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UNIQUENESS RESULTS FOR A CLASS OF HIGHER – ORDER BOUNDARY VALUE PROBLEMS

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Abstract. The classical maximum principle is utilized to obtain maximum principles for functionals which are defined on solutions of fourth, sixth and eighth-order elliptic equations. The principles derived lead to uniqueness results.

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1. Introduction. It is well known that every subharmonic function in a bounded domain Ω (i.e. $\Delta u \geq 0$ in Ω) satisfies the classical maximum principle

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{1.1}$$

The subbiharmonic function $u(x) = -x_1^4 - |x|^2$ in the ball $\Omega = \{(x_1, \ldots, x_n) | \, |x| < R\}$ (i.e. $\Delta^2 u \leq 0$ in Ω) shows that there are no classical maximum principles for the biharmonic operator $\Delta^2 u$ (and for higher-order elliptic operators at all). Still some results can be proven.

The first proof of a maximum principle for an elliptic equation of higher-order was given by Miranda [**4**]. The Miranda result was extended to semilinear equations by Schaefer [**5**] and used to deduce the non-existence of solution to certain boundary value problems. An extension of this last maximum principle to a sixth-order equation may be found in [**6**]. There are other works dealing with maximum principles for fourth and sixth-order equations and their applications (see [**1**], [**2**], [**3**], [**7**] (Chapter 10), [**9**] and the references cited therein). A general maximum principle for an equation with constant coefficients of order $2m$ was established by Tseng and Lin ([**8**]).

Our purpose here is to deduce some maximum principles for a class of fourth and sixth-order equations which do not appear to be contained in the above mentioned works. A maximum principle for an equation of eighth-order with non-constant coefficients is also presented. From these maximum principles we obtain uniqueness results for a class of higher-order equations in a plane domain Ω. If the domain is on the line, then we briefly indicate some uniqueness results for problems of order $2m$ ($m \geq 5$).

2. Maximum principles and uniqueness results. Let Ω be a bounded domain in the plane with a smooth boundary $\partial\Omega$. We denote partial derivatives $\partial u/\partial x_i$ by $u,_i$ and use the summation convention, so that, e.g., the square of the gradient of u becomes

$(\partial u/\partial x_1)^2 + (\partial u/\partial x_2)^2 = |\nabla u|^2 = u_{,i} u_{,i}$. The scalar product of the gradients of two functions, say u and v, may be written as $\nabla u \cdot \nabla v = u_{,i} v_{,i}$.

First, we consider a fourth-order equation

$$\Delta^2 u - a(x)\Delta u + b(x)u = 0 \quad \text{in } \Omega, \tag{2.1}$$

where

$$a > 0, \quad \Delta(1/a) \leq 0 \tag{2.2}$$

and

$$b \geq 0. \tag{2.3}$$

We have the following result.

LEMMA 2.1. *If* $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ *is a (classical) solution of* (2.1)*, then the functional* F_1 *given by*

$$\mathrm{F}_1 = \frac{a(x)}{2}u^2 + (|\nabla u|^2 - u\Delta u) \tag{2.4}$$

assumes its maximum value on $\partial\Omega$.

From Lemma 2.1 we now deduce a slight extension of Schaefer's result (**6**, Theorem 4).

THEOREM 2.1. *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^2 u - a(x)\Delta u + b(x)w(u) = f & \text{in } \Omega \\ u = g, \quad \dfrac{\partial u}{\partial n} = h & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where a, b *satisfy* (2.2) *and* (2.3) *and* w *is a* C^1 *function for which* $bw' > 0$ *in* Ω.

Here and throughout the paper $\partial/\partial n$ denotes the outward normal derivative operator.

It is quite obvious how Theorem 1 and Theorem 4 in [**5**] have to be modified to obtain the above stated results. The details are left to the reader.

We now prove a maximum principle for the sixth-order equation

$$\Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.6}$$

where

$$c > 0, \quad \Delta(1/c) \leq 0. \tag{2.7}$$

LEMMA 2.2. *Let* $u \in C^6(\Omega) \cap C^4(\overline{\Omega})$ *be a (classical) solution of* (2.6) *and suppose that* (2.2), (2.3) *and* (2.7) *are satisfied. Then the functional* F_2 *given by*

$$\mathrm{F}_2 = \frac{c(x)}{2}u^2 + \frac{a(x)}{2}(\Delta u)^2 + |\nabla(\Delta u)|^2 - \Delta u \Delta^2 u \tag{2.8}$$

assumes its maximum value on $\partial\Omega$.

Proof. The assumption (2.2) implies that

$$\Delta(c(x)u^2)/2 \geq c(x)u\Delta u + \sum_{i=1}^{2}\left(\frac{1}{\sqrt{c}}c,_i u + \sqrt{c}u,_i\right)^2$$
$$\geq c(x)u\Delta u.$$

Similarly,

$$\Delta(a(x)(\Delta u)^2)/2 \geq a(x)\Delta u \Delta^2 u.$$

Now

$$\Delta(|\nabla(\Delta u)|^2) = 2(\Delta u),_{ij}(\Delta u),_{ij} + 2\nabla(\Delta u)\cdot\nabla(\Delta^2 u).$$
$$\Delta(\Delta u \Delta^2 u) = (\Delta^2 u)^2 + 2\nabla(\Delta u)\cdot\nabla(\Delta^2 u) + \Delta u \Delta^3 u.$$

Hence

$$\Delta F_2 \geq 2(\Delta u),_{ij}(\Delta u),_{ij} - (\Delta^2 u)^2 + b(x)(\Delta u)^2 \quad \text{in } \Omega.$$

Since the inequality

$$2v,_{ij}v,_{ij} \geq (\Delta v)^2 \tag{2.9}$$

holds in two dimensions, we obtain

$$\Delta F_2 \geq 0 \quad \text{in } \Omega.$$ □

An immediate consequence of Lemma 2.2 is the following uniqueness result.

THEOREM 2.2. *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = f & \text{in } \Omega \\ u = g, \quad \Delta u = h, \quad \dfrac{\partial(\Delta u)}{\partial n} = i & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where a, b, c *satisfy* (2.2), (2.3) *and* (2.7).

Proof. Assume that u_1 and u_2 are solutions of (2.10) and let $v = u_1 - u_2$. The function v satisfies (2.6) and

$$v = \Delta v = \frac{\partial(\Delta v)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

Since $\Delta v = 0$ on $\partial\Omega$ we obtain that $|\nabla(\Delta v)|^2 = (\frac{\partial(\Delta v)}{\partial n})^2$ on $\partial\Omega$.
Hence, by Lemma 2.2 and (2.11),

$$F_2 \leq \max_{\partial\Omega} F_2 = 0 \quad \text{in } \Omega. \tag{2.12}$$

Consequently,

$$-\Delta v \Delta^2 v \leq 0 \quad \text{in } \Omega. \tag{2.13}$$

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Integrating (2.13) over Ω and using Green's identity we conclude that

$$|\nabla(\Delta v)| = 0 \quad \text{in } \Omega,$$

and by continuity $\Delta v \equiv 0$ in Ω. Since $v = 0$ on Ω we are led to the conclusion that $v \equiv 0$ in Ω. □

Next we deal with classical solutions (i.e. $C^8(\Omega) \cap C^6(\overline{\Omega})$) of

$$\Delta^4 u - a(x)\Delta^3 u + b(x)\Delta^2 u - c(x)\Delta u + du = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.14}$$

where

$$d > 0. \tag{2.15}$$

A uniqueness result can be inferred from the following maximum principle

LEMMA 2.3. *Let u be a classical solution of* (2.14). *Assume that* (2.2), (2.3), (2.7) *and* (2.15) *are satisfied. Then the functional*

$$F_3 = \frac{c(x)}{2}(\Delta u)^2 + \frac{a(x)}{2}(\Delta^2 u)^2 + d(|\nabla u|^2 - u\Delta u) + |\nabla(\Delta^2 u)|^2 - \Delta^2 u \Delta^3 u \tag{2.16}$$

assumes its maximum value on $\partial\Omega$.

Proof. Using (2.2) and (2.7), we have

$$\Delta(c(x)(\Delta u)^2)/2 \geq c(x)\Delta u \Delta^2 u,$$
$$\Delta(a(x)(\Delta^2 u)^2)/2 \geq a(x)\Delta^2 u \Delta^3 u.$$

By inequality (2.9) we get

$$d\Delta(|\nabla u|^2 - u\Delta u) = d(2u,_{ij} u,_{ij} - (\Delta u)^2 - u\Delta^2 u) \geq -du\Delta^2 u,$$
$$\Delta(|\nabla(\Delta^2 u)|^2 - \Delta^2 u \Delta^3 u) = 2(\Delta^2 u),_{ij} (\Delta^2 u),_{ij} - (\Delta^3 u)^2 - \Delta^2 u \Delta^4 u \geq -\Delta^2 u \Delta^4 u.$$

Hence from equation (2.14) we obtain

$$\Delta F_3 \geq 0 \quad \text{in } \Omega.$$ □

THEOREM 2.3. *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^4 u - a\Delta^3 u + b(x)\Delta^2 u - c\Delta u + du = f & \text{in } \Omega \\ u = g, \quad \Delta u = h, \quad \Delta^2 u = i, \quad \dfrac{\partial(\Delta^2 u)}{\partial n} = j & \text{on } \partial\Omega, \end{cases} \tag{2.17}$$

where a satisfies (2.2), *b satisfies* (2.3), *c satisfies* (2.7), $d > 0$, *and the curvature k of* $\partial\Omega$ *is positive.*

Proof. To establish this result, we suppose that u_1 and u_2 are two solutions of (2.17). Defining $v = u_1 - u_2$, we see that v satisfies (2.14) and

$$v = \Delta v = \Delta^2 v = \frac{\partial(\Delta^2 v)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.18}$$

Since $v = \Delta^2 v = 0$ on $\partial\Omega$, we obtain

$$\frac{\partial F_3}{\partial n} = c\Delta v \frac{\partial(\Delta v)}{\partial n} + a\Delta^2 v \frac{\partial(\Delta^2 v)}{\partial n} + d\left(2\frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial n^2} - \frac{\partial v}{\partial n}\Delta v - v\frac{\partial(\Delta v)}{\partial n}\right)$$
$$+ 2\frac{\partial(\Delta^2 v)}{\partial n}\frac{\partial^2(\Delta^2 v)}{\partial n^2} - \frac{\partial(\Delta^2 v)}{\partial n}\Delta^3 v - \Delta^2 v \frac{\partial(\Delta^3 v)}{\partial n} \quad \text{on } \partial\Omega.$$

By (2.18) we have

$$\frac{\partial F_3}{\partial n} = 2d\frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial n^2} \quad \text{on } \partial\Omega. \tag{2.19}$$

By introducing normal coordinates in the neighbourhood of the boundary, we can write

$$\Delta v = \frac{\partial^2 v}{\partial n^2} + \frac{\partial^2 v}{\partial s^2} + k\frac{\partial v}{\partial n}, \tag{2.20}$$

where $\frac{\partial v}{\partial s}$ denotes the tangential derivative of v.
Since $v = \Delta v = 0$ on $\partial\Omega$, relation (2.20) becomes

$$\frac{\partial^2 v}{\partial n^2} = -k\frac{\partial v}{\partial n},$$

which gives

$$\frac{\partial F_3}{\partial n} = -2dk\left(\frac{\partial v}{\partial n}\right)^2 \leq 0 \quad \text{on } \partial\Omega.$$

This contradicts Hopf's lemma at a point $P \in \partial\Omega$, where F_3 ($F_3 \not\equiv$ constant) assumes its maximum value. Hence F_3 is constant in $\overline{\Omega}$. Thus $\frac{\partial F_3}{\partial n} = 0$ on $\partial\Omega$ and consequently $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$. By the boundary conditions it follows that $F_3 = 0$ in $\overline{\Omega}$. Using Green's identity we arrive at $v = 0$ in $\overline{\Omega}$. □

Finally we shift our attention from the two dimensional to the one dimensional case and mention the following result (Ω denotes an open interval (α, β)).

THEOREM 2.4. *There can be at most one classical solution of the problem*

$$\begin{cases} u^{(2m)} - du^{(6)} + c(x)u^{(4)} - b(x)u'' + a(x)u = f & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u''' = g_3, \ldots, u^{(m)} = g_m & \text{on } \partial\Omega, \end{cases} \tag{2.21}$$

where $m \geq 6$ is even, $d \geq 0$ and $b \geq 0$, $a, c > 0$, $(1/a)'', (1/c)'' \leq 0$ in Ω.

The result follows since the function

$$F_4 = u''u^{(2m-2)} - 2u'''u^{(2m-3)} + 3u^{(4)}u^{(2m-4)} - \cdots + (m-3)u^{(m-2)}u^{(m+2)}$$
$$- (m-3)u^{(m-1)}u^{(m+1)}/2 - ((m-3)/2+1)\left[\left(u^{(m)}\right)^2 - u^{(m-1)}u^{(m+1)}\right]$$
$$+ \left[(u''')^2 - du''u^{(4)}\right] + c(x)(u'')^2/2 + a(x)u^2/2$$

assumes its maximum value on $\partial\Omega$, where u is a solution of

$$u^{(2m)} - du^{(6)} + c(x)u^{(4)} - b(x)u'' + a(x)u = 0 \quad \text{in } \Omega.$$

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Similarly, we can treat the problem

$$\begin{cases} u^{(2m)} + du^{(6)} - c(x)u^{(4)} + b(x)u'' - a(x)u = f & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u''' = g_3, \dots, u^{(m)} = g_m & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

where $m \geq 5$ is odd.

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