Since BC is bisected at E and KC is added to it,

therefore $BK \cdot KC + CE^2 = EK^2$.

Add EF^2 to both sides;

then $BK \cdot KC + CE^2 + EF^2 = KE^2 + EF^2$;

therefore $BK \cdot KC + CF^2 = KF^2$.

And since $MA : AB = MD : DK$,

and $MD : DK = BC : CK$;

therefore $MA : AB = BC : CK$.

Now $AL = \frac{1}{2}AB$, and $CG = 2BC$;

therefore $MA : AL = GC : CK$.

But $GC : CK = FH : HK$, on account of the parallels GF, CH;

therefore $ML : LA = FK : KH$, by composition.

Now $AL = HK$, since $AL = CF$;

therefore $ML = FK$, and therefore $ML^2 = FK^2$.

And $BM \cdot MA + LA^2 = ML^2$,

and $BK \cdot KC + CF^2 = FK^2$,

of which $AL^2 = CF^2$, for $AL = CF$;

therefore the remainder $BM \cdot MA$ = the remainder $BK \cdot KC$;

and therefore $MB : BK = KC : AM$.

But $BM : BK = CD : CK$;

therefore $CD : CK = CK : AM$.

And $DC : CK = MA : AD$;

therefore $DC : CK = CK : AM = AM : AD$.

Notes on Euclid I., 47.

By William Harvey, B. A.

ABC is a triangle, right-angled at A, and X, Y, Z are the centres of the squares described on the sides opposite the angles A, B, C; XD, XM, XN are respectively perpendicular to BC, CA, AB; MY, DY are joined, and DY meets AC at E; NZ, DZ are joined, and DZ meets AB at F.

In fig. 16 the squares are drawn outwardly, and the points X, Y, Z are on the sides of BC, CA, AB remote from the opposite angles A, B, C.

In fig. 17 the squares are drawn inwardly, and the points X, Y, Z are on the same sides of BC, CA, AB as the opposite angles A, B, C.

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The properties deperding on the points X, Y, Z are analogous in these two cases.

Properties.

(1) D is the middle point of BC, and $XD = \frac{1}{2}BC$.

(2) YD is parallel to AB, and

$= \frac{1}{2}(CA + AB)$. (fig. 16.)

$= \frac{1}{2}(CA - AB)$. (fig. 17.)

For YD joins the middle points of two sides of a triangle, and is, therefore, parallel to the third side and equal to half of it.

ZD is parallel to AC, and

$= \frac{1}{2}(CA + AB)$. (fig. 16.)

$= \frac{1}{2}(CA - AB)$. (fig. 17.)

For the above proof applies equally to YD and ZD.

(3) Hence $YD = ZD$, and the angle $ZDY =$ a right angle.

(4) E and F are the middle points of CA, AB respectively.

(5) The figure AMXN is a square, whose side

$= \frac{1}{2}(CA + AB)$. (fig. 16.)

$= \frac{1}{2}(CA - AB)$. (fig. 17.)

This follows from the congruency of the triangles CMX, BNX.*

(6) DY is equal and parallel to XM, and DXMY is a parallelogram, whose perimeter

$= \frac{1}{2}(CA + AB + BC)$. (fig. 16.)

$= \frac{1}{2}(CA - AB + BC)$. (fig. 17.)

This follows from (2) and (5).

Hence $YM = XD = \frac{1}{2}BC$.

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* My attention was drawn, after the present paper was in proof, to an anticipation of the above property in the *Educational Times*, for August 1879, (Vol. 32, p. 241), where the mathematical editor, Mr J. C. Miller, proposes and solves the following proposition:

A square is constructed on the hypotenuse of a right-angled triangle; prove that the distance of the middle of this square from each of the sides that contain the right angle of the triangle is equal to half the *difference*, or to half the *sum*, of these sides, according as the triangle and the square are on the *same* or on *opposite* sides of the hypotenuse.

DZ is equal and parallel to XN, and DXNZ is a parallelogram whose perimeter is equal to that above, and

$$ZN = XD = \tfrac{1}{2}BC.$$

(7) ZN is equal and parallel to YM, and ZNMY is a parallelogram.
Therefore ZY = NM.

(8) The points D and A are similarly situated on MN and ZY, and the triangles AMN, DZY are congruent.
For AM is equal and parallel to DZ, by (3) and (5); therefore DM is equal and parallel to ZA.

Similarly DN is equal and parallel to YA.
Hence D lies on the line MN.

(9) AX bisects the angle BAC. (fig. 16.)
AX bisects the supplement of the angle BAC. (fig. 17.)

(10) AX is equal and perpendicular to YZ,
BY is equal and perpendicular to ZX,
CZ is equal and perpendicular to XY.
For AX = MN = ZY;
and AX bisects the angle BAC or the supplement of angle BAC, by (5) and (7),
while ZY bisects the supplement of angle BAC or the angle BAC; therefore AX is perpendicular to ZY.

Again, from the congruency of the triangles BZY, ZAX, it follows that BY is equal to ZX; and since one of the sides ZY of the first triangle is perpendicular to the corresponding side AX of the second, each side of the first is perpendicular to the corresponding side of the second; and therefore BY is perpendicular to ZX.

The same is true, by parity of conditions, for CZ and XY.

(11) AX, BY, CZ are respectively perpendicular to the sides of the triangle XYZ; therefore they meet in a point O which is the orthocentre of the triangle XYZ.
This follows from (10).

(12) The area of the triangle XYZ is equal to that of the square AMXN.

For the triangle XYZ is on the same base YZ as the triangle DYZ, and the altitude of the former is twice that of the latter, since XM is equal and parallel to DY. Therefore the area of XYZ is twice that of DYZ, and DYZ is equal to AMN, by (8), which is half the square AMXN. *Hence the triangle XYZ is equal to the square AMXN.*

(13) $\triangle BCX + \triangle ABC =$ square AMXN. (fig. 16.)
$\triangle BCX - \triangle ABC =$ square AMXN. (fig. 17.)

This follows from the congruency of the triangles BNX, CMX.*

(14) $\triangle AYC + \triangle AZB + \triangle ABC =$ square AMXN. (fig. 16.)
$\triangle AYC + \triangle AZB - \triangle ABC =$ square AMXN. (fig. 17.)

For in the first case the sum of the areas is equal to the quadrilateral BCYZ, which, since D is the middle point of BC, is equal to twice the triangle DYZ, that is, to the square AMXN.

In the second case, the expression on the left side is equal to $\triangle BCY - \triangle BZY$, and $\triangle BCY = \triangle CYZ$, since BZ is parallel to CY. And since D is the middle point of BC, and B and C are on opposite sides of the base ZY

$$\triangle CYZ - \triangle BYZ = 2\triangle DYZ = \text{square AMXN}.$$

(15) In either case we prove by (13) and (14) that

$$\triangle BCX = \triangle ABZ + \triangle ACY$$

which gives the result of the 47th proposition.

(16) There are eight varieties of Euclid's figure which result from placing the squares in different positions with respect to the sides of the triangle. In the two we have considered the properties of the points X, Y, Z are the most interesting, and are strictly analogous. The properties furnished by the remaining cases are, with a few exceptions, not analogous to the above, but are of some interest in themselves.

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* From this we get a geometrical proof of the result $\Delta = s(s-a) = (s-b)(s-c)$ for the area of a right-angled triangle.

For, with the usual notation, the result of (13) becomes

$$\Delta + \tfrac{1}{4}a^2 = \tfrac{1}{4}(b+c)^2 \qquad \text{(fig. 16).}$$

$$-\Delta + \tfrac{1}{4}a^2 = \tfrac{1}{4}(b-c)^2 \qquad \text{(fig. 17).}$$

which give the two expressions for the area.