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Almost inductive limit automorphisms and embeddings into AF-algebras

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Abstract. The crossed product of an AF-algebra by an automorphism, a power of which is approximately inner, is shown to be embeddable into an AF-algebra. The proof uses almost inductive limit automorphisms, i.e. automorphisms possessing a sequence of almost invariant finite-dimensional C^*-subalgebras converging to the given AF-algebra.

In this paper we obtain results concerning the following question: when is the crossed product of an AF-algebra A by an action of $\mathbb{Z}$, isomorphic to a C^*-subalgebra of an AF-algebra? This is a natural question in view of the work concerning the corresponding question for crossed products of commutative C^*-algebras ([**13**], [**14**], [**15**], [**16**]). Moreover, our results provide the part which was missing in a construction due to B. Blackadar ([**1**]) showing that every non-type I C^*-algebra has a non-nuclear C^*-subalgebra.

Our approach uses the fact that such an embedding of the crossed product can be constructed in the case where the automorphism α generating the action of $\mathbb{Z}$ has an increasing sequence of almost invariant finite-dimensional C^*-subalgebras converging to A (such automorphisms will be called almost inductive limits). We prove the existence of an embedding of $A \times_\alpha \mathbb{Z}$ into an AF-algebra in the case when α^n is approximately inner for some $n \geq 1$. In particular if A is an UHF-algebra then any crossed product $A \times_\alpha \mathbb{Z}$ can be embedded into an AF-algebra.

In a preliminary version of the present paper (INCREST-preprint No. 2/1984, April 1984) our main result was obtained under the additional assumption that $K_0(A)\otimes\mathbb{Q}$ be ultrasimplicial in the sense of [**7**]. It is not known ([**9**]) whether this is a restriction. The argument by which ultrasimpliciality can be avoided has been pointed out to us by J. Spielberg (see remark 3.2).

The paper has three sections:

§ 1 deals with preliminaries.

§ 2 considers almost inductive limit automorphisms. There is an obvious analogy with Halmos' quasitriangularity ([**8**]). We show that almost inductive limit automorphisms are approximable by inductive limit automorphisms and use this to prove the embedding result for crossed products by almost inductive limit automorphisms.

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§ 3 contains our results on the embeddability of $A \times_\alpha \mathbb{Z}$ when α^n is approximately inner for some $n \geq 1$. An essential part in the proof is a lemma (lemma 3.1) about periodic pseudo-orbits of finite-dimensional C^*-subalgebras.

Section 1

Throughout this paper A will denote an AF-*algebra with unit.* By a nest of finite-dimensional C^*-subalgebras of A, we shall mean an increasing sequence

$$\mathbb{C}1 = A_0 \subset A_1 \subset A_2 \subset \cdots$$

of finite-dimensional C^*-subalgebras such that $A = \bigcup_{n \geq 0} A_n$.

All C^*-algebras considered will be separable and unital, and C^*-subalgebras will always be assumed to contain the unit of the bigger algebra. The set of all finite-dimensional C^*-subalgebras of A will be denoted by $\mathcal{F}(A)$.

The $*$-homomorphisms will be assumed to be unit-preserving and for a $*$-homomorphism $\psi: B \to C$ we shall denote by $\psi_*: K_0(B) \to K_0(C)$ the corresponding homomorphism between the K_0-groups. For a unitary u in a C^*-algebra, Ad u will denote the corresponding inner automorphism.

If C_1, C_2 are C^*-subalgebras of C and $\varepsilon > 0$, we shall write $C_1 \subset^\varepsilon C_2$ if

$$\sup\{\inf\{\|x-y\| \mid y \in C_2, \|y\| \leq 1\} \mid x \in C_1, \|x\| \leq 1\} < \varepsilon,$$

and $d(C_1, C_2)$ is defined by the formula

$$d(C_1, C_2) = \inf\{\varepsilon > 0 \mid C_1 \overset{\varepsilon}{\subset} C_2 \text{ and } C_2 \overset{\varepsilon}{\subset} C_1\}.$$

We shall frequently use the following standard approximation result:

If C_1, C_2 are C^*-subalgebras of C, C_1 is finite-dimensional and $\varepsilon > 0$ then there is $\delta > 0$ depending only on ε and the dimension of C_1 such that:

$$C_1 \overset{\delta}{\subset} C_2 \Rightarrow \exists u \in C \text{ unitary such that } \operatorname{Ad} u(C_1) \subset C_2 \text{ and } \|u - 1\| < \varepsilon.$$

There is also a much stronger approximation result due to E. Christensen (Thm. 6.4 in [**2**], for the history of such results see [**12**]) which we shall also need:

If C_1, C_2 are C^*-subalgebras of C, C_1 is finite-dimensional, $0 \leq \gamma < 10^{-4}$ and $C_1 \subset^\gamma C_2$, then there is a unitary $u \in C$ such that Ad $u(C_1) \subset C_2$ *and* $\|u - 1\| < 64\gamma^{\frac{1}{2}}$.

Another fact we will need is that:

If B is a finite-dimensional C^*-algebra and $\varphi_1, \varphi_2: B \to A$ are $*$-homomorphisms (A is AF), *then* $\varphi_1 = \operatorname{Ad} u \circ \varphi_2$ *if and only if we have* $\varphi_{1*} = \varphi_{2*}$ (see lemma 7.7 in [**5**]).

We also recall that an automorphism α of A is called *approximately inner* if it is a point-norm limit of inner automorphisms Ad u and (A being AF) this is equivalent to the requirement $\alpha_* = \mathrm{id}_*$ (see [**6**]).

Section 2

2.1. *Definition.* An automorphism α of A is called an *almost inductive limit automorphism* if there exists a nest of finite-dimensional C^*-subalgebras $(A_n)_{n \geq 0}$ of A, such that $\lim_{n \to \infty} d(\alpha(A_n), A_n) = 0$.

2.2. *Definition.* An automorphism α of A is called an *inductive limit automorphism* if there exists a nest of finite-dimensional C^*-subalgebras $(A_n)_{n\geq 0}$ of A, such that $\alpha(A_n) = A_n$.

It is easily seen that the following is an alternative definition of almost-inductive-limit automorphisms:

α is *an almost-inductive limit automorphism* if for every $B \in \mathcal{F}(A)$ and $\varepsilon > 0$, we can find $C \in \mathcal{F}(A)$ such that $B \subset^{\varepsilon} C$ and $d(\alpha(C), C) < \varepsilon$.

It is also easy to see that if $(A_n)_{n\geq 0}$ is a nest of finite-dimensional C^*-subalgebras then $d(\text{Ad } u(A_n), A_n) \to 0$ and hence:

α is *an almost inductive limit automorphism* if and only if $\text{Ad } u \circ \alpha$ is an almost-inductive limit automorphism.

2.3. Proposition. *If α is an almost inductive limit automorphism of A and $\varepsilon > 0$, then there is a unitary $u \in A$ such that* $\text{Ad } u \circ \alpha$ *is an inductive limit automorphism and* $\|u - 1\| < \varepsilon$.

Proof. Let $(B_n)_{n\geq 0}$ be a fixed nest of finite-dimensional C^*-subalgebras of A. We shall construct inductively C^*-subalgebras A_n, automorphisms α_n and unitaries u_n for $n \geq 0$, so that:

$A_0 = \mathbb{C}1, \quad u_0 = 1, \ \alpha_0 = \alpha;$

$A_n \subset A_{n+1}, \ \text{Ad } u_{n+1} \circ \alpha_n = \alpha_{n+1}, \ \|u_{n+1} - 1\| < \varepsilon \cdot 2^{-n-1};$

$B_j \subset^{(n+1)^{-1}} A_{n+1} \quad \text{and} \quad \alpha_{n+1}(A_j) = A_j$ for $0 \leq j \leq n+1$.

Clearly, having constructed A_n, u_n, α_n the proof of the theorem will be concluded since $u = \lim_{n\to\infty} u_n \cdots u_0$ is then well defined, $\|u - 1\| < \varepsilon$ and the nest $(A_n)_{n\geq 0}$ is invariant for $\text{Ad } u \circ \alpha$.

Thus assume we have found A_j, α_j, u_j with the desired properties for $0 \leq j \leq n$ and let us show that we can find A_{n+1}, α_{n+1}, u_{n+1}. Let

$$\gamma = \left(\frac{\varepsilon}{(1000(n+1))^{n+2}}\right)^{2^{n+1}}$$

and assume $\varepsilon < 10^{-4}$ which is no loss of generality. Since α_n is an almost inductive limit automorphism, we can find $A_{n+1} \in \mathcal{F}(A)$ such that $A_{n+1} \supset A_n$, $A_{n+1} \supset^{(n+1)^{-1}} B_j$ for $0 \leq j \leq n+1$ and $d(\alpha_n(A_{n+1}), A_{n+1}) < \gamma_0$. By Christensen's theorem there is a unitary $v_0 \in A$ so that:

$$(\text{Ad } v_0 \circ \alpha_n)(A_{n+1}) = A_{n+1} \quad \text{and} \quad \|v_0 - 1\| < \gamma_1 = 2(n+1)10^2 \gamma_0^{\frac{1}{2}}.$$

Putting $\gamma_j = (2(n+1)10^2)^j \gamma_0^{1/2^j}$, we shall find inductively unitaries $v_j \in A_{n+2-j}$ $(1 \leq j \leq n)$ such that:

$$(\text{Ad } v_j \circ \cdots \circ \text{Ad } v_0 \circ \alpha_n)(A_{n+1-j}) = A_{n+1-j} \qquad (1 \leq j \leq n)$$

and $\|v_j - 1\| < \gamma_{j+1}$.

Indeed, assume we have found v_j for $j \leq k$, then

$$d((\text{Ad } v_k \circ \cdots \circ \text{Ad } v_0 \circ \alpha_n)(A_{n-k}), A_{n-k}) \leq 2\|v_k \cdots v_0^{-1}\| \leq 2(\gamma_1 + \cdots + \gamma_{k+1})$$
$$\leq 2(n+1)\gamma_{k+1}.$$

$$(\text{Ad } v_k \circ \cdots \circ \text{Ad } v_0 \circ \alpha_n)(A_{n-k}) \subset (\text{Ad } v_k \circ \cdots \circ \text{Ad } v_0 \circ \alpha_n)(A_{n+1-k}) = A_{n+1-k},$$

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and hence by Christensen's theorem there is $v_{k+1} \in A_{n+1-k}$ such that

$$(\mathrm{Ad}\, v_{k+1} \circ \cdots \circ \mathrm{Ad}\, v_0 \circ \alpha_n)(A_{n-k}) = A_{n-k},$$

and

$$\|v_{k+1} - 1\| < 10^2(2(n+1)\gamma_{k+1})^{\frac{1}{2}} \le \gamma_{k+2},$$

which concludes the proof of the existence of the v_j's $(1 \le j \le n)$. Defining $u_{n+1} = v_n \cdots v_0$, we have

$$\|u_{n+1} - 1\| \le \gamma_1 + \cdots + \gamma_{n+1} \le (n+1)\gamma_{n+1} < \varepsilon \cdot 2^{-n}$$

and for $0 \le j \le n$

$$(\mathrm{Ad}\, u_n \circ \alpha_n)(A_{n+1-j}) = (\mathrm{Ad}\, v_n \circ \cdots \circ \mathrm{Ad}\, v_{j+1})(A_{n+1-j}) = A_{n+1-j}. \qquad \square$$

2.4. *Definition.* An automorphism α of A is called *limit periodic* if there is a nest $(A_n)_{n \ge 0}$ of finite-dimensional C^*-subalgebras and a sequence of positive integers $(d_n)_{n\ge 0}$, $d_0 = 1$ such that $\alpha(A_n) = A_n$ $(n \ge 0)$ and $(\alpha | A_n)^{d_n} = \mathrm{id}\, A_n$ $(n \ge 0)$. A sequence $(d_n)_{n \ge 0}$ with the above properties is called *a sequence of periods for* α.

2.5. PROPOSITION. *If α is an inductive limit automorphism of A with respect to a given nest $(A_n)_{n\ge 0}$ of finite dimensional C^*-subalgebras and $\varepsilon > 0$, then there is a unitary u such that* $\mathrm{Ad}\, u \circ \alpha$ *is limit periodic with respect to the nest $(A_n)_{n \ge 0}$ and $\|u - 1\| < \varepsilon$.*

Proof. Let m_n be the least positive integer such that $(\alpha | A_n)^{m_n}$ is an inner automorphism of A_n. Then we can find a sequence of integers $(d_n)_{n\ge 0}$ such that $d_0 = 1$, $d_n | d_{n+1}$, $m_n^2 | d_n$ and $\sum_{n \ge 1} d_n^{-1} < \varepsilon/10$. We shall construct inductively unitaries $u_n \in A_n$ and automorphisms α_n of A so that:

$\alpha_0 = \alpha, \quad u_0 = 1;$

$\alpha_{n+1} = \mathrm{Ad}\, u_{n+1} \circ \alpha_n, \quad u_{n+1} \in A_n' \cap A_{n+1};$

$\|u_{n+1} - 1\| < 2\pi d_{n+1}^{-1};$

$\alpha_{n+1}(A_j) = A_j, \quad (\alpha_{n+1} | A_{n+1})^{d_{n+1}} = \mathrm{id}_{A_{n+1}} \quad (n \ge 0, j \ge 0).$

Assume u_j, α_j have been constructed for $0 \le j \le n$ and let us show how one constructs u_{n+1}, α_{n+1}. Since $\alpha_n | A_{n+1}$ and $\alpha | A_{n+1}$ differ by an inner automorphism of A_{n+1}, we have that $(\alpha_n | A_{n+1})^{m_{n+1}} = \mathrm{Ad}\, v$ for some $v \in A_{n+1}$. Then we have $\alpha_n(v) = \gamma v$ where $\gamma \in Z(A_{n+1})$ the centre of A_{n+1} and

$$\gamma \alpha_n(\gamma) \cdots \alpha_n^{m_{n+1}-1}(\gamma) = 1,$$

since $\alpha_n^{m_{n+1}}(v) = \mathrm{Ad}\, v(v) = v$. It is easily seen that we can find then a unitary $\delta \in Z(A_{n+1})$ such that $(\gamma\delta^{-1}\alpha_n(\delta))^{m_{n+1}} = 1$. But then for $\tilde{v} = \delta v$ and $\tilde{\gamma} = \gamma(\delta^{-1}\alpha_n(\delta))$ we have

$$(\alpha_n | A_{n+1})^{m_{n+1}} = \mathrm{Ad}\, \tilde{v} \quad \text{and} \quad \alpha_n(\tilde{v}) = \tilde{\gamma}\tilde{v}$$

where $\tilde{\gamma} \in Z(A_{n+1})$, $\tilde{\gamma}^{m_{n+1}} = 1$. Hence for $w = \tilde{v}^{d_{n+1}/m_{n+1}}$ we have

$$(\alpha_n | A_{n+1})^{d_{n+1}} = \mathrm{Ad}\, w \quad \text{and} \quad \alpha_n(w) = \tilde{\gamma}^{d_{n+1}/m_{n+1}} w = w,$$

since $m_{n+1}^2 | d_{n+1}$.

Since $d_n | d_{n+1}$ we have $(\mathrm{Ad}\, w | A_n) = (\alpha_n^{d_{n+1}} | A_n) = \mathrm{id}_{A_n}$ and hence $w \in A_n' \cap A_{n+1}$.

Let g be a Borel function such that, defining $u_{n+1} = g(w)$, we have

$$u_{n+1}^{d_{n+1}} = w^{-1} \quad \text{and} \quad \|u_{n+1} - 1\| < 2\pi d_{n+1}^{-1}.$$

Then $u_{n+1}\in A'_n\cap A_{n+1}$, $\alpha_n(u_{n+1})=u_{n+1}$ and hence

$$\alpha_n\circ \mathrm{Ad}\, u_{n+1}=\mathrm{Ad}\, u_{n+1}\circ\alpha_n.$$

For $\alpha_{n+1}=\mathrm{Ad}\, u_{n+1}\circ\alpha_n$ we have

$$\alpha_{n+1}(A_j)=(\mathrm{Ad}\, u_{n+1}\circ\alpha_n)(A_j)=\mathrm{Ad}\, u_{n+1}(A_j)=A_j;$$

$$(\alpha_{n+1}|A_{n+1})^{d_{n+1}}=(\mathrm{Ad}\, u_{n+1}|A_{n+1})^{d_{n+1}}\circ(\alpha_n|A_{n+1})^{d_{n+1}}$$
$$=(\mathrm{Ad}\, w^{-1}|A_{n+1})\circ(\mathrm{Ad}\, w|A_{n+1})=\mathrm{id}_{A_{n+1}}.$$

This concludes the proof of the existence of the (u_n, α_n) for all $n\in\mathbb{N}$.

Now $u=\lim u_n\cdots u_1$ is well-defined and has the desired properties. □

2.6. COROLLARY. *If α is an almost inductive limit automorphism and $\varepsilon>0$ then there is a unitary u such that* $\mathrm{Ad}\, u\circ\alpha$ *is limit periodic and* $\|u-1\|<\varepsilon$.

We pass now to embedding the crossed product of A by the action of $\mathbb{Z}$ generated by α into an AF-algebra. We shall denote this crossed product by $A\times_\alpha\mathbb{Z}$.

2.7. THEOREM. *If α is an almost inductive limit automorphism of A, then $A\times_\alpha\mathbb{Z}$ can be embedded into $A\otimes B$, where B is an* UHF-*algebra.*

The theorem follows by corollary 2.6 and the following lemma.

2.8. LEMMA. *Let α be a limit periodic automorphism of A and $(d_n)_{n\geq 0}$ a sequence of periods for α such that* $\lim_{n\to\infty} d_n=\infty$. *Then $A\times_\alpha\mathbb{Z}$ can be embedded into $A\otimes B$ where B is the* UHF-*algebra* $\varinjlim \mathscr{L}(\mathbb{C}^{d_n})$.

Proof. Replacing the nest $(A_n)_{n\geq 0}$ with respect to which α is limit periodic by a subnest we may assume that $\sum d_n^{-1}<\infty$.

Consider in $B_n=\mathscr{L}(\mathbb{C}^{d_n})$ the matrix units $(e(n;i,j))_{1\leq i,j\leq d_n}$ corresponding to the canonical basis $(e_k^{(n)})_{1\leq k\leq d_n}$ of $\mathbb{C}^{d_n}$ and consider the unitary

$$s_n=e(n;1,2)+\cdots+e(n;d_n-1,d_n)+e(n;d_n,1).$$

We define injective $*$-homomorphisms $\rho_n: A_n\to A_n\otimes B_n$ by

$$\rho_n(x)=\sum_{1\leq j\leq d_n}\alpha^j(x)\otimes e(n;j,j),$$

so that

$$\rho_n(\alpha(x))=(\mathrm{Ad}\,(1\otimes s_n))(\rho_n(x)).$$

Next, we consider some special embeddings $j_n: A_n\otimes B_n\to A_{n+1}\otimes B_{n+1}$, $k_n: B_n\to B_{n+1}$ so that

$$j_n(x\otimes b)=x\otimes k_n(b)\quad\text{and}\quad k_n(b)=\sum_{1\leq j\leq m_n}W_{n,j}bW^*_{n,j},$$

where $m_n=d_{n+1}/d_n$ and the isometries $W_{n,j}:\mathbb{C}^{d_n}\to\mathbb{C}^{d_{n+1}}$ are given by the formulae

$$W_{n,j}e_s^{(n)}=\frac{1}{\sqrt{m_n}}\exp\left(\frac{2\pi sji}{d_{n+1}}\right)\sum_{0\leq k\leq m_n-1}\exp\left(\frac{2\pi jki}{m_n}\right)e^{(n+1)}_{s+kd_n}.$$

It is easily seen that the $W_{n,j}$'s $(1\leq j\leq m_n)$ have pairwise orthogonal ranges and it is easy to check that

$$W_{n,j}s_n=\exp\left(\frac{2\pi ji}{d_{n+1}}\right)s_{n+1}W_{n,j}$$

so that

$$k_n(s_n) = s_{n+1} \sum_{1 \le j \le m_n} \exp\left(\frac{2\pi j i}{d_{n+1}}\right) W_{n,j} W_{n,j}^*.$$

This implies

$$\|k_n(s_n) - s_{n+1}\| \le \frac{2\pi}{d_n},$$

and hence

$$\|j_n(1 \otimes s_n) - 1 \otimes s_{n+1}\| \le \frac{2\pi}{d_n}.$$

Since $\alpha | A_n$ has order d_n it is easy to check that the diagram

$$\begin{array}{ccc} A_n \otimes B_n & \xrightarrow{j_n} & A_{n+1} \otimes B_{n+1} \\ \uparrow{\scriptstyle \rho_n} & & \uparrow{\scriptstyle \rho_{n+1}} \\ A_n & \hookrightarrow & A_{n+1} \end{array}$$

is commutative.

Consider D the inductive limit of the $(A_n \otimes B_n, j_n)$ and $\varphi_n : A_n \otimes B_n \to D$, $\rho : A \to D$ the corresponding injective $*$-homomorphisms so that the diagram

$$\begin{array}{ccc} A_n \otimes B_n & \xrightarrow{\varphi_n} & D \\ \uparrow{\scriptstyle \rho_n} & & \uparrow{\scriptstyle \rho} \\ A_n & \hookrightarrow & A \end{array}$$

is commutative. Then $u = \lim_{n \to \infty} \varphi_n(1 \otimes s_n)$ is well-defined and $\rho(\alpha(a)) = u\rho(a)u^*$ for $a \in A$.

Since the C^*-algebra generated by $\rho_n(A_n)$ and $1 \otimes s_n$ is the crossed product $A_n \times_{\alpha_n} \mathbb{Z}/d_n\mathbb{Z}$ there is an automorphism $\beta_n(\zeta)$ of this algebra, if $\zeta^{d_n} = 1$, such that $\beta_n(\zeta) \circ \rho_n = \rho_n$ and $\beta_n(\zeta)(1 \otimes s_n) = \zeta(1 \otimes s_n)$. Hence if $\zeta^{d_n} = 1$ for some n, then there is an automorphism $\beta(\zeta)$ of the C^*-algebra generated by $\rho(A)$ and u such that $\beta(\zeta) \circ \rho = \rho$ and $\beta(\zeta)(u) = \zeta u$. Since $d_n \to \infty$ passing to the limit we see that $\beta(\zeta)$ exists for arbitrary ζ with $|\zeta| = 1$ and hence that the covariant representation $(\rho(A), u)$ gives an embedding of $A \times_\alpha \mathbb{Z}$ into D by Lanstad's imprimitivity theorem.

It is also obvious that D is isomorphic to $A \otimes B$. □

2.9. *Remark*. The construction of the approximate intertwinings $W_{n,j}$ in the proof of the preceding lemma is related to a construction in [**13**]. We could have also used the construction in [**14**], but the corresponding formulae would have been cumbersome.

Section 3

We shall prove in this section that certain automorphisms of AF-algebras are almost inductive limits. In view of theorem 2.8 this will yield results about embeddings of crossed products into AF-algebras.

3.1. LEMMA. *Let α be an automorphism of A and n a positive integer such that α^n is approximately inner. Then, given $D \in \mathcal{F}(A)$ and $m \in \mathbb{N}$ such that $n|m$, there are $B_j \in \mathcal{F}(A)$ $(1 \le j \le m)$ such that*

$$d(\alpha(B_j), B_{j+1}) < \frac{5\pi}{m} \qquad \text{for } 1 \le j \le m-1,$$

$$d(\alpha(B_m), B_1) < \frac{5\pi}{m},$$

and $B_j \supset D$ for $1 \le j \le m$.

Proof. Given $\varepsilon > 0$, we can find $B, C \in \mathcal{F}(A)$ and unitaries $u_0 = 1, u_1, \ldots, u_n, u \in A$ so that

$$(\mathrm{Ad}\, u_j \circ \alpha^j)(C) \supset D, \quad \|u_j - 1\| < \varepsilon \qquad \text{for } 0 \le j \le m$$

and

$$B \supset C, \quad (\mathrm{Ad}\, u \circ \alpha^{-m})(B) \supset C, \quad \|u - 1\| < \varepsilon.$$

Since α^m is approximately inner there is a unitary $t \in A$ so that $\mathrm{Ad}\, t|C = \alpha^m|C$. Note also that we may choose B so that there is a unitary $t' \in B$ with $\|t - t'\| < \varepsilon$. Again by the approximate innerness of α^m there is a unitary w_1 such that $\mathrm{Ad}\, u \circ \alpha^{-m}|B = \mathrm{Ad}\, w_1|B$. It follows that

$$\mathrm{Ad}\, w_1^* \circ \mathrm{Ad}\, u \circ \alpha^{-m}|B = \mathrm{id}|B$$

and hence for $x \in C \subset (\mathrm{Ad}\, u \circ \alpha^{-m})(B)$ we have

$$\begin{aligned}\|\mathrm{Ad}\, w_1^*(x) - \mathrm{Ad}\, t'(x)\| &= \|(\alpha^m \circ \mathrm{Ad}\, u^*)(x) - \mathrm{Ad}\, t'(x)\| \\ &< 2\varepsilon\|x\| + \|(\alpha^m \circ \mathrm{Ad}\, u^*)(x) - \mathrm{Ad}\, t(x)\| \\ &= 2\varepsilon\|x\| + \|\mathrm{Ad}\, u^*(x) - x\| < 4\varepsilon\|x\|.\end{aligned}$$

Since $\mathrm{Ad}\, w_1^*(x) \in B$ and $\mathrm{Ad}\, t'(x) \in B$ for $x \in C$, we infer that for $\varepsilon < \frac{1}{4}$, $\mathrm{Ad}\, w_1^*|C = (\mathrm{Ad}\, t'' \circ \mathrm{Ad}\, t')|C$ with $t'' \in B$. Thus there is $w_2 = t''t' \in B$ so that $w_2 x w_2^* = w_1^* x w_1$ for $x \in C$. Then for $w = w_1 w_2$ we have

$$\mathrm{Ad}\, w(B) = \mathrm{Ad}\, w_1(B) = (\mathrm{Ad}\, u \circ \alpha^{-m})(B) \quad \text{and} \quad \mathrm{Ad}\, w|C = \mathrm{id}_C,$$

or equivalently $w \in C' \cap A$.

Choosing a unitary $v \in C' \cap A$ such that

$$\|v^m - w\| < \varepsilon \quad \text{and} \quad \|v - 1\| < 2\pi/m$$

we define

$$B_j = (\mathrm{Ad}\, u_j \circ \alpha^j \circ \mathrm{Ad}\, v^j)(B), \qquad \text{for } 0 \le j \le m.$$

We have

$$\begin{aligned}d(\alpha(B_j), B_{j+1}) &= d(\mathrm{Ad}\, \alpha(u_j) \circ \alpha^{j+1} \circ \mathrm{Ad}\, v^j)(B), (\mathrm{Ad}\, u_{j+1} \circ \alpha^{j+1} \circ \mathrm{Ad}\, v^{j+1})(B)) \\ &< 4\varepsilon + d((\alpha^{j+1} \circ \mathrm{Ad}\, v^j)(B), (\alpha^{j+1} \circ \mathrm{Ad}\, v^{j+1})(B)) \\ &= 4\varepsilon + d(B, \mathrm{Ad}\, v(B)) \le 4\varepsilon + 4\pi/m,\end{aligned}$$

where $0 \le j \le m-1$.

Moreover

$$\begin{aligned} d(B_m, B_0) &= d((\mathrm{Ad}\, u_m \circ \alpha^m \circ \mathrm{Ad}\, v^m)(B), B) \\ &\le 2\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, v^m)(B), B) \\ &\le 4\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, w)(B), B) \\ &= 4\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, u \circ \alpha^{-m})(B), B) \end{aligned}$$

so that

$$d(\alpha(B_m), B_1) \le 6\varepsilon + d(\alpha(B_0), B_1) \le 10\varepsilon + (4\pi/m).$$

We also have

$$B_j \supset (\mathrm{Ad}\, u_j \circ \alpha^j \circ \mathrm{Ad}\, v^j)(C) = (\mathrm{Ad}\, u_j \circ \alpha^j)(C) \supset D.$$

Thus, taking $\varepsilon < (\pi/10m)$, $B_1, \ldots, B_m$ will have the desired properties. □

3.2. *Remark.* In our initial version of lemma 3.1 we required additionally that $K_0(A)$ be ultrasimplicial, this hypothesis being used in the proof of the existence of the unitary w_2. We thank J. Spielberg (Berkeley) for the nice argument given above for the existence of w_2, involving the unitaries t' and t''. This has removed the ultrasimpliciality assumption in lemma 3.1 and hence also in the main result of the present paper (theorem 3.6).

For the next lemma it will be convenient to consider for an automorphism the condition ΨO (ΨO = pseudo-orbits).

(ΨO) Given $\varepsilon > 0$ and $D \in \mathcal{F}(A)$ there is $m \in \mathbb{N}$ and there are B_j, $j \in \mathbb{Z}/m\mathbb{Z}$ such that $B_j \supset D$ and $d(\alpha(B_j), B_{j+1}) < \varepsilon$ for all $j \in \mathbb{Z}/m\mathbb{Z}$.

Consider also $\mathcal{U}$ the UHF-algebra

$$\mathcal{U} = \bigotimes_{k \ge 1} \mathcal{L}(\mathbb{C}^k)$$

and the automorphism $\sigma = \bigotimes_{k\ge 1} \mathrm{Ad}\, s_k$ where s_k is the cyclic permutation of the canonical basis of $\mathbb{C}^k$.

3.3. LEMMA. *If α satisfies ΨO then $\alpha \otimes \sigma$ is an almost inductive limit automorphism of $A \otimes \mathcal{U}$.*

Proof. Let $(e(k; i, j))_{1 \le i, j \le k}$ be the canonical matrix units in $\mathcal{L}(\mathbb{C}^k)$, so that

$$s_k = e(k; 1, 2) + \cdots + e(k; k-1, k) + e(k; k, 1).$$

Consider also

$$E(k; i, j) = I_{\mathbb{C}^1} \otimes \cdots \otimes I_{\mathbb{C}^{k-1}} \otimes e(k; i, j) \otimes I_{\mathbb{C}^k} \otimes \cdots \in \mathcal{U}.$$

Let $\mathcal{U}_k$ be the C^*-subalgebra of $\mathcal{U}$:

$$\mathcal{U}_k = \left(\bigotimes_{1 \le j \le k} \mathcal{L}(\mathbb{C}^j) \right) \otimes I_{\mathbb{C}^{k+1}} \otimes \cdots.$$

Note also that an ε-pseudo-orbit of length m, yields ε-pseudo-orbits of lengths md for every $d \ge 1$, so that there is no loss of generality in condition ΨO to require that the length m of the pseudo-orbit be greater than a given number.

Thus there is a nest of C^*-subalgebras $(D_j)_{j\ge 1}$ of A and there are integers $1 < m_1 < m_2 < \cdots$ such that there are $1/k$-pseudo-orbits $(B_j^{(k)})_{1 \le j \le m_k}$ with $B_j^{(k)} \supset D_k$.

Let

$$X_k = \sum_{1\le j\le m_k} B_j^{(k)} \otimes E(m_k; j, j) \in \mathcal{F}(A\otimes \mathcal{U}),$$

so that $X_k \supset D_k \otimes 1_{\mathcal{U}}$.

If $x \in X_k$, $\|x\| \le 1$, then $x = \sum_{1\le j\le m_k} x_j \otimes E(m_k; j, j)$ and there are $y_j \in B_j^{(k)}$ such that

$$\|y_1 - \alpha(x_2)\| < 1/k, \ldots, \quad \|y_{m_k-1} - \alpha(x_{m_k})\| < 1/k, \quad \|y_{m_k} - \alpha(x_1)\| < 1/k.$$

Thus for $y = \sum_{1\le j\le m_k} y_j \otimes E(m_k; j, j)$ we have

$$\|y - (\alpha\otimes\sigma)x\| < 1/k.$$

This shows that $(\alpha\otimes\sigma)(X_k) \subset^{1/k} X_k$. Note that $\mathcal{U}_n$ for $n < m_k$ is in the commutant of X_k. Thus for a sequence $n_1 \le n_2 \le \cdots$ such that $\lim_{j\to\infty} n_j = \infty$, $n_k < m_k$, dim $\mathcal{U}_{n_k} < k^{\frac{1}{2}}$ we have that

$$(\alpha\otimes\sigma)(X_k\mathcal{U}_{n_k}) \overset{k^{-\frac{1}{2}}}{\subset} X_k\mathcal{U}_{n_k}$$
$$X_k\mathcal{U}_{n_k} \supset D_k \otimes \mathcal{U}_{n_k}.$$

Since $X_k\mathcal{U}_{n_k} \in \mathcal{F}(A\otimes\mathcal{U})$ this implies $d((\alpha\otimes\sigma)(X_k\mathcal{U}_{n_k}), X_k\mathcal{U}_{n_k}) \to 0$ and the fact that $\alpha\otimes\sigma$ is an almost inductive limit. □

3.4. *Remark.* It is easily seen that in the preceding lemma if, instead of passing from α to $\alpha\otimes\sigma$, we had required that α satisfy some Rohlin-type condition, like the one in [**11**], then by a slight adaptation the same kind of proof would have shown that α is an almost inductive limit automorphism. This applies also to proposition 3.5.

3.5. PROPOSITION. *If α^n is approximately inner for some $n \ge 1$, then $\alpha\otimes\sigma$ is an almost inductive limit automorphism.*

Proof. Use lemma 3.1 and lemma 3.3. □

3.6. THEOREM. *If α^n is approximately inner for some $n \ge 1$, then $A \times_\alpha \mathbb{Z}$ can be embedded into $A\otimes\mathcal{U}$.*

Proof. $A \times_\alpha \mathbb{Z}$ embeds into $(A\otimes\mathcal{U}) \times_{\alpha\otimes\sigma} \mathbb{Z}$, which in turn, because of proposition 3.5 and of theorem 2.7 can be embedded into $A\otimes\mathcal{U}$. □

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