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Betti Numbers and Flat Dimensions of Local Cohomology Modules

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Abstract. Assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, and X is an R-module. In this paper, we first study the finiteness of Betti numbers of local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$. Then we give some inequalities between the Betti numbers of X and those of its local cohomology modules. Finally, we present many upper bounds for the flat dimension of X in terms of the flat dimensions of its local cohomology modules and an upper bound for the flat dimension of $\mathrm{H}^i_{\mathfrak{a}}(X)$ in terms of the flat dimensions of the modules $\mathrm{H}^j_{\mathfrak{a}}(X)$, $j \neq i$, and that of X.

1 Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity and s, t are two non-negative integers. We use the symbols $\mathfrak{a}$, X, and M as follows: $\mathfrak{a}$ denotes an ideal of R; X is an arbitrary R-module that is not necessarily finite (*i.e.,* finitely generated), and M is used for a finite R-module. The i-th local cohomology module of X with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(X)$. For a prime ideal $\mathfrak{p}$ of R, the numbers

$$\beta_i(\mathfrak{p}, X) = \dim_{\kappa(\mathfrak{p})}\big(\mathrm{Tor}_i^{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), X_{\mathfrak{p}})\big) \text{ and } \mu^i(\mathfrak{p}, X) = \dim_{\kappa(\mathfrak{p})}\big(\mathrm{Ext}^i_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), X_{\mathfrak{p}})\big)$$

are known as the i-th Betti number and the i-th Bass number, respectively, of X with respect to $\mathfrak{p}$, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. When R is local with maximal ideal $\mathfrak{m}$, we write $\beta_i(X) = \beta_i(\mathfrak{m}, X)$, $\mu^i(X) = \mu^i(\mathfrak{m}, X)$ and $k = R/\mathfrak{m}$. For basic results, notation, and terminology not given in this paper, the reader is referred to [2, 3, 11].

Section 2 is devoted to the study of finiteness of Betti numbers of local cohomology modules. We first discuss the Artinianness of local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$, extension modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$, and torsion modules $\mathrm{Tor}_i^R(R/\mathfrak{a}, X)$. In Theorem 2.1 (resp. Corollary 2.2), we observe that if $\mathrm{H}^i_{\mathfrak{a}}(X)$ (resp. $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$) is Artinian for all i, then $\mathrm{Tor}_i^R(R/\mathfrak{a}, X)$ is also Artinian for all i. In Corollary 2.3, we conclude that for a prime ideal $\mathfrak{p}$ of R, that $\beta_i(\mathfrak{p}, X)$ is finite for all i whenever $\mu^i(\mathfrak{p}, X)$ is finite for all i. As applications, we prove that if $\mathfrak{a}$ is principal or satisfies $\dim R/\mathfrak{a} \leq 1$, then all Betti numbers of the modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ with respect to any prime ideal of R are finite (Corollaries 2.4 and 2.5).

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The main ideas of Section 3 come from the article [7] by Dibaei and Yassemi, in which it is shown that there are some inequalities between the Bass numbers of a module and its local cohomology modules. Although one may expect some consistency for the Betti numbers, the similarities are far from obvious. For the local case, it was shown in [7, Theorem 2.1] that

$$\mu^t(X) \le \sum_{i=0}^{t} \mu^{t-i}(\mathrm{H}^i_{\mathfrak{a}}(X)), \tag{1.1}$$

while we show in Theorem 3.1 that

$$\beta_t(X) \le \sum_{i=0}^{\operatorname{ara}(\mathfrak{a})} \beta_{t+i}(\mathrm{H}^i_{\mathfrak{a}}(X)), \tag{1.2}$$

where $\operatorname{ara}(\mathfrak{a})$ denotes the arithmetic rank of the ideal $\mathfrak{a}$. Formula (1.1) about Bass numbers is of some interest, because Kawasaki [10, Corollary 3] and Delfino and Marley [4, Corollary 2] proved that all Bass numbers of the modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ with respect to any prime ideal are finite, where $\mathfrak{a}$ is principal or satisfies $\dim R/\mathfrak{a} \le 1$. Similarly, by our Corollaries 2.4 and 2.5, formula (1.2) is also interesting in the same situation. In [7, Theorem 2.6], it was also proved that

$$\mu^s(\mathrm{H}^t_{\mathfrak{a}}(X)) \le \sum_{i=0}^{t-1} \mu^{s+t-i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \mu^{s+t}(X) + \sum_{i=t+1}^{s+t-1} \mu^{s+t-i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)).$$

We prove, in Theorem 3.2, that

$$\beta_s(\mathrm{H}^t_{\mathfrak{a}}(X)) \le \sum_{i=0}^{t-1} \beta_{s-t+i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \beta_{s-t}(X) + \sum_{i=t+1}^{\operatorname{ara}(\mathfrak{a})} \beta_{s-t+i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)).$$

As an application, we find in Corollary 3.3 that if R is a local ring and X is an R-module such that $\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all $i \ne n$, *e.g.*, X is finite and $\mathfrak{a}$ is generated by an X-regular sequence of length n, then $\beta_i(\mathrm{H}^n_{\mathfrak{a}}(X)) = \beta_{i-n}(X)$ for all i.

In Section 4, we study the flat dimensions of local cohomology modules. In Corollary 4.1, we present many upper bounds for the projective dimension of M in terms of the flat dimensions of its local cohomology modules and, in Corollary 4.3, an upper bound for the flat dimension of $\mathrm{H}^t_{\mathfrak{a}}(X)$ in terms of the flat dimensions of the modules $\mathrm{H}^i_{\mathfrak{a}}(X)$, $i \ne t$, and that of X. In Corollary 4.6, we conclude that $\operatorname{fd}_R(\mathrm{H}^{\dim_R(M)}_{\mathfrak{m}}(M)) = \operatorname{fd}_R(M) + \dim_R(M)$ (resp. $\operatorname{fd}_R(\mathrm{H}^{\dim_R(M)}_{\mathfrak{m}}(M)) = \operatorname{depth}(R)$, $\operatorname{fd}_R(\mathrm{H}^{\dim(R)}_{\mathfrak{m}}(R)) = \dim(R)$, $\operatorname{fd}_R(E(k)) = \dim(R)$) if R is local with maximal ideal $\mathfrak{m}$ and M is Cohen–Macaulay (resp. M is Cohen–Macaulay with finite projective dimension, R is Cohen–Macaulay, R is Gorenstein).

2 Finiteness of Betti Numbers of Local Cohomology Modules

Since $\operatorname{Ext}^i_R(R/\mathfrak{a}, -)$, $\operatorname{Tor}^R_i(R/\mathfrak{a}, -)$, and $\mathrm{H}^i_{\mathfrak{a}}(-)$ are the most important functors in homological algebra, we first find out the relations between Artinianness of modules $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$, $\operatorname{Tor}^R_i(R/\mathfrak{a}, X)$, and $\mathrm{H}^i_{\mathfrak{a}}(X)$, and use them in the study of finiteness of Betti numbers of local cohomology modules.

Theorem 2.1 *Suppose that X is an arbitrary R-module such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is Artinian for all i. Then $\operatorname{Tor}^R_i(R/\mathfrak{a}, X)$ is also Artinian for all i.*

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Proof Set $c = \operatorname{ara}(\mathfrak{a})$ and $n = i + c$ where i is a non-negative integer. By [6, Lemma 2.1], there is a first quadrant spectral sequence

$$E^2_{p,q} := \operatorname{Tor}^R_p\big(R/\mathfrak{a}, \operatorname{H}^{c-q}_{\mathfrak{a}}(X)\big) \underset{p}{\Longrightarrow} \operatorname{Tor}^R_{p+q-c}(R/\mathfrak{a}, X).$$

For all r, $0 \le r \le n$, we have $E^{\infty}_{r,n-r} = E^{n+2}_{r,n-r}$, since $E^j_{r+j,n-r-j+1} = 0 = E^j_{r-j,n-r+j-1}$ for all $j \ge n+2$; so that $E^{\infty}_{r,n-r}$ is Artinian from the fact that $E^{n+2}_{r,n-r}$ is a subquotient of the Artinian R-module $E^2_{r,n-r} = \operatorname{Tor}^R_r(R/\mathfrak{a}, \operatorname{H}^{r-i}_{\mathfrak{a}}(X))$.

There exists a finite filtration

$$0 = \phi^{-1}H_n \subseteq \phi^0 H_n \subseteq \cdots \subseteq \phi^{n-1}H_n \subseteq \phi^n H_n = \operatorname{Tor}^R_i(R/\mathfrak{a}, X)$$

such that $E^{\infty}_{r,n-r} = \phi^r H_n/\phi^{r-1}H_n$ for all r, $0 \le r \le n$. Now the short exact sequences

$$0 \longrightarrow \phi^{r-1}H_n \longrightarrow \phi^r H_n \longrightarrow E^{\infty}_{r,n-r} \longrightarrow 0,$$

for all r, $0 \le r \le n$, show that $\operatorname{Tor}^R_i(R/\mathfrak{a}, X)$ is Artinian. ■

Corollary 2.2 *Suppose that X is an arbitrary R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is Artinian for all i. Then $\operatorname{Tor}^R_i(R/\mathfrak{a}, X)$ is also Artinian for all i.*

Proof This follows from [9, Proposition 2.6] (or [1, Proposition 3.3]) and Theorem 2.1. ■

The following corollary is our first result about the finiteness of Betti numbers of an R-module with respect to a prime ideal of R.

Corollary 2.3 *Suppose that X is an arbitrary R-module and that $\mathfrak{p}$ is a prime ideal of R such that $\mu^i(\mathfrak{p}, X)$ is finite for all i. Then $\beta_i(\mathfrak{p}, X)$ is also finite for all i.*

Proof It follows from Corollary 2.2. ■

In the following corollaries, for a finite R-module M, we find out when the Betti numbers of local cohomology modules $\operatorname{H}^j_{\mathfrak{a}}(M)$ with respect to all prime ideals of R are finite.

Corollary 2.4 *Let $\mathfrak{a}$ be a principal ideal of R and let M be a finite R-module. Then $\beta_i(\mathfrak{p}, \operatorname{H}^j_{\mathfrak{a}}(M))$ is finite for all integers i, j, and all prime ideals $\mathfrak{p}$.*

Proof Follows from [10, Corollary 3] and Corollary 2.3. ■

Corollary 2.5 *Let $\mathfrak{a}$ be an ideal of R with $\dim(R/\mathfrak{a}) \le 1$ and let M be a finite R-module. Then $\beta_i(\mathfrak{p}, \operatorname{H}^j_{\mathfrak{a}}(M))$ is finite for all integers i, j, and all prime ideals $\mathfrak{p}$.*

Proof This follows from [4, Corollary 2] and Corollary 2.3. ■

Let R be a local ring, let M be a finite R-module, and let r be a non-negative integer. In [5, Definition 4.4], we denoted $\mathcal{L}^r(M)$ as the set of ideals

$$\{\mathfrak{b} : \operatorname{H}^j_{\mathfrak{b}}(M) \text{ is not Artinian for some } j \ge r\}$$

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which is empty for all $r \geq \dim_R(M)$ and is non-empty for all $r < \dim_R(M)$ by [5, Corollary 4.2]. Note that from [5, Theorem 4.7 (ii)], each maximal element $\mathfrak{p}$ of the non-empty set $\mathcal{L}^r(M)$ is a prime ideal.

Corollary 2.6 *Assume that R is a local ring, M is a finite R-module, and r is a non-negative integer such that $r < \dim_R(M)$. Then for each maximal element $\mathfrak{p}$ of the non-empty set $\mathcal{L}^r(M)$, $\beta_i(\mathrm{H}^j_{\mathfrak{p}}(M))$ is finite for all i and all $j \geq r$.*

Proof It follows from [5, Theorem 4.7 (i)] and Corollary 2.3. ∎

3 Relations Between Betti Numbers of Local Cohomology Modules

In the following theorem, we compare the Betti numbers of X with those of its local cohomology modules.

Theorem 3.1 (*cf.* [7, Theorem 2.1]) *Let R be a local ring, let X be an R-module, and let t be a non-negative integer. Then*

$$\beta_t(X) \leq \sum_{i=0}^{\operatorname{ara}(\mathfrak{a})} \beta_{t+i}(\mathrm{H}^i_{\mathfrak{a}}(X)).$$

Proof We may assume that the right-hand side number is finite. Set $c = \operatorname{ara}(\mathfrak{a})$ and $n = t + c$. By [6, Lemma 2.1], there is a first quadrant spectral sequence

$$E^2_{p,q} := \operatorname{Tor}^R_p\big(k, \mathrm{H}^{c-q}_{\mathfrak{a}}(X)\big) \underset{p}{\Longrightarrow} \operatorname{Tor}^R_{p+q-c}(k, X). \tag{3.1}$$

For all r, $0 \leq r \leq n$, we have $E^{\infty}_{r,n-r} = E^{n+2}_{r,n-r}$, since $E^i_{r+i,n-r-i+1} = 0 = E^i_{r-i,n-r+i-1}$ for all $i \geq n+2$; so that $\dim_k(E^{\infty}_{r,n-r}) \leq \dim_k(\operatorname{Tor}^R_r(k, \mathrm{H}^{r-t}_{\mathfrak{a}}(X)))$ from the fact that $E^{n+2}_{r,n-r}$ is a subquotient of $E^2_{r,n-r} = \operatorname{Tor}^R_r(k, \mathrm{H}^{r-t}_{\mathfrak{a}}(X))$.

There exists a finite filtration

$$0 = \phi^{-1}H_n \subseteq \phi^0 H_n \subseteq \cdots \subseteq \phi^{n-1}H_n \subseteq \phi^n H_n = \operatorname{Tor}^R_t(k, X)$$

such that $E^{\infty}_{r,n-r} = \phi^r H_n / \phi^{r-1} H_n$ for all r, $0 \leq r \leq n$. Now the exact sequences

$$0 \longrightarrow \phi^{r-1}H_n \longrightarrow \phi^r H_n \longrightarrow E^{\infty}_{r,n-r} \longrightarrow 0,$$

for all r, $0 \leq r \leq n$, show that $\dim_k(\operatorname{Tor}^R_t(k, X)) = \sum_{r=0}^n \dim_k(E^{\infty}_{r,n-r})$. Thus we get

$$\beta_t(X) = \sum_{r=0}^{n} \dim_k(E^{\infty}_{r,n-r}) \leq \sum_{r=0}^{n} \dim_k\Big(\operatorname{Tor}^R_r\big(k, \mathrm{H}^{r-t}_{\mathfrak{a}}(X)\big)\Big) = \sum_{i=0}^{c} \beta_{t+i}\big(\mathrm{H}^i_{\mathfrak{a}}(X)\big),$$

which completes the proof. ∎

In the following theorem, for a given non-negative integer t, we compare the Betti numbers of local cohomology module $\mathrm{H}^t_{\mathfrak{a}}(X)$ with the Betti numbers of X and those of some other local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$ where $i \neq t$.

Theorem 3.2 (*cf.* [7, Theorem 2.6]) *Let R be a local ring, let X be an R-module, and let s, t be non-negative integers. Then*

$$\beta_s(\mathrm{H}^t_{\mathfrak{a}}(X)) \le \sum_{i=0}^{t-1} \beta_{s-t+i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \beta_{s-t}(X) + \sum_{i=t+1}^{\operatorname{ara}(\mathfrak{a})} \beta_{s-t+i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)).$$

Proof We may assume that the right-hand side number is finite. Set $c = \operatorname{ara}(\mathfrak{a})$, $u = c - t$ and $n = s + u$, and consider the first quadrant spectral sequence (3.1). For all $r \ge 2$, let $Z^r_{s,u} = \ker(E^r_{s,u} \longrightarrow E^r_{s-r,u+r-1})$ and $B^r_{s,u} = \operatorname{Im}(E^r_{s+r,u-r+1} \longrightarrow E^r_{s,u})$. Thus, we have the exact sequences:

$$0 \longrightarrow Z^r_{s,u} \longrightarrow E^r_{s,u} \longrightarrow E^r_{s,u}/Z^r_{s,u} \longrightarrow 0,$$
$$0 \longrightarrow B^r_{s,u} \longrightarrow Z^r_{s,u} \longrightarrow E^{r+1}_{s,u} \longrightarrow 0,$$

which show that

$$\begin{aligned}\dim_k(E^r_{s,u}) &= \dim_k(E^r_{s,u}/Z^r_{s,u}) + \dim_k(E^{r+1}_{s,u}) + \dim_k(B^r_{s,u})\\ &\le \dim_k(E^r_{s-r,u+r-1}) + \dim_k(E^{r+1}_{s,u}) + \dim_k(E^r_{s+r,u-r+1})\\ &\le \dim_k(E^2_{s-r,u+r-1}) + \dim_k(E^{r+1}_{s,u}) + \dim_k(E^2_{s+r,u-r+1}).\end{aligned}$$

As we have $E^r_{s+r,u-r+1} = 0 = E^r_{s-r,u+r-1}$ for all $r \ge s + u + 2$, we obtain $E^\infty_{s,u} = E^{s+u+2}_{s,u}$. To complete the proof, it is enough to show that $\dim_k(E^\infty_{s,u}) \le \beta_{s-t}(X)$, because we have

$$\begin{aligned}\beta_s(\mathrm{H}^t_{\mathfrak{a}}(X)) &= \dim_k(E^2_{s,u})\\ &\le \dim_k(E^2_{s-2,u+1}) + \dim_k(E^3_{s,u}) + \dim_k(E^2_{s+2,u-1})\\ &\le \sum_{i=t-2}^{t-1} \dim_k(E^2_{s-t+i-1,c-i}) + \dim_k(E^4_{s,u}) + \sum_{i=t+1}^{t+2} \dim_k(E^2_{s-t+i+1,c-i})\\ &\le \cdots\\ &\le \sum_{i=0}^{t-1} \dim_k(E^2_{s-t+i-1,c-i}) + \dim_k(E^{s+u+2}_{s,u}) + \sum_{i=t+1}^{c} \dim_k(E^2_{s-t+i+1,c-i})\\ &= \sum_{i=0}^{t-1} \beta_{s-t+i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \dim_k(E^{s+u+2}_{s,u}) + \sum_{i=t+1}^{c} \beta_{s-t+i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)).\end{aligned}$$

There exists a finite filtration

$$0 = \phi^{-1}H_n \subseteq \phi^0 H_n \subseteq \cdots \subseteq \phi^{n-1}H_n \subseteq \phi^n H_n = \operatorname{Tor}^R_{s-t}(k, X)$$

such that $E^\infty_{r,n-r} = \phi^r H_n/\phi^{r-1}H_n$ for all r, $0 \le r \le n$. Thus, $E^\infty_{s,u} = \phi^s H_n/\phi^{s-1}H_n$ and so

$$\dim_k(E^\infty_{s,u}) \le \dim_k(\phi^s H_n) \le \dim_k(\operatorname{Tor}^R_{s-t}(k, X)) = \beta_{s-t}(X),$$

as we desired. ■

A straightforward application of Theorems 3.1 and 3.2 is to find the Betti numbers of local cohomology modules for certain cases.

Corollary 3.3 *Let R be a local ring, let X be an R-module, and let n be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all i, $i \ne n$* (e.g., *X is finite and $\mathfrak{a}$ may be generated by an X-regular sequence of length n). Then, for all $i \ge 0$, $\beta_i(\mathrm{H}^n_{\mathfrak{a}}(X)) = \beta_{i-n}(X)$.*

Proof For all $i \geq 0$, apply Theorem 3.1 with $t = i - n$ (resp. Theorem 3.2 with $s = i$ and $t = n$) to get $\beta_{i-n}(X) \leq \beta_i(\mathrm{H}^n_{\mathfrak{a}}(X))$ (resp. $\beta_i(\mathrm{H}^n_{\mathfrak{a}}(X)) \leq \beta_{i-n}(X)$). ∎

We close this section by identifying the Betti numbers of the non-zero top local cohomology modules of X with respect to an ideal $\mathfrak{a}$ when $\mathrm{cd}(\mathfrak{a}, X) \leq 2$. Recall that cohomological dimension of X with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}(\mathfrak{a}, X)$, is the largest integer i in which $\mathrm{H}^i_{\mathfrak{a}}(X)$ is not zero (see [8]).

Corollary 3.4 *Let R be a local ring and let X be an R-module.*

(i) *If* $\mathrm{cd}(\mathfrak{a}, X) = 0$, *then* $\beta_i(\Gamma_{\mathfrak{a}}(X)) = \beta_i(X)$ *for all* i.
(ii) *If* $\mathrm{cd}(\mathfrak{a}, X) = 1$, *then* $\beta_i(\mathrm{H}^1_{\mathfrak{a}}(X)) = \beta_{i-1}(X/\Gamma_{\mathfrak{a}}(X))$ *for all* i.
(iii) *If* $\mathrm{cd}(\mathfrak{a}, X) = 2$, *then* $\beta_i(\mathrm{H}^2_{\mathfrak{a}}(X)) = \beta_{i-2}(\mathrm{D}_{\mathfrak{a}}(X))$ *for all* i.

Proof (i) This is clear from Corollary 3.3.

(ii) For all $i \neq 1$, $\mathrm{H}^i_{\mathfrak{a}}(X/\Gamma_{\mathfrak{a}}(X)) = 0$ by assumption. Again, use Corollary 3.3.

(iii) By [2, Corollary 2.2.8], $\mathrm{H}^i_{\mathfrak{a}}(\mathrm{D}_{\mathfrak{a}}(X)) = 0$ for all $i \neq 2$. Now, the assertion follows from Corollary 3.3. ∎

4 Flat Dimensions of Local Cohomology Modules

In the course of the remaining parts of the paper, for an arbitrary R-module X, we denote by $\mathrm{pd}_R(X)$ and $\mathrm{fd}_R(X)$ the projective dimension and the flat dimension of X, respectively.

As an application of Theorem 3.1, the following corollary gives us many upper bounds for the projective dimension of a finite R-module M.

Corollary 4.1 *Let R be a local ring and let M be a finite R-module. Then the inequality*

$$\mathrm{pd}_R(M) \leq \sup\left\{ \mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) - i : 0 \leq i \leq \mathrm{ara}(\mathfrak{a})\right\} \tag{4.1}$$

holds for every ideal $\mathfrak{a}$ of R.

Proof We may assume that the right-hand side number is finite and that t is an integer such that $t > \sup\left\{ \mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) - i : 0 \leq i \leq \mathrm{ara}(\mathfrak{a})\right\}$. Then $\beta_{t+i}(\mathrm{H}^i_{\mathfrak{a}}(M)) = 0$ for all i, $0 \leq i \leq \mathrm{ara}(\mathfrak{a})$. The result follows by Theorem 3.1. ∎

Assume that R is a local ring, X is an arbitrary R-module, and t is a non-negative integer such that $\mathrm{H}^t_{\mathfrak{a}}(X)$ is finite. Then we can use Theorem 3.2, with a similar argument as in the proof of Corollary 4.1, to find an upper bound for the projective dimension of $\mathrm{H}^t_{\mathfrak{a}}(X)$ in terms of the flat dimensions of modules $\mathrm{H}^i_{\mathfrak{a}}(X)$, $i \neq t$, and that of X. More precisely, we get

$$\begin{aligned}\mathrm{pd}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sup\{&\mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + t - i + 1 : i < t\} \cup \{\mathrm{fd}_R(X) + t\} \\ &\cup \{\mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + t - i - 1 : i > t\}.\end{aligned} \tag{4.2}$$

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Since the above inequality holds in the strong conditions, our next aim is to prove the inequality

$$\begin{aligned}(4.3)\quad \operatorname{fd}_R(\mathrm{H}^t_{\mathfrak a}(X)) \le \sup\{\operatorname{fd}_R(\mathrm{H}^i_{\mathfrak a}(X)) + t - i + 1 : i < t\} \cup \{\operatorname{fd}_R(X) + t\}\\ \cup \{\operatorname{fd}_R(\mathrm{H}^i_{\mathfrak a}(X)) + t - i - 1 : i > t\}\end{aligned}$$

with no restrictions on R or $\mathrm{H}^t_{\mathfrak a}(X)$. We need the following useful lemma for this purpose.

Lemma 4.2 *For two arbitrary R-modules N and X, there are first quadrant spectral sequences*

(i) ${}^{I}E^2_{p,q} := \operatorname{Tor}^R_p(N, \mathrm{H}^{\operatorname{ara}(\mathfrak a)-q}_{\mathfrak a}(X)) \underset{p}{\Longrightarrow} H_n$ *and*

(ii) ${}^{II}E^2_{p,q} := \mathrm{H}^{\operatorname{ara}(\mathfrak a)-p}_{\mathfrak a}(\operatorname{Tor}^R_q(N, X)) \underset{p}{\Longrightarrow} H_n$.

Proof By assuming $c = \operatorname{ara}(\mathfrak a)$, there exist elements $x_1, \dots, x_c$ of R such that $\sqrt{\mathfrak a} = (x_1, \dots, x_c)$. Let $F_\bullet$ be a free resolution of N and consider the first quadrant bicomplex $\mathcal T = \{C(F_p \otimes_R X)^{c-q}\}$, where $C(F_p \otimes_R X)^\bullet$ is the Čech complex of $F_p \otimes_R X$ with respect to $x_1, \dots, x_c$. We denote the total complex of $\mathcal T$ by $\operatorname{Tot}(\mathcal T)$.

(i) The first filtration has ${}^{I}E^2$ term the iterated homology $H'_p H''_{p,q}(\mathcal T)$. By [2, Theorem 5.1.19], we have

$$H''_{p,q}(\mathcal T) = H^{c-q}(C(F_p \otimes_R X)^\bullet) = \mathrm{H}^{c-q}_{\mathfrak a}(F_p \otimes_R X) = F_p \otimes_R \mathrm{H}^{c-q}_{\mathfrak a}(X).$$

Hence,

$${}^{I}E^2_{p,q} = H_p(F_\bullet \otimes_R \mathrm{H}^{c-q}_{\mathfrak a}(X)) = \operatorname{Tor}^R_p(N, \mathrm{H}^{c-q}_{\mathfrak a}(X)),$$

which yields the first quadrant spectral sequence

$${}^{I}E^2_{p,q} := \operatorname{Tor}^R_p(N, \mathrm{H}^{c-q}_{\mathfrak a}(X)) \underset{p}{\Longrightarrow} H_n(\operatorname{Tot}(\mathcal T)).$$

(ii) The second filtration has ${}^{II}E^2$ term the iterated homology $H''_p H'_{q,p}(\mathcal T)$. We have

$$\begin{aligned}H'_{q,p}(\mathcal T) &= H_q(C(R)^{c-p} \otimes_R F_\bullet \otimes_R X) = C(R)^{c-p} \otimes_R H_q(F_\bullet \otimes_R X)\\ &= C(\operatorname{Tor}^R_q(N, X))^{c-p}.\end{aligned}$$

Thus, again by [2, Theorem 5.1.19],

$${}^{II}E^2_{p,q} = H^{c-p}(C(\operatorname{Tor}^R_q(N, X))^\bullet) = \mathrm{H}^{c-p}_{\mathfrak a}(\operatorname{Tor}^R_q(N, X))$$

which gives the first quadrant spectral sequence

$${}^{II}E^2_{p,q} := \mathrm{H}^{c-p}_{\mathfrak a}(\operatorname{Tor}^R_q(N, X)) \underset{p}{\Longrightarrow} H_n(\operatorname{Tot}(\mathcal T)).$$ ■

Corollary 4.3 *Let X be an arbitrary R-module and let t be a non-negative integer. Then the inequality (4.3) holds true.*

Proof Assume that $t \le \operatorname{ara}(\mathfrak a)$ and that the right-hand side number is finite. Assume also that N is an arbitrary R-module and that s is an integer that is bigger than the right hand side number. Set $c = \operatorname{ara}(\mathfrak a)$, $u = c - t$ and $n = s + u$, and consider the first quadrant spectral sequence Lemma 4.2(i). For all $r \geq 2$, let ${}^IZ^r_{s,u} = \ker({}^IE^r_{s,u} \longrightarrow {}^IE^r_{s-r,u+r-1})$ and ${}^IB^r_{s,u} = \operatorname{Im}({}^IE^r_{s+r,u-r+1} \longrightarrow {}^IE^r_{s,u})$, so that we have the exact sequences:

$$0 \longrightarrow {}^IZ^r_{s,u} \longrightarrow {}^IE^r_{s,u} \longrightarrow {}^IE^r_{s,u}/{}^IZ^r_{s,u} \longrightarrow 0,$$
$$0 \longrightarrow {}^IB^r_{s,u} \longrightarrow {}^IZ^r_{s,u} \longrightarrow {}^IE^{r+1}_{s,u} \longrightarrow 0.$$

Since ${}^IE^2_{s-r,u+r-1} = 0 = {}^IE^2_{s+r,u-r+1}$, ${}^IE^r_{s-r,u+r-1} = 0 = {}^IE^r_{s+r,u-r+1}$. Thus, ${}^IE^r_{s,u}/{}^IZ^r_{s,u} = 0 = {}^IB^r_{s,u}$, which shows that ${}^IE^r_{s,u} = {}^IE^{r+1}_{s,u}$ and so

$$\operatorname{Tor}^R_s(N, \mathrm{H}^t_{\mathfrak{a}}(X)) = {}^IE^2_{s,u} = {}^IE^3_{s,u} = \cdots = {}^IE^\infty_{s,u}.$$

To complete the proof, it is enough to show that ${}^IE^\infty_{s,u} = 0$.

From the first quadrant spectral sequence Lemma 4.2(ii), there is a finite filtration

$$0 = \phi^{-1}H_n \subseteq \phi^0 H_n \subseteq \cdots \subseteq \phi^{n-1}H_n \subseteq \phi^n H_n = H_n$$

such that ${}^{II}E^\infty_{r,n-r} = \phi^r H_n/\phi^{r-1}H_n$ for all r, $0 \leq r \leq n$. Note that, for all r, $0 \leq r \leq n$, we have ${}^{II}E^\infty_{r,n-r} = {}^{II}E^{n+2}_{r,n-r}$, since ${}^{II}E^i_{r+i,n-r-i+1} = 0 = {}^{II}E^i_{r-i,n-r+i-1}$ for all $i \geq n+2$. Thus ${}^{II}E^\infty_{r,n-r} = 0$ because ${}^{II}E^{n+2}_{r,n-r}$ is a subquotient of ${}^{II}E^2_{r,n-r} = \mathrm{H}^{c-r}_{\mathfrak{a}}(\operatorname{Tor}^R_{n-r}(N, X)) = 0$. Hence, $\phi^r H_n/\phi^{r-1}H_n = 0$ for all r, $0 \leq r \leq n$, and so we get

$$0 = \phi^{-1}H_n = \phi^0 H_n = \cdots = \phi^{n-1}H_n = \phi^n H_n = H_n.$$

Again from the first quadrant spectral sequence Lemma 4.2(i), there exists a finite filtration

$$0 = \psi^{-1}H_n \subseteq \psi^0 H_n \subseteq \cdots \subseteq \psi^{n-1}H_n \subseteq \psi^n H_n = H_n$$

such that ${}^IE^\infty_{r,n-r} = \psi^r H_n/\psi^{r-1}H_n$ for all r, $0 \leq r \leq n$. Since $H_n = 0$, $\psi^s H_n = 0$ and so $\psi^s H_n/\psi^{s-1}H_n = 0$. Thus, ${}^IE^\infty_{s,u} = 0$, which yields the assertion. ∎

By considering the inequalities (4.1), (4.2), and (4.3), it is natural to raise the following question.

Question 4.4 *Let X be an arbitrary R-module. Does the inequality*

$$\operatorname{fd}_R(X) \leq \sup\left\{ \operatorname{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - i : 0 \leq i \leq \operatorname{ara}(\mathfrak{a})\right\}$$

hold true for every ideal $\mathfrak{a}$ *of R?*

Corollary 4.5 *Let R be a local ring, let M be a finite R-module, and let n be a non-negative integer such that* $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for all i,* $i \neq n$ (e.g., $\mathfrak{a}$ *may be generated by an M-regular sequence of length n). Then we have*

$$\operatorname{fd}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) = \operatorname{fd}_R(M) + n.$$

Proof By Corollaries 4.3 and 4.1, we get

$$\operatorname{fd}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) \leq \operatorname{fd}_R(M) + n \quad \text{and} \quad \operatorname{pd}_R(M) + n \leq \operatorname{fd}_R(\mathrm{H}^n_{\mathfrak{a}}(M)),$$

which yield the assertion. ∎

Corollary 4.6 *Let R be a local ring with maximal ideal* $\mathfrak{m}$. *Then the following statements hold true.*

(i) $\mathrm{fd}_R(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)) = \mathrm{fd}_R(M) + \dim_R(M)$ *whenever M is a Cohen–Macaulay R-module.*

(ii) $\mathrm{fd}_R(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)) = \mathrm{pd}_R(M) + \mathrm{depth}_R(M)$ *whenever M is a Cohen–Macaulay R-module.*

(iii) $\mathrm{fd}_R(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)) = \mathrm{depth}(R)$ *whenever M is a Cohen–Macaulay R-module with* $\mathrm{pd}_R(M) < \infty$.

(iv) $\mathrm{fd}_R(\mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)) = \dim(R)$ *whenever R is Cohen–Macaulay.*

(v) $\mathrm{fd}_R(E(k)) = \dim(R)$ *whenever R is Gorenstein.*

Proof This follows from Corollary 4.5 and the Auslander–Buchsbaum formula. ■

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