DECOMPOSITIONS OF MODULES OVER A DISCRETE VALUATION RING

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Abstract

Let N be a direct summand of a module which is a direct sum of modules of torsion-free rank one over a discrete valuation ring. Then there is a torsion module T such that $N \oplus T$ is also a direct sum of modules of torsion-free rank one.

Subject classification (*Amer. Maths. Soc.* (*MOS*) *1970*): primary 13 C 05; secondary 13 H 99, 20 K 25, 18 E 15.

1. Introduction

R will denote a fixed discrete valuation ring, and module will mean R-module. The generator of the unique maximal ideal of R will be denoted p. The quotient ring of R will be represented by Q. If X is a subset of the module M, then $[X]$ represents the submodule generated by X. $E(M)$ will denote the endomorphism ring of the module M. A module M has *torsion-free rank one* if, given elements m and n of infinite order in M, there are nonzero elements r and s in R such that $rm = sn$. A module is *completely decomposable* if it is a direct sum of modules of torsion-free rank one. Two modules M and N are *almost isomorphic* if there are torsion modules S and T such that $M \oplus S$ is isomorphic to $N \oplus T$. We may now state the main theorem of the paper.

THEOREM 1. *Let N be a direct summand of a completely decomposable module $M = \bigoplus_I M_i$, where each M_i has torsion-free rank one. Then there is a subset J of I such that N is almost isomorphic to $\bigoplus_J M_i$.*

A lengthy direct proof of this theorem appeared in the author's doctoral dissertation (Stanton (1973)), which was written under the direction of Professor

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Elbert Walker. Categorical methods developed by C. Walker and Warfield (1976), along with a category defined by E. Walker, have resulted in substantial simplifications. The latter approach will be followed here. The special case of Theorem 1 in which M is a simply presented module is a consequence of the following result of Warfield.

THEOREM 2. (Warfield (1975), Theorem 5.2.) *Let M and N be summands of simply presented modules. Then $M \simeq N$ if and only if $f(\alpha, M) = f(\alpha, N)$ for all ordinals α and for $\alpha = \infty$, and $S(e, M) = S(e, N)$ for all equivalence classes e of indicators.*

In the above theorem, f represents the Ulm invariants while S represents the invariants in Stanton (1975).

In the next section, a proof of Theorem 1 is given and some consequences of the theorem are discussed. Lemmas 5 and 6, as well as Corollaries 7, 9 and 10, originally appeared in the author's dissertation (Stanton (1973)).

2. The proof of Theorem 1

In the following discussion, we will be working in an additive category (see MacLane (1967), pp. 249–253). The three definitions below, as well as Theorem 3, are due to C. Walker and Warfield (1976).

DEFINITION. An additive category satisfies a *weak Grothendieck condition* if for every index set I and every nonzero monic $A \to \bigoplus_I B_i$, there is a finite subset J of I and a commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \bigoplus_J B_i \\
\downarrow & & \downarrow \\
A & \longrightarrow & \bigoplus_I B_i
\end{array}
$$
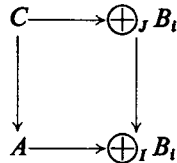
with the map $C \to A$ nonzero.

DEFINITION. An object S in an additive category is *small* if every map $f: S \to \bigoplus_I M_i$ factors through a subsum $\bigoplus_F M_i$, for some finite subset F of I.

DEFINITION. An object M is *countably finitely approximable* if there is a countable family $\{f_i: S_i \to M\}_{i=1}^{\infty}$ of maps with each S_i small, with the property that a monic $f: L \to M$ is an isomorphism if and only if each f_i can be factored through f.

Note that a small object is trivially countably finitely approximable.

THEOREM. 3. (C. Walker, Warfield (1976), Theorem 5.) *Let $\mathscr{C}$ be an additive category with kernels and infinite sums which satisfies a weak Grothendieck condition.*

If $M = \bigoplus_I M_i = N \oplus K$, with each M_i countably finitely approximable, and with the endomorphism ring of each M_i a local ring, then N is isomorphic to a direct sum $\bigoplus_J M_i$, for some subset J of I. Consequently, any two direct decompositions of M have isomorphic refinements.

We now define a category which will allow the use of Theorem 3 to prove Theorem 1. Let $\mathscr{C}$ be the category (defined by E. Walker) whose objects are modules, and whose morphisms are given by

$$\operatorname{Hom}(M, N) = \operatorname{Hom}_R(M, N)/I(M, N)$$

where A is a class of modules closed under submodules, homomorphic images and direct sums, and

$$I(M, N) = \{\varphi \in \operatorname{Hom}_R(M, N) \colon \varphi(M) \in A\}.$$

$\mathscr{C}$ is easily seen to be an additive category with a weak Grothendieck condition. Any object in A is equivalent to 0 in $\mathscr{C}$. In the sequel, the notation $\operatorname{Hom}(M, N)$ will always refer to the group of morphisms of this category. $\operatorname{Hom}(M, M)$ may be considered as a ring in the usual way.

THEOREM 4. (E. Walker.) *Let M and N be modules. Then M and N are equivalent as objects of $\mathscr{C}$ if and only if there are modules S and T in A such that $M \oplus S \simeq N \oplus T$ as modules.*

PROOF. The latter condition clearly implies the former. Suppose that M and N are equivalent. Then there are homomorphisms $\alpha\colon M \to N$, $\beta\colon N \to M$, $\rho\colon M \to M$ and $\sigma\colon N \to N$ such that

$$\beta\alpha = 1_M + \rho, \quad \alpha\beta = 1_N + \sigma$$

and such that the image T of ρ and the image S of σ are in A. We claim that $M \oplus S = N \oplus T$. Let $\zeta\colon M \oplus S \to N \oplus T$ and $\xi\colon N \oplus T \to M \oplus S$ be defined by

$$\zeta(m, s) = (\alpha m - s, \rho m - \beta s),$$

$$\xi(n, t) = (\beta n - t, \sigma n - \alpha t).$$

The identities $\alpha\rho = \sigma\alpha$ and $\beta\sigma = \rho\beta$ can be used to show the ranges of ζ and ξ are as claimed. Routine calculations show that ζ and ξ are inverses of each other.

LEMMA 5. *Let M be a module of torsion-free rank one such that the divisible part of M is a torsion module and let $\varphi \in E(M)$. Then there is an element m of infinite order in M such that φ restricted to $[m]$ is multiplication by an element of R.*

PROOF. Pick x of infinite order in M and let $\varphi(x) = y$. If y is a torsion element, then $\varphi(p^k x) = 0$ for some $k \geqslant 0$, so we may select m to be $p^k x$. If y is not torsion, there is a unit a of R, and $j, k \geqslant 0$ such that $p^k a x = p^j y$. Then $\varphi(p^k a x) = \varphi(p^j y) =$

$p^k ay$, and a comparison of heights guarantees that $k \geqslant j$. Letting $m = p^j ax$, φ restricted to $[m]$ is multiplication by $p^{k-j} a$.

Henceforth, the class A in the definition of Walker's category $\mathscr{C}$ will consist of the class of all torsion modules.

LEMMA 6. *Let M be a module of torsion-free rank one such that the divisible part of M is a torsion module. Considered as an R-module, $E(M) = R \oplus S$, where R represents multiplication by elements of R, and $S = I(M, M)$.*

PROOF. It is easily seen that S is an R-module and that $R \cap S = 0$. If $\varphi \in E(M)$, then there is $r \in R$ and $m \in M$ such that m has infinite order and $\varphi(m) = rm$, by Lemma 5. So $\varphi = r + (\varphi - r)$, where the first term is in R and the second is in S.

COROLLARY 7. *If M is a module of torsion-free rank one, then $\mathrm{Hom}(M, M)$, its endomorphism ring in $\mathscr{C}$, is local.*

PROOF. If the divisible part of M is torsion, this follows from Lemma 6. Otherwise $M = Q \oplus T$, where T is torsion, and again $\mathrm{Hom}(M, M)$ is local.

LEMMA 8. *If M is a module of torsion-free rank one, then M is small as an object of $\mathscr{C}$.*

PROOF. Let $\varphi: M \to \bigoplus_I M_i$ be a homomorphism, and let $\pi_i: \bigoplus_I M_i \to M_i$ be the projection for each i. Select an element $m \in M$ of infinite order, and let J be a subset of I such that $I \setminus J$ is finite and $\pi_i \varphi m = 0$ for all $i \in J$. Then for all $i \in J$, $\pi_i \varphi M$ is torsion, and thus equivalent to 0 in $\mathscr{C}$. Hence M is small.

The proof of Theorem 1 can now be completed by observing that the hypotheses of Theorem 3 are satisfied and that equivalence in the category $\mathscr{C}$ is given by almost isomorphism by Theorem 4.

We now give some consequences of Theorem 1. The definition of 'nice' is found in Fuchs (1973), p. 73, while 'decomposition basis' is defined in Rotman and Yen (1961), p. 251.

COROLLARY 9. *Let N be a direct summand of a completely decomposable module $M = \bigoplus_I M_i$, where each M_i has torsion-free rank one. Then*

(i) *N has a nice decomposition basis.*

(ii) *There is a subset J of I such that $S(e, N) = S(e, \bigoplus_J M_i)$, for every equivalence class e of indicators.*

PROOF. By Theorem 1, $N \oplus T_1 = (\bigoplus_J M_i) \oplus T_2$, for some subset J of I and torsion modules T_1 and T_2. If x_i is an element of infinite order in M_i, then $X = \{x_i : i \in J\}$ is a decomposition basis for $(\bigoplus_J M_i) \oplus T_2$. For each $i \in J$, there is a nonzero

element $r_i \in R$ such that $r_i x_i \in N$. Then $\{r_i y_i : i \in J\}$ is a decomposition basis for N, and is nice by Wallace (1971), Lemma 1, proving (i). The proof of (ii) follows from the fact that the S-invariants of torsion modules are all 0.

E. Walker (1974) has pointed out that a significant simplification of the proof of Warfield's result (Theorem 2) is possible by using Corollary 9. The following result is a consequence of Theorem 2.

COROLLARY 10. *Let M and N be direct summands of simply presented modules. Then M is almost-isomorphic to N if, and only if, for each equivalence class e of indicators, $S(e, M) = S(e, N)$.*

PROOF. Let $S(e, M) = S(e, N)$. Then there are torsion modules T_1 and T_2 such that the Ulm invariants of $M \oplus T_1$ and $M \oplus T_2$ are equal. By Theorem 2, $M \oplus T_1 \simeq M \oplus T_2$.

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