{|(\xi,\eta)|}{\|\xi\|\,\|\eta\|}$$

or equivalently

$$\sin(\vartheta/2) = \inf \frac{\|\xi+\eta\|}{\|\xi\|+\|\eta\|}.$$

The angle can also be measured from the projections p and q:

$$\begin{aligned}\vartheta &= \sup\{\alpha \mid \sin(\alpha)\|\eta\| \leq \|p^{\perp}\eta\|, \eta \in N\ominus(M\cap N)\}\\ &= \sup\big\{\alpha \bigm| \|p\eta\| \leq \cos(\alpha)\|\eta\|, \eta \in N\ominus(M\cap N)\big\}\\ &= \sup\big\{\alpha \bigm| \big(\cos^2(\alpha), 1\big) \text{ avoids the spectrum of } pqp\big\}.\end{aligned}$$

The equality of the last two quantities follows from Raeburn and Sinclair [19, Lemma 1.8].

THEOREM 5.5. *The following are equivalent:*

(i) The map $x \longmapsto s^{1/2}xs^{1/2}\colon L^{\bullet} \longrightarrow L_t$ *is surjective.*

(ii) $L_t = R_1 + R_2$, *where* R_1 *and* R_2 *are respectively the closed right ideals of* L_t *generated by* $f_A^t(A)$ *and* $f_B^t(B)$.

(iii) There is $\epsilon > 0$ *such that* $(1-\epsilon, 1)$ *does not meet the spectrum of* $|t|$.

(iv) The angle between p_t *and* q_t *is positive.*

PROOF. The equation $\begin{pmatrix}1 & 0\\ t^* & \lambda\end{pmatrix}\begin{pmatrix}\lambda & -t\\ -t^* & \lambda\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & \lambda\end{pmatrix} = \lambda\begin{pmatrix}1 & 0\\ 0 & \lambda^2 - t^*t\end{pmatrix}$ shows that $\mathrm{Sp}(s) = \{1 \pm \mu \mid \mu \in \mathrm{Sp}(|t|) \cup \mathrm{Sp}(|t^*|)\}$. Hence $(1-\epsilon, 1) \cap \mathrm{Sp}(|t|)$ is empty if and only if $(0,\epsilon) \cap \mathrm{Sp}(s)$ is empty. Thus (i) is equivalent to (iii) by Corollary 3.3. Now we know that $p_tq_tp_t = \begin{pmatrix}|t^*|^2 & 0\\ 0 & |t|^2\end{pmatrix} p_t$ so the angle between p_t and q_t is positive if and only if there is $\epsilon > 0$ such that $(1-\epsilon, 1)$ avoids the spectrum of $|t|$ (or of $|t^*|$). Thus (iii) and (iv) are equivalent.

Let us show that (i) and (ii) are equivalent. Note that

$$R_1 = \left(f_A^t(A)L_t\right)^- = \left(s^{1/2}\begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix}s^{1/2}\right)^-$$
$$R_2 = \left(f_B^t(B)L_t\right)^- = \left(s^{1/2}\begin{pmatrix} 0 & 0 \\ X^* & B \end{pmatrix}s^{1/2}\right)^-.$$

So if $x \longmapsto s^{1/2}xs^{1/2}$ is onto then clearly $L_t = R_1 + R_2$. Since $f_A^t(A)$ is hereditary, $f_A^t(A)L_t$ is already closed; for if $c \in \left(f_A^t(A)L_t\right)^-$ then $cc^* \in \left(f_A^t(A)L_tf_A^t(A)\right)^- \subseteq f_A^t(A)$. So $|c|^{1/2} \in f_A^t(A)$ and by Pedersen [18, 1.4.5] there is $b \in L_t$ such that $c = |c|^{1/2}b \in f_A^t(A)L_t$. Similarly $f_B^t(B)L_t$, $L_tf_A^t(A)$, and $L_tf_B^t(B)$ are closed. Suppose $L_t = R_1 + R_2$, then

$$\begin{aligned} L_t &= f_A^t(A)L_t + f_B^t(B)L_t \\ &= L_tf_A^t(A) + L_tf_B^t(B) \quad \text{(by taking adjoints)} \\ &= f_A^t(A)L_tf_A^t(A) + f_A^t(A)L_tf_B^t(B) + f_B^t(B)L_tf_A^t(A) + f_B^t(B)L_tf_B^t(B) \\ &= s^{1/2}Ls^{1/2}. \end{aligned}$$

REMARK 5.6. Let I and J be closed right ideals in a C^*-algebra C, and p and q the corresponding open projections in C''. Then $I+J$ is closed if and only if the angle between p and q is positive. Since the open projections for $I \cap \mathrm{her}(A \cup B)$ and $J \cap \mathrm{her}(A \cup B)$ in $\mathrm{her}(A \cup B)''$ are p and q respectively, it is not hard to see that it is enough to suppose that $I + J$ is dense in C or equivalently that $p \vee q = 1$. Let $A = pC''p \cap C$, $B = qC''q \cap C$, and $X = pC''q \cap C$. Then X is a Hilbert A-B-bimodule and we have an embedding of (A, X, B) into C with associated quasi-multiplier $t = pq \in X''$. By Theorem 4.3 $C \simeq L_t$ and under this correspondence I and J get sent to the ideals R_1 and R_2 respectively of Theorem 5.5. Now $(1 - \epsilon, 1)$ avoids the spectrum of $|t| = \sqrt{qpq}$ if and only if $(1 - \epsilon, 1)$ avoids the spectrum of $|t^*| = \sqrt{pqp}$ if and only if the angle between p and q is positive. So we just have to apply Theorem 5.5.

REMARK 5.7. (a) Let s be a positive quasi-multiplier of a C^*-algebra A, $B = (s^{1/2}As^{1/2})^-$, and $X = (As^{1/2})^-$. Then X is a Hilbert A-B-bimodule, $I_B = B$, and $I_A = (AsA)^-$. It is easy to see that $I_A = A$ if and only if s is not contained in any $I'' \subset A''$ for a proper closed two sided ideal I in A, *i.e.* s has central support 1 in A''. If $I_A = A$ then A is strongly Morita equivalent to B via X. If A is σ-unital and s is one-to-one then A is isomorphic to B. To verify this last remark, let e be a strictly positive element of A. It was shown in [5, Theorem 4.9 and following remark] that $\{(ese)^{1/2}(e^2 + n^{-1})^{-1}\}$ converges strictly to a left multiplier l of A with $l^*l = s$. Moreover if s is one-to-one then l is one-to-one with dense range. Thus $(lAl^*)^- = A$. By Proposition 3.1 the map $s^{1/2}as^{1/2} \longmapsto lal^*: (s^{1/2}As^{1/2})^- \longrightarrow A$ is bounded. Hence B is isomorphic to A.

(b) Let X be a Hilbert A-B-bimodule and $t \in QM(X)$ with $\|t\| \leq 1$. Let $s = \begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix}$. Clearly s is not contained in any I'' for any proper closed two sided ideal of L. So L_t is strongly Morita equivalent to L. If A and B are σ-unital and t does not achieve the norm 1 then L is σ-unital and so L_t and L are isomorphic.

REMARK 5.8. It is possible to generalize the basic theory to the case of n hereditary C^*-algebras or even infinitely many hereditary C^*-algebras. To explain this we first need a couple of definitions. If X is a Hilbert A-B-bimodule and Y is a Hilbert B-C-bimodule, then $Z = X \otimes_B Y$ is a Hilbert A-C-bimoddule. The construction of Z was given in [20, Theorem 5.9] (one can also find this in [2, §13.5]). The basic definitions are

$$
{}_A(x_1 \otimes y_1, x_2 \otimes y_2) = {}_A\big(x_1\, {}_B(y_1, y_2), x_2\big) \text{ and}
$$
$$
(x_1 \otimes y_1, x_2 \otimes y_2)_C = \big(y_1, (x_1, x_2)_B y_2\big)_C.
$$

These inner products define a semi-norm on the algebraic tensor product, and one mods out by the elements of norm 0 and completes to obtain Z. If X and Y are Hilbert A-B-bimodules, then a map $\varphi: X \longrightarrow Y$ will be called a *morphism* if it is a bimodule homomorphism which preserves the inner products. Then φ is an isomorphism of X with a closed sub-bimodule of Y.

For the general embedding problem we are given C^*-algebras $A_1, A_2, A_3\ A_4, \ldots$ and Hilbert A_i-A_j-bimodules X_{ij} such that $X_{ii} = A_i$ (with the obvious bimodule structure) and $X_{ji} = X_{ij}^*$ for $i < j$. We are also given multiplications

$$
\mu_{ijk}: X_{ij} \otimes_{A_j} X_{jk} \longrightarrow X_{ik},
$$

which are morphisms in the sense above, such that μ_{iik} is just the left A_i-module structure of X_{ik} and μ_{ikk} is just the right A_k-module structure of X_{ik}, μ_{iji} is the A_i-valued inner product structure on X_{ij} or X_{ji}, the associative law holds, and $[\mu_{ijk}(x \otimes y)]^* = \mu_{kji}(y^* \otimes x^*)$. An *embedding* of $\{X_{ij}\}$ into a C^*-algebra C is a collection $f = \{f_{ij}\}$, where $f_{ij}: X_{ij} \longrightarrow C$, is such that f_{ii} is a *-isomorphism of A_i onto a hereditary C^*-subalgebra of C, C is hereditarily generated by $\bigcup f_{ii}(A_i), f_{ij}(X_{ij}) = [f_{ii}(A_i)Cf_{jj}(A_j)]^-, f_{ji}(x^*) = [f_{ij}(x)]^*$, and f is a homomorphism for the multiplications μ_{ijk} (this last includes part (iii) of Definition 2.1).

An embedding into the linking algebra always exists. One defines L (or L_n if there are more than n A_i's) as the set of $n \times n$ matrices whose ij-entries are in X_{ij}. Then L is an involutive algebra, and it is given a C^*-norm as in Definition 2.2: one uses the representations $\pi_i: L \longrightarrow \mathcal{L}(\oplus_j X_{ji})$. If there are infinitely many A_i's, L is then defined as the inductive limit of the L_n's.

Given any embedding, Proposition 4.2 yields t_{ij} in $QM(X_{ij})$ such that $t_{ii} = 1 = 1_{A_i}$ and $t_{ji} = t_{ij}^*$. The $n \times n$ matrix (t_{ij}) is then a positive element s, or s_n, of $QM(L)$. The proof that s is positive uses the fact that a matrix (l_{ij}) in L'' is positive if and only if $\Sigma_{ij} x_i^* l_{ij} x_j \geq 0$ for each $x_1, x_2, x_3, \ldots, x_n$ in X_{ik}.

It can be shown that the embeddings are thus classified by the positive quasi-multiplier matrices with 1's on the main diagonal. When there are infinitely many A_i's, s need not be bounded but each s_n is bounded and positive. In this case L_s is the direct limit of the L_{s_n}'s. The main differences between the proof of the general result and the proof for the case $n = 2$ have been sketched, and the details are left to the reader.

REMARK 5.9. There is a theory of the relative position of two closed submodules of a (right) Hilbert C^*-module which closely parallels the theory of the relative position of two subspaces of Hilbert space and the theory of embeddings of Hilbert C^*-bimodules. Since this theory can be easily derived from our main theorem (Theorem 4.3), we shall sketch the argument. If X and Y are right Hilbert A-modules, an *embedding* of (X, Y) into a Hilbert A-module Z is a pair (g, h) such that

(i) $g: X \longrightarrow Z$ and $h: Y \longrightarrow Z$ are (isometric) isomorphisms from X and Y onto closed submodules of Z.

(ii) $[g(X) + h(Y)]^- = Z$

In a natural way $\mathcal{K}(Y, X)$ is a Hilbert $\mathcal{K}(X)$-$\mathcal{K}(Y)$-bimodule, and the linking algebra of the bimodule can be identified with $\mathcal{K}(X \oplus Y)$. Any embedding (g, h) induces an embedding (f_1, f_2, f_3) of $\big(\mathcal{K}(X), \mathcal{K}(Y, X), \mathcal{K}(Y)\big)$ into $\mathcal{K}(Z)$ by the formulas $f_1(\theta_{x_1,x_2}) = \theta_{g(x_1),g(x_2)}, f_2(\theta_{x,y}) = \theta_{g(x),h(y)}$, and $f_3(\theta_{y_1,y_2}) = \theta_{h(y_1),h(y_2)}$. Assumption (ii) above implies that $f_1\big(\mathcal{K}(X)\big) \cup f_3\big(\mathcal{K}(Y)\big)$ hereditarily generates $\mathcal{K}(Z)$ (*cf.* [5, Theorem 2.5]). In addition to the conditions of Definition 2.1, we also have the compatibility conditions: $f_2(T)h(y) = g(Ty)$ and $f_2(T)^*g(x) = h(T^*x)$ for T in $\mathcal{K}(Y, X)$, x in X, and y in Y. By Proposition 4.2 there is t in $QM\big(\mathcal{K}(Y, X)\big)$, with $\|t\| \leq 1$, such that $f_2(btc) = f_1(b)f_3(c)$ for all b in $\mathcal{K}(X)$ and c in $\mathcal{K}(Y)$. Because $[\mathcal{K}(X)X]^- = X$ and $[\mathcal{K}(Y)Y]^- = Y$, it is then routine to check that

$$(*) \qquad \big(g(x)|h(y)\big)_A = x^*ty, \quad \forall x \in X,\ y \in Y,$$

where the multiplication on the right takes place in L'', and L is the linking algebra of $X \oplus Y$.

There is an easy extension of Proposition 3.1. If W is a right Hilbert A-module and s a positive quasi-multiplier in $QM\big(\mathcal{K}(W)\big)$, we can define a new A-valued inner product by $\langle w_1, w_2\rangle_A = w_2^* s w_1$. The Hausdorff completion of $(W, \langle\cdot,\cdot\rangle_A)$ is a right Hilbert A-module which can be identified with $\overline{s^{1/2}W}$. If we apply this with $W = X \oplus Y$ and $s = \begin{pmatrix} 1 & t^* \\ t & 1 \end{pmatrix}$, we see that every t in $QM\big(\mathcal{K}(Y, X)\big)$ with $\|t\| \leq 1$ arises from an embedding (g, h) so that $(*)$ is satisfied, and moreover (g, h) is uniquely determined up to isomorphism. In other words we have the following analogue of Theorem 4.3:

(i) If X and Y are right Hilbert A-modules, then the isomorphism classes of embed-

https://doi.org/10.4153/CJM-1994-065-5 Published online by Cambridge University Press

dings of (X, Y) are in one-to-one correspondence with elements t of $QM\big(\mathcal{K}(Y,X)\big)$ such that $\|t\| \leq 1$. Here (g,h) is isomorphic to (g',h') is there is an isomorphism $\vartheta: Z \longrightarrow Z'$ such that $\vartheta g = g'$ and $\vartheta h = h'$.

We can use a weaker equivalence relation: (g,h) is *equivalent* to (g',h') if there is an isomorphism $\vartheta: Z \longrightarrow Z'$ such that $\vartheta\big(g(X)\big) = g'(X)$ and $\vartheta\big(h(Y)\big) = h'(Y)$. Since the automorphisms of X are in one to one correspondence with the unitaries in $\mathcal{L}(X)$, or in $M\big(\mathcal{K}(X)\big)$, we have the following analogue of Theorem 5.1.

(ii) The equivalence classes of embeddings of (Y, X) are in one-to-one correspondence with the equivalence classes of $\big\{t \in QM\big(\mathcal{K}(Y,X)\big) : \|t\| \leq 1\big\}$, where t is *equivalent* to t' if there are unitaries U, V in $M\big(\mathcal{K}(X)\big)$, $M\big(\mathcal{K}(Y)\big)$, respectively, such that $t' = UtV$.

ACKNOWLEDGEMENTS. Some of the first author's work was done while he was on sabbatical at the University of Copenhagen. The second author would like to thank Professors Claire Anantharaman-Delaroche and Jean Renault at l'Université d'Orléans for making his sabbatical visit possible and for their warm hospitality.

REFERENCES

1. C. Akemann and G. K. Pedersen, *Complications of semi-continuity in C^*-algebra theory*, Duke Math. J. **40**(1973), 785–795.
2. B. Blackadar, *K-theory for Operator Algebras*, Springer-Verlag, New York, 1986.
3. B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Analysis **45**(1982), 297–340.
4. L. G. Brown, *Stable Isomorphism of Hereditary Subalgebras of C^*-algebras*, Pacific J. Math. **71**(1977), 335–348.
5. ———, *Close hereditary C^*-subalgebras and the structure of quasi-multipliers*, M.S.R.I. (1211-85), 1985, preprint.
6. ———, *Semicontinuity and Multipliers of C^*-algebras*, Canad. J. Math. **XL**(1988), 865–988.
7. L. G. Brown, P. Green, and M. A. Rieffel, *Stable Isomorphism and Strong Morita Equivalence of C^*-algebras*, Pacific J. Math. **71**(1977), 349–363.
8. M.-D. Choi and E. G. Effros, *Injectivity and Operator Spaces*, J. Funct. Analysis **24**(1977), 156–209.
9. C. Davis and W. M. Kahane, *The rotation of eigenvalues by a perturbation III*, SIAM J. Numer. Anal. **7**(1970), 1–46.
10. J. Dixmier, *Position relative de deux variétés linéaires fermées dans un espace de Hilbert*, Rev. Sci. **86**(1948), 387–399.
11. ———, *C^*-algebras*, North Holland, Amsterdam, 1977.
12. P. Halmos, *Two Subspaces*, Trans. Amer. Math. Soc. **144**(1969), 381–389.
13. R. V. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. II, Academic Press, Orlando, 1986.
14. G. G. Kasparov, *Hilbert C^*-modules; Theorems of Stinespring and Voiculescu*, J. Operator Theory **4**(1980), 133–150.
15. M. G. Krein, M. A. Krasnosel'skiĭ, and D.P.Mil'man, *Defect Numbers of linear operators in Banach space and some geometrical problems*, Sb. Trudov Inst. Mat. Akad. Nauk Ukr.SSR **11**(1948), 97–112, (in Russian).
16. W. Paschke, *Inner Product Modules over B^*-algebras*, Trans. Amer. Math. Soc. **182**(1973), 443–468.
17. G. K. Pedersen, *C^*-algebras and Their Automorphisms*, Academic Press, London, 1979.
18. ———, *Measure theory for C^*-algebras II*, Math. Scand. **22**(1968), 63–74.
19. I. Raeburn and A. Sinclair, *The C^*-algebra generated by two projections*, Math. Scand. **65**(1989), 278–290.
20. M. A. Rieffel, *Induced Representations of C^*-algebras*, Adv. in Math. **13**(1974), 176–257.

21. S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
22. N.-T. Shen, *Embeddings of Hilbert Bimodules*, Doctoral dissertation, Purdue University, 1982.

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