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CERTAIN SUBSETS OF PRODUCTS OF METACOMPACT SPACES AND SUBPARACOMPACT SPACES ARE REALCOMPACT

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We will say that a space X has property $(*)$ if and only if each discrete subset of X is realcompact; i.e., the cardinality of each discrete subset of X is nonmeasurable. In [**8**], Shirota shows that a completely regular T_1-space X is realcompact if and only if X has property $(*)$ and X is complete with respect to some uniformity. In [**7**], Moran, using measure theoretic techniques, shows that any normal metacompact T_1-space with property $(*)$ is realcompact.

Let $\mathscr{M}$ denote the class of T_3-spaces which are either metacompact or subparacompact. It is the purpose of this paper to establish the following result:

THEOREM. *The normal space X is realcompact if and only if X has property $(*)$ and X can be embedded as a closed subspace in the product of a collection of members of $\mathscr{M}$.*

Recall that a space X is *realcompact* if it can be embedded as a closed subspace in the product of a collection of copies of the reals. For basic theorems and notation concerning realcompact spaces and ultrafilters, the reader is referred to [**3**].

The space X is said to be *metacompact* if whenever $\mathscr{U}$ is an open cover of X, there is a point finite open refinement of $\mathscr{U}$

A space is said to be *subparacompact* if whenever $\mathscr{U}$ is an open cover of X, there is a refinement $\mathscr{H} = \bigcup_{i=1}^{\infty} \mathscr{H}_i$ of $\mathscr{U}$ that covers X such that, for each i, $\mathscr{H}_i$ is a discrete collection of closed sets. The name "subparacompact" is due to Burke [**1**]. In [**5**], McAuly shows that every semi-metric space is subparacompact and in [**2**] Creede shows that every semi-stratifiable space is subparacompact.

It should be pointed out that the normality condition of our Theorem cannot be removed since the space Ψ given in Exercise 51 of [**3**] is a completely regular Moore space that is not realcompact.

For convenience, if $\mathscr{E}$ is a class of spaces and X is a space, then $C(X, \mathscr{E})$ will denote the class of continuous functions on X with range in $\mathscr{E}$. If f is a function, then Range(f) will denote the range of f.

LEMMA 1. *Suppose that X is a T_1-space and $\mathscr{E}$ is a class of T_3-spaces such that the topology on X is the weak topology induced by $C(X, \mathscr{E})$. Then X can be*

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embedded as a closed subspace in the product of a collection of members of $\mathscr{E}$ *if if and only if it is true that, if* $\mathscr{F}$ *is a free ultrafilter of closed subsets of* X*, then there is a member* f *of* $C(X, \mathscr{E})$ *and an open cover* $\mathscr{U}$ *of* Range(f) *such that* $\{f^{-1}(U) : U \in \mathscr{U}\}$ *refines* $\{(X - F) : F \in \mathscr{F}\}$.

Proof. Suppose that $\mathscr{E}'$ is a collection of members of $\mathscr{E}$ and T is a homeomorphism taking X onto a closed subspace of $\prod \mathscr{E}'$. For each $E \in \mathscr{E}'$, P_E will denote the projection of $\prod \mathscr{E}'$ onto E; thus, for each $E \in \mathscr{E}'$, $P_E \cdot T$ is in $C(X, \mathscr{E})$. Let $\mathscr{F}$ be a free ultrafilter of closed subsets of X. For each E in $\mathscr{E}'$, let $\mathscr{F}_E$ denote the set, $\{H : H$ is a closed subset of E and $(P_E \cdot T)^{-1}(H) \in \mathscr{F}\}$.

Claim 1. If $E \in \mathscr{E}'$, then there is at most one point in $\bigcap \mathscr{F}_E$: since E is regular, if there were two points p and q in $\bigcap \mathscr{F}_E$, then there would be open sets U and V in E containing p and q respectively such that $U^- \cap V^- = \emptyset$; thus, $(P_E \cdot T)^{-1}(U^-)$ and $(P_E \cdot T)^{-1}(V^-)$ would be disjoint members of $\mathscr{F}$, which would be impossible.

Claim 2. Suppose that $E \in \mathscr{E}'$ and x is a point of E such that if U is an open set in E that contains x, then U^- is in $\mathscr{F}_E$. Then x is in each member of $\mathscr{F}_E$. To see that this is true, suppose the contrary; i.e., suppose that there is a member H of $\mathscr{F}_E$ that does not contain x. By the regularity of E, there is an open set U containing x such that $U^- \cap H = \emptyset$, but this is impossible since $\mathscr{F}_E$ is centered.

Claim 3. There is a member E of $\mathscr{E}'$ such that $\bigcap \mathscr{F}_E = \emptyset$: suppose otherwise. Then there is a point x of $\prod \mathscr{E}'$ such that, for each E, $P_E(x) = \bigcap \mathscr{F}_E$. We will show that x is a limit point of $T(X)$; and so, $T^{-1}(x)$ will be a member of $\mathscr{F}$, which will be a contradiction. To this end, let $\mathscr{E}''$ be a finite subset of $\mathscr{E}'$ and for each E in $\mathscr{E}''$, let U_E be an open set in E containing $P_E(x)$. It will suffice to show that there is a point y of X such that $P_E(T(y))$ is in U_E for each E in $\mathscr{E}''$. For each E in $\mathscr{E}''$, let V_E be an open set in E containing $P_E(x)$ such that $V_E^- \subset U_E$. Then

$$\bigcap_{E \in \mathscr{E}''} (P_E \cdot T)^{-1}(V_E^-) \neq 0;$$

and so, let y be a point of this intersection. It must be the case that $P_E(T(y))$ is in U_E for each E in $\mathscr{E}''$.

Let E be a member of $\mathscr{E}'$ such that $\bigcap \mathscr{F}_E = \emptyset$. According to Claim 2, there is an open cover $\mathscr{U}$ of E such that $\{(P_E \cdot T)^{-1}(U^-) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$, which completes this part of the proof.

Suppose now that it is true that: if $\mathscr{F}$ is a free ultrafilter of closed sets in X, then there are a member f of $C(X, \mathscr{E})$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let $\mathscr{E}'$ be a collection of members of $\mathscr{E}$ such that : (*i*) $C(X, \mathscr{E}')$ determines the topology on X and (ii) if $\mathscr{F}$ is a free ultrafilter of closed sets in X, then there are a member f of $C(X, \mathscr{E}')$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let T be the natural embedding of X into

$$\prod_{f \in C(X, \mathscr{E}')} \text{Range}(f)$$

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given by $P_f(T(x)) = f(x)$ (see the Embedding Lemma, [**4**, p. 116]). Let y be a limit point of $T(X)$ in

$$\{\prod_{f \in C(X, E')} \operatorname{Range}(f)\} - T(X).$$

Let $\mathscr{F}' = \{f^{-1}(V^-) : V$ is an open set in Range (f) containing $P_f(y), f \in C(X, \mathscr{E}')\}$. Then $\mathscr{F}'$ is a centered collection in X. Since y is not in $T(X)$, it must be the case that $\bigcap \mathscr{F}' = \emptyset$. Let $\mathscr{F}$ be an ultrafilter of closed sets in X that contains $\mathscr{F}'$. $\mathscr{F}$ must be free; and so, there is a member f of $C(X, \mathscr{E}')$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let V be an open set in Range(f) containing $P_f(y)$ such that V^- is a subset of some member of $\mathscr{U}$. Then $f^{-1}(V^-)$ is not in $\mathscr{F}$, which is a contradiction from which the lemma follows.

LEMMA 2 [**6**, Theorem 18]. *If $\mathscr{U}$ is an open cover of the space X, then there is a discrete subset H of X such that*

(1) *$\{st(x, \mathscr{U}) : x \in H\}$ covers X*

and

(2) *No member of $\mathscr{U}$ contains two points of H.*

In the following theorem, $\mathscr{M}_1$ will denote the class of regular metacompact T_1-spaces; $\mathscr{M}_2$ will denote the class of regular subparacompact T_1-spaces; and $\mathscr{M}_3 = \mathscr{M}_1 \cup \mathscr{M}_3$.

THEOREM. *The following conditions on a normal T_1-space are equivalent:*

(1) *X is realcompact.*

(2) *X has property $(*)$ and X can be embedded as a closed subspace in the product of a collection of members of $\mathscr{M}_3$.*

(3) *X has property $(*)$ and if $\mathscr{F}$ is a free ultrafilter of closed subsets of X, then there is a point finite open cover of X that refines $\{X - F : F \in \mathscr{F}\}$.*

Proof. (1) $\Rightarrow$ (2): This is obvious since every closed subset of a realcompact space is realcompact and the real line is a member of $\mathscr{M}_3$.

(2) $\Rightarrow$ (1): Let $\mathscr{Z}$ be a free z-ultrafilter in X. We will show that X is realcompact by showing that there is a countable subcollection of $\mathscr{Z}$ with empty intersection.

Let $\mathscr{F}$ be an ultrafilter of closed subsets of X that contains $\mathscr{Z}$. Since X can be embedded as a closed subset of the product of a collection of members of $\mathscr{M}_3$, by Lemma 1, there is a member M of $\mathscr{M}_3$, a map f taking X onto M, and an open cover $\mathscr{U}$ of M such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$.

Case 1. M is in $\mathscr{M}_1$: Let $\mathscr{U}'$ be a point finite open refinement of $\mathscr{U}$ that covers M. Applying Lemma 2, there is a discrete subset H of X such that

(i) $\{st(x, f^{-1}(\mathscr{U}')) : x \in H\}$ covers X and

(ii) no member of $f^{-1}(\mathscr{U}')$ contains two points of H.

Let $\mathscr{W}$ be the subcollection of $f^{-1}(\mathscr{U}')$ consisting of all W which contain a point of H. Now, since $\mathscr{W}$ is both point finite and infinite, it follows that

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the cardinality of $\mathscr{W}$ is the same as the cardinality of H; and so, there is a one-to-one function φ from H to $\mathscr{W}$. For each F in $\mathscr{F}$, let $H(F) = \{x \in H : \varphi(x) \cap F = \emptyset\}$. Clearly, $\{H(F) : F \in \mathscr{F}\}$ has the finite intersection property; and so, we may let $\mathscr{K}$ denote an ultrafilter of subsets of H that contains $\{H(F) : F \in \mathscr{F}\}$. Since, for each $x \in H$ there is an F_x in $\mathscr{F}$ such that $\varphi(x) \cap F_x = \emptyset, \mathscr{K}$ must be a free ultrafilter in H. Thus, since H is realcompact, there is a countable subcollection $\{K_1, K_2, \ldots\}$ of $\mathscr{K}$ with $\bigcap K_i = \emptyset$.

Claim 1. If $K \in \mathscr{K}$, then there is a member F of $\mathscr{F}$ such that $F \subset \bigcup_{x \in K}(\varphi(x))$: To see that this is so, suppose otherwise; i.e., suppose that there is a K in $\mathscr{K}$ such that $X - \bigcup_{x \in K}(\varphi(x))$ is in $\mathscr{F}$. Then $H(X - \bigcup_{x \in K} (\varphi(x)))$ is a member of $\mathscr{K}$ that fails to intersect K which is a contradiction and hence Claim 1 follows.

Claim 2. $\bigcap_{i=1}^{\infty}((\bigcup_{x \in K_i}(\varphi(x))) = \emptyset$: To see that this is true, let $y \in X$ and observe that there is a finite subset S of H such that $\varphi(x)$ contains y if and only if $x \in S$. Thus, since $\bigcap K_i = \emptyset$, there is an integer n such that $K_n \cap S = \emptyset$. For that n, it must be the case that $\bigcup_{x \in K_n}(\varphi(x))$ does not contain y.

Now, By Claim 1, for each integer n there is a member F_n of $\mathscr{F}$ such that $F_n \subset \bigcup_{x \in K_n}(\varphi(x))$. Since X is normal, for each n there is a zero-set, Z_n, in X such that $F_n \subset Z_n \subset \bigcup_{x \in K_n}(\varphi(x))$. $\{Z_1, Z_2, \ldots\}$ is a subcollection of $\mathscr{Z}$ with $\bigcap Z_i = \emptyset$.

Case 2. M is in M_2: Let $\mathscr{H} = \bigcup_{i=1}^{\infty}\mathscr{H}_i$ be a closed cover of M that refines $\mathscr{U}$ such that, for each $i, \mathscr{H}_i$ is a discrete collection of sets. For each n, let $\mathscr{K}_n = \{f^{-1}(H) : H \in \mathscr{H}_n\}$ and let $\mathscr{K} = \bigcup_{n=1}^{\infty}\mathscr{K}_n$. Then for each $n, \mathscr{K}_n$ is a discrete collection of closed sets in X. Furthermore, if n is a positive integer and $K \in \mathscr{K}_n$, then K is not in $\mathscr{F}$.

We have two cases to consider:

Subcase 1. There is an integer n such that $\bigcup K_n$ is in $\mathscr{F}$: Let A be an indexing set for $\mathscr{H}_n$ so that we have $\mathscr{H}_n = \{H(a) : a \in A\}$. We will view A as a topological space with the discrete topology. For each a in A, $K(a)$ will denote the set $f^{-1}(H(a))$ so that A is also an indexing set for $\mathscr{K}_n$. If $B \subset A$, then let $K(B) = \bigcup_{a \in B}K(a)$ and $H(B) = \bigcup_{a \in B}H(a)$. Let $\mathscr{B} = \{B \subset A: K(B) \in \mathscr{F}\}$. It is clear that $\mathscr{B}$ has the finite intersection property. Furthermore, since it is true that if $B \subset A$, then either $K(B) \in \mathscr{F}$ or $K(A - B) \in \mathscr{F}$, it follows that $\mathscr{B}$ is an ultrafilter of subsets of A. Note that if $a \in A$, then $K(\{a\})$ is not in $\mathscr{F}$. Thus, $\mathscr{B}$ is a free ultrafilter in A. Since A has the same cardinality as a discrete subset of X, it follows that A is realcompact; and so, there is a countable subset $\{B_1, B_2, \ldots\}$ of $\mathscr{B}$ with $\bigcap B_i = \emptyset$. It follows that $\{K(B_1), K(B_2), \ldots\}$ is a countable subset of $\mathscr{F}$ with empty intersection and $\{H(B_1), H(B_2), \ldots\}$ is also a centered collection of closed subsets of M with empty intersection. It is true then that $\{M - H(B_1), M - H(B_2), \ldots\}$ is a countable open cover of X. Since M is subparacompact, M is countably metacompact; thus, there is a point finite open cover $\{U_1, U_2, \ldots\}$ of M such that, for each k, $U_k \subset (M - H(B_k))$. For each k, let $V_k = \bigcup_{j>k}U_j$. For each k, it follows that $\bigcap_{j=1}^{k}H(B_j) \subset f^{-1}(V_k)$. Since X is normal, there is, for each k,

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a zero-set Z_k such that $\bigcap_{j=1}^{k} K(B_j) \subset Z_k \subset f^{-1}(V_k)$. Thus, $\{Z_1, Z_2, \ldots\}$ is a countable subset of $\mathscr{Z}$ with $\bigcap Z_i = \emptyset$.

Subcase 2. For each n, it is true that $\bigcup \mathscr{K}_n$ is not in $\mathscr{F}$. In this case, for each n, let F_n be a member of $\mathscr{F}$ that does not intersect $\bigcup \mathscr{K}_n$. Since X is normal, there is, for each n, a zero-set Z_n such that $F_n \subset Z_n \subset X - \bigcup \mathscr{K}_n$. It follows that $\{Z_1, Z_2, \ldots\}$ is a countable subcollection of $\mathscr{Z}$ with empty intersection. This completes the proof that (2) implies (1).

(1) $\Rightarrow$ (3): This is a corollary to Lemma 1.

(3) $\Rightarrow$ (1): The argument for this implication is the same as that for Case 1 in the proof that (2) implies (1).

COROLLARY 1. *Every normal subparacompact T_1-space with property* (∗) *is realcompact.*

COROLLARY 2. (Moran [**7**]). *Every normal metacompact T_1-space with property* (∗) *is realcompact.*

In view of our Theorem, two questions seem to be of interest:

Question 1. Is every normal subparacompact T_1-space complete with respect to a uniformity on the space? Indeed, must every normal Moore space be complete with respect to some uniform structure?

Question 2. Must every metacompact normal T_1-space be complete with respect to some uniform structure on the space?

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