FULL INDIVIDUAL AND CLASS DIFFERENTIATION THEOREMS IN THEIR RELATIONS TO HALO AND VITALI PROPERTIES

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Introduction. In his article **(22)**, de Possel laid the foundations for an *abstract theory of differentiation of set functions*, the term "abstract" being meant in the sense of Fréchet-Nikodym, that is, without reference to a euclidean, metric, or topological background. In 1.1, we adopt, substantially, his notion of *derivation basis*. De Possel considered two Vitali properties for a derivation basis. The *strong* or classical Vitali property asserts the existence of an enumerable disjointed p.p. covering family; it implies the *full differentiation* theorem for integrals, that is, the existence almost everywhere of the derivative and its equality with a Radon-Nikodym integrand. The *weak* Vitali property asserts the existence of a p.p. covering family with arbitrarily small *overlap*; it is equivalent to the *density property* or the *full differentiability of Lipschitzian integrals*. One of us, in **(10)** and **(11)**, introduced in Morse's setting of **(14)**, two variations of the weak Vitali property. In the *pseudo-strength* the overlap refers to any Radon measure; the $\mathfrak{L}^{(p)}$*-overlap property* ($p > 1$) involves the pth power of the excess of covering function. The pseudo-strength implies the existence almost everywhere of the derivative of any Radon measure. The $\mathfrak{L}^{(p)}$-overlap property does the same for the integrals of $\mathfrak{L}^{(q)}$-functions ($p^{-1} + q^{-1} = 1$). In §1, these Vitali properties are transferred to a general derivation basis $\tilde{B}$. The missing topology is replaced by the pretopology **(21)** which is derived from $\tilde{B}$. Individual and class differentiation theorems are established for integrals and Radon measures. The technique at first follows de Possel's trend. Section 2 deals with the following converse problem: Do full differentiation assertions for sigma-additive set functions imply covering properties of Vitali types? It suffices to refer to the proof of the Zygmund theorem in **(29)**, and of the Zygmund-Marcinkiewitz-Jessen theorem in **(25)** concerning the interval basis, to realize that not all differentiation proofs rest on Vitali properties. By an adaptation of de Possel's proof of his equivalence theorem, we show that the full differentiability of integrals or Radon measure is equivalent to a Vitali property (2.2). Here again we proceed from individual to class assumptions. We prove (2.4) that the full differentiability of integrals of $\mathfrak{L}^{(q)}$-functions ($q > 1$) implies the $\mathfrak{L}^{(p')}$-overlap property for $p' < p$, when $\tilde{B}$ is a $\mathfrak{D}$-basis. Thus the interval basis possesses the $\mathfrak{L}^{(p)}$-overlap property for any (finite) $p \geqslant 1$. In the classical proof by Carathéodory of the Lebesgue differentiation theorem for the cube basis, the preliminary Vitali theorem is deduced from a *halo property* of cubes,

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namely: If for any cube V_0 (*nucleus*), $H(V_0)$ (*halo*) denotes the union of those cubes V which are not greater than V_0 and intersect V_0, then the *dilatation*, that is, the ratio of the measure of the halo to the measure of V_0, is uniformly bounded for all V_0, in fact equals 3^n, where n denotes the dimension of the euclidean space. Halo properties differ by the requirements for the $\tilde{B}$-sets constituting the halo, mainly by the *incidence* requirements; in the example just given, the non-vacuity of $V \cdot V_0$ was demanded. In **(1)**, Busemann and Feller gave, for the special euclidean bases considered by them, a *weak halo property equivalent to the density property.* In 2.5, we give an individual differentiability criterion of Busemann-Feller type, thus shedding light on a second converse problem: Do full differentiation properties for σ-additive set functions imply halo properties? In this connection we mention that a halo property of Busemann-Feller type creeps into the proof of 2.4. Morse, in his fundamental memoir **(14)**, formulated halo conditions, securing the strong Vitali property for his *blankets.* He assumes that the $\tilde{B}$-sets are closed; but he also shows that his differentiation theorems remain valid when this assumption is dropped. In §3 we prove that in our setting, the pointwise halo condition implies the Vitali property for integrals or Radon measures (3.2). We give two examples (3.3 and 3.4), where the surrender of the closeness of the $\tilde{B}$-sets leads to the substitution in the assertions of the new Vitali property (pseudo-strength) in place of the strong Vitali property. In §4 we tackle the differentiation of functions λ defined on the $\tilde{B}$-sets. Our main tool is the *Vitali integration* to transform the "interval" function λ into a set function which is expected to turn out to be an integral ψ on the measurable sets. When this proves true, then the differentiability study of λ reduces to that of ψ and our methods of §1 are applicable. Our results contain as a special case those published by Morse in **(14)** on the differentiation of *addivelous functions.* The authors wish to acknowledge with thanks helpful suggestions made by K. O. Househam in the course of many discussions.

§1. Differentiation of σ-Additive Set Functions under Covering Assumptions of Vitali Type

1.1. Setting. R denotes a set of points, which is our universe. $\mathbf{S}$ denotes the Boolean σ-algebra[1] of all subsets of R.

For two sets X and Y belonging to $\mathbf{S}$, $X \supset Y$ means ordinary inclusion, permitting the equality $X = Y$.

We use both the lattice-theoretical symbols $\cup$, $\cap$, $\bigcup$, $\bigcap$, and the algebraic symbols $+$, $-$, and ., in Stone's sense. However, we generally use the latter only when Stone's and Hausdorff's (set-theoretical) meaning coincide.

$\mathbf{M}$ denotes a Boolean σ-algebra of subsets of R with R as unit; μ represents a fixed σ-finite measure defined on $\mathbf{M}$; μ^* is the completion in $\mathbf{S}$ of μ, defined on $\mathbf{M}^*$. Also, $\bar{\mu}$ represents the outer measure derived from μ (or, equivalently,

[1]Definition of Boolean σ-algebras and other related terms may be found in **(4, pp. 19–26)**.

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from μ^*), defined on **S**, namely $\bar{\mu}(S) = \inf \mu(M)$, where the infimum is taken over all sets M such that $S \subset M$ and $M \in \mathbf{M}$. Similarly, we define $\tilde{\mu}$ on **S** by $\tilde{\mu}(S) = \sup \mu(M)$, where the supremum is taken over all sets M such that $S \supset M$ and $M \in \mathbf{M}$.

N denotes the family of the μ-nullsets, which is a σ-ideal in **M** (regarded as a Boolean σ-ring); $\mathbf{N}^*$ is the family of the μ^*-nullsets, which is a σ-ideal in **S** (envisaged as a Boolean σ-ring).

By $X \supset Y$ (mod **N**) we shall mean $Y - X \cdot Y \in \mathbf{N}$; $X = Y$ (mod **N**) will be understood to mean that Stone's difference $X - Y = [(X - X\cdot Y) + (Y - X\cdot Y)] \in \mathbf{N}$.

For $S \in \mathbf{S}$, a *μ-cover* $\bar{S}$ of S is any **M**-set for which $\bar{S} \supset S$, and $\bar{\mu}(S\cdot M) = \mu(\bar{S}\cdot M)$ for any $M \in \mathbf{M}$. Similarly (**3**, p. 68), a *μ-kernel* $\underline{S}$ of S is any **M**-set such that $\underline{S} \subset S$, and $\tilde{\mu}(S\cdot M) = \mu(\underline{S}\cdot M)$ for any $M \in \mathbf{M}$.

Two sets S' and S'' are said to be *μ^*-entangled* if they have positive outer measure and common μ-cover.

We define a *derivation basis* $\tilde{B}$ as follows. We assume that to each point x of a fixed subset E of R, there correspond sequences, in the sense of Moore-Smith, of **M**-sets of finite positive measure, called *constituents*, which are said to *converge* to x, and are denoted generically by $M_\iota(x)$. Further, we assume de Possel's heredity (or Fréchet's convergence) axiom[2]; namely, every (cofinal) subsequence of an x-converging sequence itself converges to x. The family of the sequences $M_\iota(x)$ is our derivation basis $\tilde{B}$. The elements of $\tilde{B}$ are thus converging sequences, together with corresponding convergence points. (This notion involves a basic measure μ. The correspondence of converging sequences to points is called *prebasis* by Haupt and Pauc in (**9**). A prebasis defines a *pretopology* (**21**, §2). Some pretopological notions involve a σ-ideal of nullsets). The definition just given does not exclude the possibility that two distinct points correspond to the same converging sequence. We denote by **D** the family of sets occurring in the sequences $M_\iota(x)$ for all $x \in E$. If λ is a numerical function defined on the **D**-sets, and $x \in E$, then we define

$$D^* \lambda(x) = \sup\left[\limsup \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right],$$

where the expression in brackets denotes the limit superior for any one x-converging sequence $M_\iota(x)$, and the supremum is taken among all sequences converging to x. In exactly similar fashion we define

$$D_* \lambda(x) = \inf\left[\liminf \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right].$$

We call $D^*\lambda(x)$ and $D_*\lambda(x)$ the *upper* and *lower $\tilde{B}$-derivates* at x, respectively. If $D^*\lambda(x) = D_*\lambda(x)$ (finite or infinite), we say that the *$\tilde{B}$-derivative* $D\lambda(x) =$

[2]This is introduced in (**22**, p. 397). The limitation to ordinary sequences $i = 1, 2, \ldots$ is irrelevant throughout de Possel's paper.

$D^*\lambda(x) = D_*\lambda(x)$ *exists at* x, or that λ is $\tilde{B}$*-differentiable at* x. If the sequences $M_\iota(x)$ are subsequences of one universal sequence **(14)**, then we can drop the prefixes "sup" and "inf" in the expressions for $D^*\lambda(x)$ and $D_*\lambda(x)$.

By *subbasis of* $\tilde{B}$, we mean any subfamily $\tilde{B}^*$ of $\tilde{B}$, containing all subsequences of any of its sequences, retaining the corresponding convergence points. The family of the constituents occurring in the $\tilde{B}^*$*-sequences* is called the *spread* of $\tilde{B}^*$; the set of points $D(\tilde{B}^*)$, each of which is a convergence point of at least one $\tilde{B}^*$-sequence, is called the *domain* of $\tilde{B}^*$. The spread $\mathbf{V} = \mathbf{V}(X)$ of any subbasis $\tilde{B}^*$ with $D(\tilde{B}^*) \supset X$ (mod $\mathbf{N}^*$) is called a $\tilde{B}$*-fine covering* of X. A $\tilde{B}$-fine covering $\mathbf{V} = \mathbf{V}(X)$ of a set X may also be defined as a family of constituents containing, for almost every x (that is, everywhere but on an $\mathbf{N}^*$-set) in X, the sets of at least one sequence $M_\iota(x)$.

The importance of the latter notion for the theory of differentiation results from the following considerations. If $X \subset [D^*\lambda > \alpha]$, then the family of those constituents M satisfying $\lambda(M) > \alpha\mu(M)$ is a $\tilde{B}$-fine covering of X. The same is true if "D^*" is replaced by "D_*" and "$>$" by "$<$" in the preceding sentence.

In the second definition of a $\tilde{B}$-fine covering, the requirement of the existence of at least one sequence $M_\iota(x)$ may be replaced by the following stronger one: Every x-converging sequence admits of a subsequence, the sets to which belong to $\mathbf{V}$. When this condition holds, we shall say **(21**, p. 74**)** that $\mathbf{V}$ is a *full* $\tilde{B}$*-fine covering of* X. This new requirement is equivalent to the apparently stronger one: For every x-converging sequence S consisting of the sets M_ι there exists an index $\iota' = \iota'(S)$ such that $\iota > \iota'$ implies $M_\iota \in \mathbf{V}$. The intersection of two full $\tilde{B}$-fine coverings of X is again a full $\tilde{B}$-fine covering of X; the intersection of a $\tilde{B}$-fine covering of X and a full $\tilde{B}$-fine covering of X is a $\tilde{B}$-fine covering of X.

With the same notation as above, the family of those constituents M satisfying $\lambda(M) > \alpha\mu(M)$ is a full $\tilde{B}$-fine covering of any set $X \subset [D_*\lambda > \alpha]$. The same is true if "$D_*$" and "$>$" are replaced by "$D^*$" and "$<$", respectively, in the foregoing sentence.

A point x is termed *totally interior* (with respect to $\tilde{B}$) to a subset X of R, if, for every x-converging sequence S consisting of the sets M_ι, there exists **(18)** an index $\iota' = \iota'(S, x)$, such that $\iota > \iota'$ implies $M_\iota \subset X$. We represent by $I(X)$ the set of points x which are totally interior to X. $I(X)$ need not be a subset of X. (In the case of a blanket F, $I(X)$ is $F \odot X$ in Morse's notation **(14**, p. 217**)**.) If G is such a subset of R that $E{\cdot}G \subset I(G)$, (mod $\mathbf{N}^*$), then G is called an *external* $\mathfrak{D}$*-open set* (with respect to $\tilde{B}$ and $\mathbf{N}^*$). We use G as a generic name; $\mathbf{G}$ will denote their family. $\mathfrak{D}$ refers to Denjoy, who introduced the internal $\mathfrak{D}$-open sets under the name *ensembles-enveloppes*, for his special bases, and used them as approximation sets **(2**; 21, p. 84**)**.

From the above follows the G-pruning principle: If $\mathbf{V}$ is a $\tilde{B}$-fine covering of X, and if the external $\mathfrak{D}$-open set G includes X (mod $\mathbf{N}^*$), then the family $\mathbf{V}_G$ of the $\mathbf{V}$-constituents in G is still a $\tilde{B}$-fine covering of X.

Remarks. In Morse's differentiation theory **(14)**, the space R is a metric space, with, however, the slight relaxation that two distinct points may have zero distance apart. R is provided with a Carathéodory outer measure ϕ, finite on bounded sets. To each point of a subset A of R there corresponds a family $F(x)$ of sets such that every (spherical) neighborhood of x includes an $F(x)$-set. The function F is called a *blanket.* (Blankets are special cases of prebases **(9)**.) In all blankets studied by Morse and Hayes **(10; 11; 12; 13; 14)**, the sets occurring in the families $F(x)$ are Borelian. In order to subsume Morse's blankets under the general derivation bases, we must reduce the domain A of definition of F to the set E of points x without 0-sequences, that is, sequences $M_1, M_2, \ldots, M_i, \ldots$ with $M_i \in F(x)$, $\phi(M_i) = 0$ $(i = 1, 2, \ldots)$, which converge metrically to x; under Morse's assumptions, the set A-E of points x with 0-sequences is a ϕ-nullset **(14**, p. 218**)**. Then we correlate to each M in $F(x)$ the index $\rho = \rho(M, x) =$ diameter $(M \cup \{x\})$, and define the x-converging sequences as subsequences of the *universal* sequence $M_\rho(x)$. The restriction of ϕ to the Borelian sets is taken as our fundamental measure μ.

1.2. Comparison lemmas. For $S \subset R$, we denote by $S{\cdot}\mathbf{M}$ the family of sets $S{\cdot}M$, where $M \in \mathbf{M}$, and by μ_S the restriction of $\bar{\mu}$ to $S{\cdot}\mathbf{M}$; thus, for $M \in \mathbf{M}$,

$$\mu_S(S{\cdot}M) = \bar{\mu}(S{\cdot}M) = \mu(\bar{S}{\cdot}M).$$

For $X \subset S$, we have $\mu_S(X) = \bar{\mu}(X)$. A real-valued function h defined on S is said to be *μ_S-measurable* if the Lebesgue sets $[h < \alpha]$ $(-\infty < \alpha < \infty)$ belong to $S{\cdot}\mathbf{M}$.

LEMMA 1.21 We suppose that:

(A1) f and g are real-valued functions defined on P and Q, respectively, where $Q \subset P \subset R$.

(A′2) Whenever A and B are μ^*-entangled sets of finite outer measure for which $A \cup B \subset Q$, then there exist no two numbers α and β such that $\alpha < \beta$, $A \subset [f < \alpha]$, and $B \subset [g > \beta]$.

Then $f \geqslant g$ (mod $\mathbf{N}^*$) *on* Q, *that is*, $Q{\cdot}[f < g] \in \mathbf{N}^*$.

Proof. We assume the assertion to be false; thus $\bar{\mu}(Q{\cdot}[f < g]) > 0$. There exist two (rational) numbers α and β such that $\bar{\mu}(Q{\cdot}[f < \alpha < \beta < g]) > 0$. We take for A and B two equal subsets of positive finite outer measure of $[f < \alpha < \beta < g]{\cdot}Q$. Clearly $A \subset [f < \alpha]{\cdot}Q$, $B \subset [\beta < g]{\cdot}Q$, $A = B$. Since $\bar{\mu}(A) = \bar{\mu}(B) > 0$, we have a contradiction with (A′2).

LEMMA 1.22. We assume that (A1) holds and in addition:

(A″2) Whenever A and B are any two μ^*-entangled sets of finite outer measure for which $A \cup B \subset Q$, then there exist no two numbers α and β such that $\alpha < \beta$, $A \subset [f > \beta]$, and $B \subset [g < \alpha]$.

Then $f \leqslant g$ (mod $\mathbf{N}^*$) *on* Q, *that is*, $Q{\cdot}[f > g] \in \mathbf{N}^*$.

Proof. Replace f and g in Lemma 1.21 by $-f$ and $-g$, respectively.

LEMMA 1.23. We assume (A1) holds and also:

(A2) There exist no two μ^*-entangled sets A and B of finite outer measure with $A \cup B \subset Q$ such that the convex closure of $f(A)$ and $g(B)$ have positive distance apart.[3]

Then $f = g$ (mod $\mathbf{N}^*$) *on* Q, *that is,* $Q \cdot [f \neq g] \in \mathbf{N}^*$; *also the restriction* $f|Q$ *of* f *to* Q, *and* g, *are both* $\mu^*{}_Q$*-measurable.*

Proof. It is readily seen that (A′2) and (A″2) together are equivalent to (A2); application of Lemmas 1.21 and 1.22 completes the proof of the first part.

We attend to the second part. Since μ is σ-finite, $R = \bigcup R_n$, where $R_n \in \mathbf{M}$ and $\mu(R_n) < \infty$ for $n = 1, 2, \ldots$. Hence $Q = \bigcup Q_n$, where $Q_n = Q \cdot R_n$. Since the $\mu^*{}_{Q_n}$-measurability of $g|Q_n$ (restriction of g to Q_n), for all n, implies the $\mu^*{}_Q$-measurability of $g|Q = g$, we can limit ourselves to the case where $\bar{\mu}(Q)$ is finite. We assume that the $\mu^*{}_Q$-measurability of g does not hold; hence, there exists a (rational) number δ such that $D = [g \leqslant \delta]$ is not $\mu^*{}_Q$-measurable. We denote by $\bar{D}$ and $\underline{D}$ a $\mu^*{}_Q$-cover and a $\mu^*{}_Q$-kernel of D, respectively. We let $D' = D - \underline{D}$, $D'' = \bar{D} - D$. The $\mu^*{}_Q$-non-measurability of D implies that $\mu^*{}_Q(D') = \bar{\mu}(D')$ and $\mu^*{}_Q(D'') = \bar{\mu}(D'')$ are both positive. Thus, for a suitable $\beta > \delta$, the set $S = [g > \beta] \cdot D''$ is of positive outer measure. The difference

$$D^\circ = \bar{D} - \underline{D} = D' + D'' \in Q \cdot \mathbf{M}^*;$$

hence there exists a $\mu^*{}_Q$-cover $\bar{S}$ of S which is included in D°, so that $\bar{S} = \bar{S} \cdot D' + \bar{S} \cdot D''$. Since $f = g$ (mod $\mathbf{N}^*$) in Q, then $D = [f \leqslant \delta] \cdot Q$ (mod $\mathbf{N}^*$); defining $A = \bar{S} \cdot D' \cdot [f \leqslant \delta]$, then $A = \bar{S} \cdot D'$ (mod $\mathbf{N}^*$). Due to the definition of D'', $\underline{\mu}^*{}_Q(D'') = 0$, thus $\bar{S} \cdot D''$ contains no $\mu^*{}_Q$-measurable set of positive $\mu^*{}_Q$-measure. Since $\bar{S} = A + \bar{S} \cdot D''$ (mod $\mathbf{N}^*$) and $A \subset \bar{S}$, it follows that $\bar{S}$ is a $\mu^*{}_Q$-cover for A. Let $B = S$. Then A and B are $\mu^*{}_Q$-entangled, hence μ^*-entangled. If α denotes a (rational) number between δ and β, we have $A \subset [f < \alpha]$, $B \subset [g > \beta]$, contradicting (A′2), implied by (A2).

COROLLARY 1.24. *If* $P = Q = R$ (mod $\mathbf{N}^*$), (A1) *and* (A2) *imply* $f = g$ (mod $\mathbf{N}^*$) *and the* μ^**-measurability of* f *and* g.

Remarks. Lemmas 1.21 and 1.22 will be used when f is a Radon-Nikodym μ^*-integrand and g a derivate. They are analogous to de Possel's lemma (**22**, p. 394). Lemma 1.23 can be used when f and g are the (extreme) derivates. If we know somehow that both f and g are μ^*-measurable, we can formulate (A′2) and (A″2) considering only μ^*-measurable sets A and B. The μ^*-entanglement condition then means $A = B$ (mod $\mathbf{N}^*$) and $\mu^*(A) = \mu^*(B) > 0$.

[3]This formulation, which may seem unnecessarily sophisticated for numerical functions, is intended for the more general case where f and g take their values in a separable Banach space.

1.3. The individual Vitali assumption.

PRELIMINARY DEFINITIONS 1.31. By **M**-*function* we shall mean a real-valued function defined on **M**; by **M**-measure, a non-negative σ-additive **M**-function; by *signed* **M**-*measure*, a σ-additive **M**-function of variable sign.

μ-*finiteness* means finiteness on the **M**-sets of finite measure. Hence, a μ-*finite* μ-*integral* is a μ-integral $\psi(M) = \int_M f(x)\, d\mu$, finite on the **M**-sets of finite measure.

We say that the *property* (G_σ) holds if and only if R is the union of enumerably many **G**-sets G°_n such that $\bar{\mu}(G^\circ_n) < \infty$, $n = 1, 2, \ldots$.

If such a sequence G°_n exists, then a set X is said to be *bounded* if it is included in one of the sets G°_n. Thus, our notion of boundedness depends upon the special sequence of **G**-sets occurring in the formulation of (G_σ).

When (G_σ) holds, we adopt the following definitions. A *Radon* μ-*integral* is any (indefinite) μ-integral $\psi(M) = \int_M f(x)\, d\mu$, bounded in the sets G°_n; that is, there exists, for $n = 1, 2, \ldots$, a number $\beta(n)$ such that if $M \in \mathbf{M}$ and $M \subset G^\circ_n$, then $|\psi(M)| \leqslant \beta(n)$. A *Radon measure* is an **M**-measure bounded in the sets G°_n; a *signed Radon measure* is a σ-additive **M**-function bounded in the sets G°_n. A σ-*bounded function* is any real-valued function defined on R and bounded on each set G°_n.

We state some useful classical decomposition theorems. Any μ-finite signed **M**-measure is the sum of a μ-finite integral and a finite singular part. Any signed Radon measure is the sum of a Radon μ-integral and a singular part. Also, any signed Radon measure ψ is the difference of two Radon measures ψ^+ and ψ^-; the sum $\tau = \psi^+ + \psi^-$ is the *total variation* of ψ. If (G_σ) is not assumed, "Radon" can be replaced by "μ-finite".

Henceforth, when any concept involving boundedness is considered, it will be tacitly understood that (G_σ) is presupposed.

Remarks. In the formulation of Lemmas 1.21, 1.22, and 1.23, the phrase "of finite outer measure" may be replaced by "bounded," when (G_σ) holds.

In the subsequent sections we state "full differentiation theorems" for functions ψ of the type just described, namely, theorems asserting the existence almost everywhere (that is, mod **N***) on E of the $\tilde{B}$-derivative $D\psi$ and its coincidence on E with a Radon-Nikodym μ^*-integrand. We avoid the use of such terms as "R-N derivative" (**4**, p. 133) and "pseudo-dérivée" (**22**, p. 396), reserving "derivate" and "derivative" for functions defined by means of a convergence process, either pointwise, as usual, or globally, as in (**2**) under "L-dérivée." In the (G_σ) case such an assertion will be proved if we establish it for any G°_n as universe and the G°_n-pruned basis as derivation basis. The sets G°_n play the part of autonomous domains of differentiation. Thus, assuming (G_σ), we reduce the case of a finite basic measure μ, in which the Radon assumption on ψ implies μ-finiteness.

We do not assume the sets G°_n to be μ- or μ^*-measurable, so the μ-covers of subsets of G°_n, in particular, of $E \cdot G^\circ_n$, need not be included in G°_n (mod **N***) (see Proposition 1.48). The constituents of the G°_n-pruned basis, being μ-measurable, are included (mod $\mathbf{N}^*$) in any measure kernel of G°_n.

Actually, for our purposes, a weaker form of (G_σ) suffices, as follows: There exist enumerably many sets R°_n of finite outer measure such that $R = \cup I(R^{\circ}_n)$ (mod $\mathbf{N}^*$). This property is weaker, since $I(R^{\circ}_n)$ need not be a $\mathbf{G}$-set. A set X is then said to be bounded if for some n, $X \subset I(R^{\circ}_n)$. Similarly, R°_n-pruning of a $\tilde{B}$-fine covering $\mathbf{V}$ of a set X means discarding all $\mathbf{V}$-sets not included in R°_n. The remaining $\mathbf{V}$-sets form a $\tilde{B}$-fine covering of $X \cdot I(R^{\circ}_n)$.

DEFINITIONS 1.32. By M-*family* we mean an enumerable family of sets, each with an associated multiplicity (**27**, p. 277). Equivalently, an M-family may be defined by any sequence of sets, the multiplicity associated with a set coinciding with its number of appearances in the sequence. In the latter formulation, abstraction is made of the order of appearance of any set. Certain advantages arise from the use of M-families instead of ordinary families in the work to follow. For instance, the *frequency* (defined a few lines farther on) is additive: thus, if $\mathbf{E}$ and $\mathbf{F}$ are M-families and $\mathbf{G}$ is the M-family obtained by uniting them, then $\phi_{\mathbf{E}} + \phi_{\mathbf{F}} = \phi_{\mathbf{G}}$. However, it is only subadditive for ordinary families. Also, any μ-measurable function on R, taking only positive integral values, may be regarded as the frequency function of a measurable M-family covering R. Awkward limitations occur if we restrict ourselves to families without repetition. In natural fashion, we may define the limit of a sequence $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n, \ldots$ of M-families as the M-family $\mathbf{E}$, if it exists, such that $\lim \phi_{\mathbf{E}_n} = \phi_{\mathbf{E}}$. So defined, $\mathbf{E}$ has an *overlap* (defined just below) which is conveniently represented by use of the Lebesgue convergence theorem.

If $\mathbf{E}$ is an M-family, then $\sigma\mathbf{E}$ will denote the union of the sets occurring in $\mathbf{E}$. By $\mathbf{E}$-*frequency* $\phi_{\mathbf{E}}(x)$ *at the point* x we shall mean the number of $\mathbf{E}$-sets (possibly ∞) to which x belongs; by $\mathbf{E}$-*excess function* we shall mean that function $\epsilon_{\mathbf{E}}$ defined on $\sigma\mathbf{E}$ by $\epsilon_{\mathbf{E}}(x) = \phi_{\mathbf{E}}(x) - 1$. We define $\theta\mathbf{E} = [\epsilon_{\mathbf{E}}(x) > 0] = [\phi_{\mathbf{E}}(x) \geqslant 1]$, and call $\theta\mathbf{E}$ the $\mathbf{E}$-*overlap set.*

Henceforth we assume that the $\mathbf{E}$-sets belong to $\mathbf{M}$. Then $\phi_{\mathbf{E}}$ and $\epsilon_{\mathbf{E}}$ are μ-measurable. If ψ is any $\mathbf{M}$-measure, we define the ψ-*overlap of* $\mathbf{E}$ by

$$\omega(\mathbf{E}, \psi) = \int_{\sigma\mathbf{E}} \epsilon_{\mathbf{E}}(x)\, d\psi.$$

In case $\psi(\sigma\mathbf{E})$ is finite, we note that

$$\omega(\mathbf{E}, \psi) = \sum_{M \epsilon \mathbf{E}} \psi(M) - \psi(\sigma\mathbf{E}).$$

In the particular case $\psi = \mu$, the foregoing equations define the μ-*overlap of* $\mathbf{E}$, which is of somewhat special importance (**8**, p. 193).

If $X \subset R$, M is a μ-cover for X, and ψ is any $\mathbf{M}$-measure, then the ψ-*overflow of* $\mathbf{E}$ *with respect to* X *and* M is defined as $\psi(\sigma\mathbf{E} - M \cdot \sigma\mathbf{E})$. If ψ is μ-absolutely continuous, then the quantity just defined will not depend upon the particular μ-cover M, but will be the same for every set $\bar{X}$, and the terminal phrase "and M" may be dropped. In particular, if $\psi = \mu$, then $\mu(\sigma\mathbf{E} - \bar{X}\cdot\sigma\mathbf{E})$ is the *μ-overflow of* $\mathbf{E}$ *with respect to* X.

If $X \subset R$, then we define the *μ-defect of covering of* X as the number $\bar{\mu}(X - X\cdot\sigma\mathbf{E})$, and we denote this by the notation $\gamma(\mathbf{E}, X, \mu)$. $\mathbf{E}$ is said to be an *ϵ-covering in measure* of X if $\gamma(\mathbf{E}, X, \mu) < \epsilon$; it is said to be an **0**-*covering in measure* of X if $X \subset \sigma\mathbf{E}$ (mod $\mathbf{N}^*$).

DEFINITIONS 1.33. If ψ denotes a non-negative $\mathbf{M}$-measure, we say that the basis $\tilde{B}$ possesses the *Vitali ψ-property* if, and only if, for any $X \subset E$ of finite outer measure, any $\tilde{B}$-fine covering $\mathbf{V}$ of X, any μ-cover M of X, and any $\epsilon > 0$, there exists an (enumerable) M-family $\mathbf{E}$ of $\mathbf{V}$-sets such that, for $S = \sigma\mathbf{E}$:

(V1) $X - X\cdot S \in \mathbf{N}^*$ ($\mathbf{E}$ is an 0-covering of X);

(V2) $\psi(S - S\cdot M) < \epsilon$ (the ψ-overflow of $\mathbf{E}$ with respect to X and M is less than ϵ);

(V3) $\omega(\mathbf{E}, \psi) < \epsilon$ (the ψ-overlap of $\mathbf{E}$ is less than ϵ).

(K. O. Househam has suggested the term *ψ-redundancy* of $\mathbf{E}$ with respect to X and M for the sum of the ψ-overflow and the ψ-overlap.)

In case only (V1) and (V3) hold, we say that $\tilde{B}$ possesses the *reduced Vitali ψ-property.*

Remarks. If $\tilde{B}$ possesses the Vitali property corresponding to ψ, then it evidently possesses the Vitali property corresponding to all $\psi' \leqslant \psi$; that is, the Vitali ψ-property has a hereditary character. In particular, if ψ is a Radon or a μ-finite $\mathbf{M}$-measure, $\tilde{B}$ possesses the Vitali property corresponding to the μ-absolutely continuous part of ψ.

Some equivalent formulations of the Vitali ψ-property are possible. The requirement (V1) may be replaced by an ϵ-covering condition; simultaneously, "enumerable" may be replaced by "finite." That such an ϵ-covering version implies the original version can be shown by an exhaustion process. The requirement that X be of finite outer measure may be dropped. In the (G_σ) case, the phrase "of finite outer measure" may be replaced by "bounded".

DEFINITION 1.34. We define the upper *μ-approximation property of the* $\mathbf{M}$-*sets by the* $\mathbf{G}$-sets (abbreviated (UG)) as follows: Corresponding to any $\mathbf{M}$-set of M of finite measure, and any $\eta > 0$, there exists (**21**, p. 83) a $\mathbf{G}$-set G for which $M \subset G$ and $\bar{\mu}(G - M) < \eta$.

We note that (UG) implies (G_σ). (UG) is not altered if the condition "of finite measure" is waived.

PROPOSITION 1.35. *If* (UG) *holds, ψ is a non-negative μ-finite* (*resp., Radon*) *μ-integral, and $\tilde{B}$ possesses the reduced Vitali ψ-property, then $\tilde{B}$ enjoys the Vitali ψ-property.*

Proof. We let X denote any subset of E of finite outer measure (resp., bounded), $\mathbf{V}$ any $\tilde{B}$-fine covering of X, ϵ any positive number. We use (UG) to find a **G**-set $G' \supset \bar{X}$ with $\bar{\mu}(G' - \bar{X}) < 1$. Since ψ is μ-absolutely continuous and $\psi(\bar{X})$ is finite, there exists $\eta = \eta(X, \psi, \epsilon) > 0$ such that $|\psi(\bar{X}) - \psi(M)| < \epsilon$ whenever $M \in \mathbf{M}$, $M \subset G'$ and $\mu(M - \bar{X}) < \eta$, where $M - \bar{X}$ denotes Stone's difference. Invoking (UG), and the fact that the product of two $\mathfrak{D}$-open sets is again $\mathfrak{D}$-open, we find a **G**-set G with $G' \supset G \supset \bar{X}$ and $\bar{\mu}(G - \bar{X}) < \eta$. We apply the reduced Vitali ψ-property to the G-pruned family $\mathbf{V}_G$, to obtain an M-family **E** satisfying (V1) and (V3). Since the **E**-sets lie in G, we have $\mu(S - S\cdot\bar{X}) \leqslant \bar{\mu}(G - G\cdot\bar{X}) < \eta$. Thus $\psi(S) < \psi(\bar{X}) + \epsilon$, whence $\psi(S - S\cdot\bar{X}) < \epsilon$; (V2) holds, as required.

DEFINITION 1.36. We say that *Haupt's adaptation property* holds if and only if there exists a σd-family $\mathbf{G}^\circ$ of **G**-sets which is a Borel generator for **M** (that is, **M** is the smallest $\sigma\delta$-family including $\mathbf{G}^\circ$) (**5**, p. 173).

PROPOSITION 1.37. *Haupt's adaptation property implies the following (which includes* (UG)*): For any Radon measure ψ, any* **M***-set M, and any $\epsilon > 0$, there exists a* **G***-set G such that $G \supset M$ and $\bar{\psi}(G - M) < \epsilon$.*

Proof. For the case $\psi(R) < \infty$, the proof is given in (**7**, p. 27). We shall establish the theorem assuming $\psi(R) = \infty$. We introduce the sequence $G^\circ_1, G^\circ_2, \ldots, G^\circ_n, \ldots$, associated with the property (G_σ), and for any set $M \in \mathbf{M}$ and any positive integer n, we define $\psi_n(M) = \bar{\psi}(G^\circ_n\cdot M)$.

We let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ denote a sequence of positive numbers whose sum is less than ϵ. If $M \in \mathbf{M}$, then we may apply the theorem to ψ_n, since $\psi_n(R) < \infty$, to find a $\mathbf{G}^\circ$-set G'_n such that $M \subset G'_n$ and

$$\psi_n(G'_n - M) = \bar{\psi}(G^\circ_n \cdot (G'_n - M)) < \epsilon_n.$$

The set $G^\circ_n\cdot G'_n = G''_n$ is thus a **G**-set (not necessarily a $\mathbf{G}^\circ$-set) including $G^\circ_n\cdot M$, such that $\bar{\psi}(G''_n - G^\circ_n\cdot M) < \epsilon_n$. We let S denote the union of the sets G''_n; then $S \supset M$, and

$$S - M = \bigcup G''_n - \bigcup G^\circ_n \cdot M \subset \bigcup (G''_n - G^\circ_n \cdot M),$$

hence

$$\bar{\psi}(S - M) \leqslant \sum \bar{\psi}(G''_n - G^\circ_n \cdot M) < \epsilon.$$

Since S is a **G**-set, the proposition is proved.

Remarks. The property described in Proposition 1.37 is called the *universal upper approximation property for* $\mathbf{G}^\circ$*-sets.* It holds (**16**, pp. 244–245) in the special case where R is a metric space, ψ is a classical finite Radon measure, and $\mathbf{G}^\circ$ is the family of the open sets.

PROPOSITION 1.38. *If ψ is a μ-finite* **M***-measure, Haupt's adaptation property and the reduced Vitali ψ-property both hold, then the Vitali ψ-property holds.*

Proof. This follows closely the proof of Proposition 1.35, except that we take a μ-cover M of X, and use Proposition 1.37 directly to find a **G**-set $G \supset M$ with $\bar{\psi}(G - M) < \epsilon$. As before, we find an M-family **E** satisfying

(V1) and (V3), with members lying in G. Thus $\bar{\psi}(S - S\cdot M) \leqslant \psi(G - M) < \epsilon$, and (V2) holds.

1.4. The individual full differentiation theorem for Radon or μ-finite μ-integrals.

PROPOSITION 1.41. *If ψ is a non-negative Radon (or μ-finite) μ-integral $\int f(x)\,d\mu$, and $\tilde{B}$ possesses the Vitali μ-property, then $D_*\psi \geqslant f(\text{mod } \mathbf{N}^*)$ on E.*

Proof. According to the Remarks following Definition 1.31, we need treat only the case where ψ is μ-finite. We shall obtain a contradiction from the assumed existence of two μ^*-entangled subsets A and B of E of finite outer measure and two numbers α, β such that $\alpha < \beta$, $A \subset [f > \beta]$, $B \subset [g < \alpha]$, where $g = D_*\psi$. Since $[f > \beta] \in \mathbf{M}$, $A' = \bar{A}\cdot[f > \beta]$ is a μ-cover of A; since $\mu(\bar{A}) > 0$, we have

1.411 $$\psi(A') > \beta\,\mu(A').$$

On the other hand, the family $\mathbf{V}$ of the constituents V satisfying

1.412 $$\psi(V) < \alpha\,\mu(V)$$

is a $\tilde{B}$-fine covering of $B \subset [g < \alpha]$. Thus, by virtue of the Vitali μ-property, for any natural number n, there exists an M-family $\mathbf{E}_n$ of $\mathbf{V}$-sets V_{ni}, such that if $S_n = \sigma\mathbf{E}_n$, then

1.413 $B - B\cdot S_n \in \mathbf{N}^*$; $\mu(S_n - S_n\cdot\bar{B}) < 2^{-n}$; μ —overlap of $\mathbf{E}_n$ is less than 2^{-n}.

Using 1.413 and 1.412 we obtain

1.414 $$\psi(\bar{B}) \leqslant \psi(S_n) \leqslant \sum_i \psi(V_{ni}) < \alpha\sum_i \mu(V_{ni}) < \alpha(\mu(S_n) + 2^{-n}),$$

and $\lim \mu(S_n) = \mu(\bar{B})$. Combining, we obtain

1.415 $$\psi(\bar{B}) \leqslant \alpha\mu(\bar{B}),$$

which, since $\psi(A') = \psi(\bar{B})$ and $\alpha < \beta$, is a contradiction of 1.411. From Lemma 1.22 follows the assertion $f \leqslant g$ (mod $\mathbf{N}^*$).

Remarks. If an ϵ-covering version of the Vitali μ-property is used in place of the 0-covering version, then the first statement in 1.413 is replaced by $\mu(\bar{B} - \bar{B}\cdot S_n) < \eta_n$, and because of the μ-absolute continuity of ψ, η_n can be so chosen that $\psi(\bar{B}) \leqslant \psi(S_n) + 2^{-n}$; 1.414 has to be altered accordingly.

An example of a blanket possessing the Vitali μ-property, and a function $g \in \mathfrak{L}^{(p)}$ for every $p > 1$, such that its integral ψ has $D\psi = \infty$ everywhere, is known (**11**, p. 293).

PROPOSITION 1.42. *If ψ is a non-negative Radon (or μ-finite) μ-integral $\int f(x)\,d\mu$, and $\tilde{B}$ possesses the Vitali ψ-property, then $D^*\psi \leqslant f(\text{mod } \mathbf{N}^*)$ on E.*

Proof. As in the preceding proposition, we may and do assume that ψ is μ-finite. We assume that A and B are two μ^*-entangled sets of finite outer measure, α and β two numbers such that $\alpha < \beta$, $A \subset [f < \alpha]$, $B \subset [g > \beta]$, where $g = D^*\psi$. Since $[f < \alpha] \in \mathbf{M}$, $A' = \bar{A}\cdot[f < \alpha]$ is a μ-cover for A. Since $\mu(\bar{A}) > 0$, we obtain

1.421 $$\psi(A') < \alpha\mu(A').$$

The family $\mathbf{V}$ of the constituents V satisfying

1.422 $$\psi(V) > \beta\mu(V)$$

is a $\tilde{B}$-fine covering of $B \subset [g > \beta]$. We use the Vitali ψ-property to determine for each natural number n an M-family $\mathbf{E}_n$ of $\mathbf{V}$-sets V_{ni} such that if $S_n = \sigma\mathbf{E}_n$, then

1.423 $B - B\cdot S_n \in \mathbf{N}^*$; $\psi(S_n - S_n\cdot\bar{B}) < 2^{-n}$; ψ —overlap of $\mathbf{E}_n$ is less than 2^{-n}.

The ψ-overlap condition yields

$$\psi(S_n) > \sum_i \psi(V_{ni}) - 2^{-n};$$

hence, using 1.422,

$$\psi(S_n) > \beta\sum_i \mu(V_{ni}) - 2^{-n} \geqslant \beta\,\mu(S_n) - 2^{-n}.$$

This last and 1.423 together yield, for $n = 1, 2, \ldots,$

1.424 $$\psi(\bar{B}) + 2^{-n} > \beta\mu(\bar{B}) - 2^{-n};$$

hence $\psi(\bar{B}) \geqslant \beta\mu(\bar{B})$, which contradicts 1.421, since $\alpha < \beta$ and $\psi(A') = \psi(\bar{B})$. Thus, Lemma 1.21 applies and $D^*\psi \leqslant f$ (mod $\mathbf{N}^*$) on E.

Remark. With the ϵ-covering version of the Vitali ψ-property, we replace the first statement in 1.423 by $\mu(\bar{B} - \bar{B}\cdot S_n) < 2^{-n}$, hence $\mu(S_n) > \mu(\bar{B}) - 2^{-n}$, and in 1.424, we replace $\mu(\bar{B})$ by $\mu(\bar{B}) - 2^{-n}$.

THEOREM 1.43. *If ψ is a non-negative Radon (or μ-finite) μ-integral and $\tilde{B}$ possesses the Vitali μ-property and the Vitali ψ-property, then the $\tilde{B}$-derivative $D\psi$ exists almost everywhere on E and is equal,* mod $\mathbf{N}^*$, *to $f|E$, where f denotes any Radon-Nikodym μ^*-integrand of ψ.*

Proof. This is an immediate consequence of Propositions 1.41 and 1.42.

DEFINITION 1.44. An $\mathbf{M}$-function ψ is said to be *majorized* or *dominated* by the $\mathbf{M}$-function $\psi°$ if $|\psi(M)| \leqslant \psi°(M)$ for every $M \in \mathbf{M}$.

We note that a signed Radon measure (resp., μ-finite $\mathbf{M}$-measure) dominated by a μ-integral is itself a μ-integral; also a finitely additive $\mathbf{M}$-function dominated by a Radon measure (resp., μ-finite $\mathbf{M}$-measure) is a signed Radon measure (resp., signed μ-finite $\mathbf{M}$-measure).

THEOREM 1.45. *If $\psi°$ is a non-negative Radon (or μ-finite) μ-integral and $\tilde{B}$ possesses the Vitali μ-property and the Vitali $\psi°$-property, then for any signed Radon measure (or μ-finite signed $\mathbf{M}$-measure) ψ dominated by $\psi°$, the $\tilde{B}$-derivative $D\psi$ exists* p.p. *on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ.*

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Proof. This follows immediately upon decomposing ψ into ψ^+ and ψ^- and using the hereditary character of the Vitali ψ-property.

THEOREM 1.46. *If ψ° is a non-negative Radon μ-integral, $\tilde{B}$ possesses the Vitali μ-property and the Vitali ψ°-property, ψ is a signed Radon measure, and there corresponds to each G°_n a positive finite number $\kappa(n)$ such that $|\psi(M)| \leqslant \kappa(n) \cdot \psi^\circ(M)$ for any* **M***-set $M \subset G^\circ_n$ (ψ°-Lipschitz condition), then $D\psi$ exists almost everywhere on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ.*

Proof. Apply Theorem 1.45 to each G°_n used as an autonomous domain of $\tilde{B}$-differentiation, with $\kappa(n)\cdot\psi^\circ$ as majorant.

Remark. If we know that the extreme derivates are μ^*-measurable, then from the Remarks under Corollary 1.24, it follows that Theorems 1.43, 1.45, and 1.46 remain valid, if, in the definition of the Vitali property, X is taken from **M**.

DEFINITION 1.47. The special case of the Vitali property, wherein $\psi = \mu$, is the so-called *weak Vitali property*, and a basis $\tilde{B}$ possessing it is called a *weak* derivation basis.

Remarks. By Theorem 1.46, such a basis differentiates (in de Possel's sense) the uniformly μ-Lipschitzian integrals; explicitly, if ψ is a σ-additive **M**-function for which $|\psi(M)| \leqslant \kappa\, \mu(M)$ for $M \in$ **M**, where κ is a constant, then $D\psi$ exists almost everywhere on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ. In de Possel's version, a weak derivation basis differentiates the integral of any essentially bounded μ-measurable function, and in the Radon case, the integrals of functions which are measurable and essentially bounded on each G°_n. Five equivalent properties defining these bases in the case $E = R$ (mod **N***) are given in (**22**, pp. 403–405).

PROPOSITION 1.48. *If each B-fine covering of any subset X of E admits an enumerable subfamily covering X* (mod **N***), *then for any* **G***-set G, we have* $\overline{E\cdot G} \subset G$ (mod **N***) *or equivalently* $E\cdot G \subset \underline{G}$ (mod **N***). *If in addition, $E = R$, then the* **G***-sets are μ^*-measurable.*

Proof. The family **W** of the B-constituents included in G is a B-fine covering of $E\cdot G$; thus there exists an M-family $\mathbf{E} \subset \mathbf{W}$ with $E\cdot G \subset \sigma\mathbf{E}$ (mod **N***). Hence $\overline{E\cdot G} \subset \sigma\mathbf{E}$ (mod **N**), and $\overline{E\cdot G} \subset G$ (mod **N***). If $E = R$, then $\bar{G} \subset G$ (mod **N***), hence $G = \bar{G}$ (mod **N***).

1.5. The individual full differentiation theorem for Radon measures.

LEMMA 1.51. *If ψ is a Radon measure (or a μ-finite* **M***-measure), $\tilde{B}$ possesses the Vitali ψ-property, $Q \subset E$, $\mu(Q) < \infty$, $0 < \eta < \infty$, and there exists a $\tilde{B}$-fine covering* **V** *of Q such that for all* **V***-sets V,*

1.511 $$\psi(V) \geqslant \eta\, \mu(V)$$

then $\psi(M) \geqslant \eta\, \bar{\mu}(Q)$ for any $M \in$ **M** *with $Q \subset M$.*

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Proof. The Remarks following Definitions 1.31 permit us to consider only the case of a μ-finite **M**-measure. We may also assume $\mu(M) < \infty$. We let ϵ denote an arbitrary positive number, let T denote a μ-cover of Q for which $Q \subset T \subset M$, and invoke the Vitali ψ-property to obtain an M-family **E** of sets V_i, $i = 1, 2, \ldots$, for which, putting $\sigma\mathbf{E} = S$, we have

1.512 $\quad Q - Q{\cdot}S \in \mathbf{N}^*$; ψ-overlap of **E** is less than ϵ; $\psi(S - S{\cdot}T) < \epsilon$.

From 1.511 and the first two conditions of 1.512 we obtain

$$\psi(S) > \sum_i \psi(V_i) - \epsilon \geqslant \eta \cdot \sum_i \mu(V_i) - \epsilon \geqslant \eta\, \mu(S) - \epsilon \geqslant \eta\bar{\mu}(Q) - \epsilon;$$

this result, combined with the last inequality of 1.512, yields

$$\psi(M) \geqslant \psi(T) \geqslant \psi(T{\cdot}S) > \psi(S) - \epsilon \geqslant \eta\, \bar{\mu}(Q) - 2\epsilon,$$

which, since ϵ is arbitrary, gives the desired relation.

THEOREM 1.52. *If ψ is a Radon measure (or a μ-finite **M**-measure) and $\tilde{B}$ possess the Vitali μ- and ψ-properties, then ψ possesses almost everywhere in E a $\tilde{B}$-derivative $D\psi$ which is equal to a Radon-Nikodym integrand of ψ.*

Proof. We decompose ψ into the μ-regular part ψ_r and the μ-singular part ψ_s, denoting by N_0 an **N**-set on which ψ_s is concentrated, that is, $\psi_s(R-N_0)=0$. $\tilde{B}$ possesses the Vitali μ- and ψ-properties, hence, in accordance with the Remarks under Definitions 1.33, $\tilde{B}$ also has the Vitali μ- and ψ_r-properties. Due to Theorem 1.43, we need prove only that $D^*\psi_s = 0$ (mod $\mathbf{N}^*$).

We let $A_n = [D^*\psi_s > n^{-1}]{\cdot}(R - N_0)$. The family of $\tilde{B}$-constituents V for which $\psi_s(V) \geqslant n^{-1}\,\mu(V)$ is a $\tilde{B}$-fine covering of A_n. In accordance with Lemma 1.51, $\psi_s(M) \geqslant n^{-1}\,\bar{\mu}(A_n)$ for any $M \in \mathbf{M}$ with $A_n \subset M$; in particular, this holds for $M = R - N_0$, thus $0 \geqslant n^{-1}\,\bar{\mu}(A_n)$, hence $\bar{\mu}(A_n) = 0$, and A_n is an $\mathbf{N}^*$-set. Now

$$[D^*\psi_s > 0]\cdot(R - N_0) = \bigcup_n [D^*\psi_s > n^{-1}]\cdot(R - N_0) = \bigcup_n A_n,$$

which is therefore also an $\mathbf{N}^*$-set. Finally,

$$[D^*\psi_s > 0] = [D^*\psi_s > 0]{\cdot}N_0 + [D^*\psi_s > 0]{\cdot}(R - N_0)$$

is an $\mathbf{N}^*$-set, and the proof is complete.

The following results are immediate consequences of Theorems 1.52, 1.45, and 1.46.

THEOREM 1.53. *If ψ° is a Radon measure (resp., μ-finite **M**-measure) and $\tilde{B}$ possesses the Vitali μ- and ψ°-properties, then $\tilde{B}$ differentiates any signed Radon (resp., μ-finite) **M**-measure dominated by ψ°.*

THEOREM 1.54. *If ψ° is a Radon measure, $\tilde{B}$ possesses the Vitali μ- and ψ°-properties, and ψ is such a signed Radon measure that there corresponds to any*

G°_n a positive (finite) number $\kappa(n)$ such that $|\psi(M)| \leqslant \kappa(n)\cdot\psi^\circ(M)$ for any **M**-*set $M \subset G^\circ_n$, then $\tilde{B}$ differentiates ψ.*

The remark following Theorem 1.46 applies here also.

1.6. Class differentiation theorems.

DEFINITION 1.61. If $\tilde{B}$ possesses the Vitali ψ-property for *every* non-negative Radon (resp., μ-finite) μ-integral ψ, then we say that $\tilde{B}$ has the *Vitali property for non-negative Radon* (resp., *μ-finite*) *μ-integrals.*

THEOREM 1.62. *If $\tilde{B}$ has the Vitali property for non-negative Radon (resp., μ-finite) μ-integrals, then $\tilde{B}$ differentiates every Radon (resp., μ-finite) μ-integral.*

Proof. This follows from Theorem 1.43.

DEFINITION 1.63. If $\tilde{B}$ possesses the Vitali ψ-property for *every* Radon (resp., μ-finite) **M**-measure ψ, then we say that $\tilde{B}$ has the *Vitali property for Radon* (resp., *μ-finite*) **M**-measures.

THEOREM 1.64. *If $\tilde{B}$ has the Vitali property for Radon (resp., μ-finite)* **M**-*measures, then $\tilde{B}$ differentiates every Radon (resp., μ-finite)* **M**-*measure.*[4]

Proof. This follows from Theorem 1.52.

Remark. De Possel **(23; 24)** defines a "système dérivant généralisé" as a correspondence to each point x of a filter F_x of non-negative μ-measurable summable real functions f, vanishing outside a measurable set of finite measure (depending on f), and with $\int_R f d\mu > 0$, ψ denotes any function on **M** into a Banach space, enumerably additive and of bounded variation. The derivative $D\psi(x)$ is defined as

$$\lim_{F_x}\left(\int_R f d\psi \Big/ \int_R f d\mu\right).$$

Conditions are stated for F_x to differentiate Lipschitz, μ-absolutely continuous, and general functions ψ.

DEFINITIONS 1.65. We shall introduce a chain of properties between the Vitali μ-property and the Vitali property for non-negative Radon μ-integrals, under the assumption that (G_σ) holds. We let p and q denote two numbers, both greater than 1, for which $p^{-1} + q^{-1} = 1$. By *$\mu^{(q)}$-functions* we shall mean those Radon μ-integrals ψ of the form $\psi(M) = \int_M f(x)\, d\mu$, for $M \in$ **M**, where f is such a function that for any given positive integer n, $\int_M |f(x)|^q d\mu$ is (uniformly) bounded on the **M**-sets M included in G°_n; by $\mathfrak{L}^{(q)}$-functions we shall mean those functions f which are integrands of $\mu^{(q)}$-functions. We shall say that $\tilde{B}$ is an *$S^{(p)}$-basis* if and only if for each subset $X \subset E$ of finite outer

[4]In case $\tilde{B}$ is a blanket, the Vitali property for classical Radon measures is the "pseudo-strength" of **(10)**, which also is referred to as "Vitalische-Hayes'sche Eigenschaft" in **(21**, p. 91**)**. The existence p.p. of the derivative of any classical Radon measure is established in **(10)**.

measure, each $\tilde{B}$-fine covering $\mathbf{V}$ of X, and each $\epsilon > 0$, there exists an M-family $\mathbf{E}$ of $\mathbf{V}$-sets for which, putting $\sigma\mathbf{E} = S$,

(I) $\mathbf{E}$ is an 0-covering of X;

(II) the μ-overflow of $\mathbf{E}$ with respect to X is less than ϵ;

(III) $\int_S\{\epsilon_{\mathbf{E}}(x)\}^p d\mu < \epsilon$ (the $\mathfrak{L}^{(p)}$*-overlap* of $\mathbf{E}$ is less than ϵ).

Statements (I), (II), and (III) are meaningful for $p = 1$; we accordingly define an $S^{(1)}$*-basis* as one having properties (I), (II), and (III), with $p = 1$. We define as μ^∞*-functions* all integrals of μ-measurable functions which are essentially bounded on each set G°_n.

Remarks. Comparison with Definitions 1.33 shows that for any $p \geqslant 1$, (I) and (II) are the same as (V1) and (V2), while (III) is at least as strong as (V3) for $\psi = \mu$; hence every $S^{(p)}$-basis, $p \geqslant 1$, possesses the Vitali μ-property, and, in accordance with the Remarks following Definition 1.47, differentiates the μ^∞-functions. The following is an extension of this result.

THEOREM 1.66. *If $p > 1$ and $\tilde{B}$ is an $S^{(p)}$-basis, then $\tilde{B}$ differentiates the $\mu^{(q)}$-functions.*

Proof. We let $\tilde{B}$ denote any $S^{(p)}$-basis. From the property (G_σ), it follows that we may restrict our proof to the case where the domain E of $\tilde{B}$ lies within one set G°_N; that is, E may be assumed to be bounded. Furthermore, it follows from the remarks just above, and from Theorem 1.45, that we need prove only that for each non-negative $\mu^{(q)}$-function ψ, defined by $\psi(M) = \int_M f(x)\cdot d\mu$ for $M \in \mathbf{M}$, $\tilde{B}$ possesses the Vitali ψ-property.

Accordingly, we let X denote any subset of E (necessarily of finite outer measure), $\mathbf{V}$ any $\tilde{B}$-fine covering of X, and ϵ any positive number. We put $G^\circ_N = G$ and define ϵ' as any positive number such that

1.661 $$(\epsilon')^{1/p}\left(\int_{\underline{G}}\{f(x)\}^q\cdot d\mu\right)^{1/q} < \epsilon.$$

From the μ-absolute continuity of ψ on the $\mathbf{M}$-subsets of G, it follows that there exists a positive number η for which

1.662 $$|\psi(M') - \psi(M'')| < \epsilon$$

whenever $\mu(M' - M'') < \eta$, where $M' \in \mathbf{M}$, $M'' \in \mathbf{M}$, $M' \subset G$, $M'' \subset G$, and $M' - M''$ denotes Stone's difference. We may and do assume that $\eta < \epsilon'$.

We may assume $\mathbf{V}$ to be G-pruned. We use the $S^{(p)}$-properties of $\tilde{B}$ to find an M-family $\mathbf{E}$ of $\mathbf{V}$-sets for which, putting $S = \sigma\mathbf{E} \subset G$, we have

1.663 $$X - X\cdot S \in \mathbf{N}^*;\ \mu(S - S\cdot\bar{X}) < \eta;\ \int_S\{\epsilon_{\mathbf{E}}(x)\}^p\cdot d\mu < \eta.$$

Evidently $\mathbf{E}$ satisfies (V1) of Definition 1.33. From the first relation in 1.663 and Proposition 1.48, we see that $S \subset \underline{G}$ (mod $\mathbf{N}^*$) and $\bar{X} \subset G$ (mod $\mathbf{N}^*$); hence, because of 1.662,

$$|\psi(S) - \psi(\bar{X}\cdot S)| < \epsilon;$$

therefore $\psi(S - S\cdot\bar{X}) < \epsilon$, and (V2) holds.

Using Hölder's inequality, 1.661, and the last relation in 1.663, we have

$$\int_{\sigma \mathbf{E}} \epsilon_{\mathbf{E}}(x)\, d\psi = \int_S \epsilon_{\mathbf{E}}(x) f(x)\, d\mu$$
$$\leqslant \left(\int_S \{\epsilon_{\mathbf{E}}(x)\}^p d\mu\right)^{1/p} \left(\int_S \{f(x)\}^q d\mu\right)^{1/q}$$
$$< \eta^{1/p} \left(\int_{\underline{G}} \{f(x)\}^q d\mu\right)^{1/q} < \epsilon.$$

Hence (V3) holds, and the proof is complete.

Remark. In **(11)**, there is given an example of an $S^{(p)}$-basis ($p > 1$) and a function which is a $\mu^{(q')}$-function for each q', $q' < p/(p-1)$, whose derivative is infinite everywhere. In this example, as in all counter-examples known to us in the theory of differentiation, a derivate is infinite on a set of positive measure. In this connection it is interesting to observe that Zygmund's proof **(29)** depends upon the summability of the derivates, which prevents a "flight to infinity" on a set of positive measure.

THEOREM 1.67. *In the definition of an $S^{(p)}$-basis, the 0-covering condition may be replaced by an ϵ-covering condition; simultaneously* $\mathbf{E}$ *may be required to be finite.*

Proof. Since we are merely relaxing the initial definition of an $S^{(p)}$-basis, we have to prove only that any $S^{(p)}$-basis under the ϵ-covering definition is an $S^{(p)}$-basis under the 0-covering definition. We thus assume that for any subset $X \subset E$ of finite outer measure, any $\tilde{B}$-fine covering $\mathbf{V}$ of X, and any $\epsilon > 0$, there exists a finite family $\mathbf{F}$ of $\mathbf{V}$-constituents such that

1.671 $$\bar{\mu}(X - X \cdot \sigma\mathbf{F}) < \epsilon;\ \mu(\sigma\mathbf{F} - \bar{X} \cdot \sigma\mathbf{F}) < \epsilon;\ \int_{\sigma\mathbf{F}} \{\epsilon_{\mathbf{F}}(x)\}^p d\mu < \epsilon.$$

We take a subset X of E of finite outer measure, a $\tilde{B}$-fine covering of X, and a positive number ϵ. We choose a sequence of positive numbers, $\eta_1, \eta_2, \dots, \eta_n, \dots$ whose sum is less than ϵ.

We shall determine inductively a sequence of finite families $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n, \dots$ of $\mathbf{V}$-constituents, such that, for $n = 1, 2, \dots$:

(a) $\mathbf{F}_1 \subset \mathbf{F}_2 \subset \dots \subset \mathbf{F}_n$;

(b) $\mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{F}_n) < \eta_n$;

(c) $\mu(\sigma\mathbf{F}_n - \bar{X} \cdot \sigma\mathbf{F}_n) < \sum_{i=1}^{n} \eta_i = \zeta_n$;

(d) $\int_{\sigma\mathbf{F}_n} \{\epsilon_{\mathbf{F}_n}(x)\}^p d\mu < \sum_{i=1}^{n} \eta_i = \zeta_n$.

The existence of a family $\mathbf{F}_1$ satisfying (b), (c), and (d), for $n = 1$, follows from our hypotheses as expressed in 1.671. We assume the existence of a nested sequence of families $\mathbf{F}_1, \mathbf{F}_2, \dots \mathbf{F}_n$, satisfying (a), (b), (c), and (d), and proceed to find $\mathbf{F}_{n+1}$ also satisfying them.

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We put $\sigma\mathbf{F}_n = S$, $X - X\cdot S = Y$; then $\bar{Y} = \bar{X} - \bar{X}\cdot S$ is a μ-cover for Y. From (d) and the fact that $\mu(S) < \infty$ it follows that

$$\int_S \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \infty;$$

thus we may find a positive number $\gamma = \gamma(\eta_{n+1})$ such that

$$1.672 \qquad \int_M \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \eta_{n+1}/2^p$$

whenever M is an $\mathbf{M}$-set, $M \subset S$, and $\mu(M) < \gamma$. We may and do assume that $\gamma < \eta_{n+1}/2^p$.

Again recalling 1.671, we find a finite subfamily $\mathbf{H}$ of $\mathbf{V}$ for which, putting $\sigma\mathbf{H} = T$,

$$1.673 \qquad \mu(\bar{Y} - \bar{Y}\cdot T) < \gamma;\ \mu(T - \bar{Y}\cdot T) < \gamma;\ \int_T \{\epsilon_{\mathbf{H}}(x)\}^p d\mu < \gamma.$$

Noting that $S\cdot T \subset T - \bar{Y}\cdot T$, using 1.672, and the second relation of 1.673, we obtain

$$1.674 \qquad \int_{S\cdot T} \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \eta_{n+1}/2^p.$$

We define $\mathbf{F}_{n+1} = \mathbf{F}_n \cup \mathbf{H}$ and let $\sigma\mathbf{F}_{n+1} = U$. We observe that $\bar{X} - \bar{X}\cdot U = \bar{Y} - \bar{Y}\cdot T$ and $(U - \bar{X}\cdot U) \subset (S - \bar{X}\cdot S) + (T - \bar{Y}\cdot T)$, whence from 1.673 and (c), we obtain

$$\mu(\bar{X} - \bar{X}\cdot U) < \gamma < \eta_{n+1};\ \mu(U - \bar{X}\cdot U) < \zeta_n + \gamma < \zeta_n + \eta_{n+1} = \sum_{i=1}^{n+1} \eta_i,$$

which establishes (b) and (c) as applied to $\mathbf{F}_{n+1}$.

Next,

$$1.675 \qquad \int_U \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu = \int_{U-T} \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu + \int_T \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu.$$

Since $U - T \subset S$, and $\epsilon_{\mathbf{F}n+1} = \epsilon_{\mathbf{F}n}$ on $U - T$, then by (d),

$$1.676 \qquad \int_{U-T} \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu \leqslant \int_S \{\epsilon_{\mathbf{F}_n}(x)\}^p d\mu < \zeta_n.$$

Using 1.673, 1.674, and Minkowski's inequality, we also obtain

$$\begin{aligned}\int_T \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu &= \int_T \{\phi_{\mathbf{F}_n}(x) + \phi_{\mathbf{H}}(x) - 1\}^p d\mu \\ &\leqslant \left[\left(\int_T \{\phi_{\mathbf{F}_n}(x)\}^p d\mu\right)^{1/p} + \left(\int_T \{\phi_{\mathbf{H}}(x) - 1\}^p d\mu\right)^{1/p}\right]^p \\ &= \left[\left(\int_{T\cdot S} \{\phi_{\mathbf{F}_n}(x)\}^p d\mu\right)^{1/p} + \left(\int_T \{\epsilon_{\mathbf{H}}(x)\}^p d\mu\right)^{1/p}\right]^p < \eta_{n+1}.\end{aligned}$$

Putting this last inequality and 1.676 into 1.675, we establish (d) as applied to $\mathbf{F}_{n+1}$.

Finally, we define

$$\mathbf{E} = \bigcup_{n=1}^{\infty} \mathbf{F}_n.$$

It is clear from (a), (b), (c), and (d) that $\mathbf{E}$ is an M-family of $\mathbf{V}$-constituents satisfying conditions (I), (II), and (III) of Definitions 1.65, which completes our proof.

Remarks. No change in the definition of an $S^{(p)}$-basis results if X is required to be a bounded subset of E. For then any subset $Y \subset E$ of finite outer measure may be decomposed into a countable sum of bounded disjoint subsets of E, for each of which the $S^{(p)}$-property holds, and by the method of the preceding theorem a countable family $\mathbf{F}$ of $\mathbf{V}$-sets may be found which is an 0-covering of Y, with $\mathfrak{L}^{(p)}$-overlap and (μ, Y)-overflow of $\mathbf{F}$ both as small as desired.

If, in the proof of Theorem 1.66, we adopt the ϵ-covering version for the definition of $S^{(p)}$-bases, the first relation in 1.663 is replaced by $\mu(\bar{X} - \bar{X}\cdot S) < \epsilon$. The Vitali ψ-property is established in the ϵ-covering version.

1.7. Relation to Younovitch's differentiation theorem. Younovitch (**28**, Theorem III) assumes that $E = R$, μ is complete, and $\mu(R)$ is finite. The sets consisting of a single point belong to $\mathbf{M}$. The basis $\tilde{B}$ is a special de Possel basis such that each point x there corresponds to one ordinary sequence $A_1(x), A_2(x), \ldots, A_n(x), \ldots$ of "neighborhoods" such that $x \in A_n(x)$. The x-converging sequences are the subsequences of the basic sequence $A_1(x)$, $A_2(x), \ldots, A_n(x), \ldots$.

YOUNOVITCH'S DIFFERENTIATION THEOREM 1.71. *$\tilde{B}$ differentiates the (finite) μ-integrals if there exists a positive constant α such that:*

(Y1) *Corresponding to any set M in $\mathbf{M}$ of positive μ-measure, any $\tilde{B}$-fine covering $\mathbf{V}$ of M, and any posiive ϵ, there exists a finite disjoint subfamily V_1, $V_2, \ldots, V_n$, satisfying the inequalities*

$$1.711 \qquad \sum_{i=1}^{n} \mu(V_i - V_i \cdot M) < \epsilon, \quad \mu\left(M \cdot \sum_{i=1}^{n} V_i\right) > \alpha\mu(M).$$

In addition, each set V_i, $i = 1, 2, \ldots, n$, must belong to the basic sequence corresponding to a point p_i of M; that is, for a certain index n_i,

$$V_i = A_{n_i}(p_i).$$

The second inequality of 1.711 may be expressed by saying that the *exhaustion power* of the V_i with respect to M is greater than α.

Remarks. If ψ denotes a non-negative μ-integral, (Y1) holds, and M, $\mathbf{V}$, and ϵ are regarded as prescribed, then we can, by suitable finite iteration of the exhaustion process under ψ-overflow control, produce a finite system $q_1, \ldots, q_k$ of points of M and indices $n_1, \ldots, n_k$ such that

(1) the μ-defect of covering of M by the sets

$$A^j = A_{n_j}(q_j)$$

is less than ϵ;

(2) the ψ-excess of covering of M by the sets A^j is less than ϵ;

(3) the ψ-overlap of the sets A^j is less than ϵ.

Consequently, (Y1) implies the Vitali property for integrals, (i) expressed in the ϵ-covering version, (ii) with reference to anchorage points, (iii) restricted to **M**-measurable sets. As noted previously, the ϵ-covering version with finite M-family **E** is equivalent to our original one. Actually, by infinite iteration of the exhaustion process outlined above, we can obtain an (enumerable) M-family satisfying (2) and (3), and covering M (mod **N**).

Younovitch, like de Possel, formulates Vitali-covering properties with reference to points, thus strengthening the Vitali assumptions. Condition (iii), however, appears to be substantially weaker than the Vitali property for μ-integrals. If, by any means, one can prove under Younovitch's assumptions that the (extreme) derivates are μ-measurable (see Remark after Theorem 1.46), or that any $\tilde{B}$-fine covering of a set X is a $\tilde{B}$-fine covering of any measure cover for X, then Younovitch's theorem follows from Theorem 1.62. Younovitch's note, unfortunately, contains neither proofs nor even a hint of them.

§2. The Converse Problem; Covering Properties deduced from Differentiation Properties of σ-Additive Set Functions

2.1. De Possel's equivalence theorem. (22, pp. 403–405).

Definition 2.11. A derivation basis $\tilde{B}$ possesses the *density property* if it differentiates the integrals of the characteristic functions of μ-measurable sets, that is, if, for any **M**-set M, the density of x, defined as the limit of $\mu(M_\iota(x)\cdot M)/\mu(M_\iota(x))$, exists and equals C_M (characteristic function of M) almost everywhere on E.

Theorem 2.12. *The density property and the Vitali μ-property are equivalent.*

Proof. As noted in the Remarks after Definition 1.47, the Vitali μ-property implies the density property. We have to prove the converse.

We assume that the density property holds and let X denote a subset of E of finite outer measure, **V** a $\tilde{B}$-fine covering of X, and ϵ a positive number. We select α, $0 < \alpha < 1$, so that

2.121 $$0 < (\alpha^{-1} - 1)\,\mu(\tilde{X}) < \epsilon.$$

If Y is any subset of X such that $\bar{\mu}(Y) > 0$, then we define $\mathbf{V}(Y, \alpha)$ as the family of **V**-sets V for which

2.122 $$\mu(\tilde{Y}\cdot V) > \alpha\mu(V),$$

and μ_Y as the supremum of the numbers $\mu(V)$, for $V \in \mathbf{V}(Y, \alpha)$.

Now, from the density property, if $\bar{\mu}(Y) > 0$, it follows that the density of Y equals 1 for at least one point $y \in Y$. Hence there exists at least one y-converging sequence in $\tilde{B}$ whose constituents belong to $\mathbf{V}$ and satisfy 2.122. The family $\mathbf{V}(Y, \alpha)$ is thus non-vacuous, hence $\mu_Y > 0$. In case $Y \subset X$ and $\bar{\mu}(Y) = 0$, we put $\mu_Y = 0$.

We fix a number κ, $0 < \kappa < 1$. By the definition of μ_X, there exists a $\mathbf{V}$-set V_1, such that

$$\mu(V_1) > \kappa\, \mu_X, \quad \mu(\bar{X}\cdot V_1) > \alpha\mu(V_1).$$

We let $X_1 = X$, $X_2 = X_1 - V_1 \cdot X_1$. From this point we proceed inductively, assuming that sets V_i have been defined in $\mathbf{V}$ for $i = 1, 2, \ldots, n$, satisfying the relations

2.123 $$\mu(\bar{X}_i \cdot V_i) > \alpha\mu(V_i), \; \mu(V_i) > \kappa\mu_{X_i},$$

where

$$X_{i+1} = X_1 - X_1 \cdot \bigcup_{j=1}^{i} V_j.$$

In case $\bar{\mu}(X_{n+1}) = 0$, we stop the process; in case $\bar{\mu}(X_{n+1}) > 0$, we define a new $\mathbf{V}$-constituent V_{n+1} such that

$$\mu(V_{n+1}) > \kappa\, \mu_{X_{n+1}}, \quad \bar{\mu}(X_{n+1} \cdot V_{n+1}) > \alpha\mu(V_n).$$

The process just described leads to the construction of an M-family $\mathbf{E}$ consisting of a finite or infinite sequence of sets V_i taken from $\mathbf{V}$, satisfying 2.123 $(i = 1, 2, \ldots)$. Since also the sets $X_i \cdot V_i$ are disjoint (mod $\mathbf{N}^*$), we have

$$\mu(\bar{X}) \geqslant \mu\Big(\bar{X} \cdot \bigcup_i V_i\Big) \geqslant \mu\Big(\bigcup_i \bar{X}_i \cdot V_i\Big) = \sum_i \mu(\bar{X}_i \cdot V_i) > \alpha \sum_i \mu(V_i);$$

consequently,

2.124 $$\sum_i \mu(V_i) < \alpha^{-1}\mu\Big(\bar{X} \cdot \bigcup_i V_i\Big).$$

Putting $S = \sigma\mathbf{E}$ and combining 2.121 with 2.124, we obtain

2.125 $$\Big\{\sum_i \mu(V_i) - \mu(S)\Big\} + \mu(S - S\cdot\bar{X}) = \sum_i \mu(V_i) - \mu(S\cdot\bar{X}) < (\alpha^{-1} - 1)\,\mu(S\cdot\bar{X}) < \epsilon.$$

Hence conditions (V2) and (V3) of Definition 1.33, with $\psi = \mu$, hold.

To show that (V1) holds for our family $\mathbf{E}$, we note that if the sequence of sets V_i is finite, then for some positive integer N, we have $\mathbf{E} = \{V_1, V_2, \ldots, V_N\}$ and $\mu(\bar{X}_{N+1}) = 0$. Thus $\bar{\mu}(X - X\cdot\sigma\mathbf{E}) = 0$, as required by (V1). If the sequence is infinite, then from 2.123 and 2.124 we see that

$$\kappa \sum_i \mu_{X_i} < \sum_i \mu(V_i) < \alpha\mu(\bar{X}) < \infty; \text{ hence } \lim \mu_{X_i} = 0.$$

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We let $X_\infty = X - X \cdot \bigcup V_i$. Since $X_\infty \subset X_n$ and hence $\mathbf{V}(X_\infty, \alpha) \subset \mathbf{V}(X_n, \alpha)$, for $n = 1, 2, \ldots$, then

$$\mu_{X_\infty} = 0.$$

This means that $\sigma\mathbf{E} = \bigcup V_i \supset X$ (mod $\mathbf{N}^*$), as required, and the proof is complete.

Remarks. If we wish, as does de Possel, to "anchor" the V_n to points of X, we can extract V_n from an x_n-converging sequence, whose constituents belong to $\mathbf{V}(X_n, \alpha)$, and such that $x_n \in X_n$.

As is known (**25**, p. 129), in any Euclidean space the interval basis possesses the density property, therefore, by Theorem 2.12, it is a weak derivation basis. There exists (**17**) an example of a summable function f in the plane whose indefinite integral is I-differentiable (*strongly* differentiable), although the integral of $|f|$ is not.

2.2 A necessary and sufficient condition for a weak derivation basis to differentiate a μ-finite M-measure (Radon measure) ψ. We assume that $\tilde{B}$ is a weak derivation basis; that is, $\tilde{B}$ possesses the Vitali μ-property.

We let f denote a μ-measurable, non-negative, almost everywhere finite function with domain R; by f_n we shall mean that function for which $f_n(x) = 0$ if $f(x) \geqslant n$ and $f_n(x) = f(x)$ if $f(x) < n$. We further define $r_n(x) = f(x) - f_n(x)$, and for $M \in \mathbf{M}$,

$$\psi(M) = \int_M f(x)\, d\mu; \quad \psi_n(M) = \int_M f_n(x)\, d\mu; \quad \rho_n(M) = \int_M r_n(x)\, d\mu.$$

Since f_n is a μ-measurable bounded function, $\tilde{B}$ differentiates its integral ψ_n; that is, $D\psi_n$ exists almost everywhere in E and equals f_n(mod $\mathbf{N}^*$). We have

$$D^*\psi = D\psi_n + D^*\rho_n,$$

hence

$$D^*\psi = f_n + D^*\rho_n \text{ (mod } \mathbf{N}^*)$$

on E. In accordance with the definition of f_n and the finiteness (mod $\mathbf{N}^*$), we have $\lim f_n = f$ almost everywhere on E. This leads to the following result.

LEMMA 2.21. *A necessary and sufficient condition for a weak derivation basis to differentiate ψ is*

$$\lim\, [D^*\rho_n > 0] = 0.$$

In particular, if $\bar{\mu}(E)$ is finite, this is equivalent to the condition that $\lim \bar{\mu}[D^*\rho_n > \epsilon] = 0$ *for each positive* ϵ.

COROLLARY 2.22. *If $\tilde{B}$ differentiates the non-negative μ-integral ψ, it differentiates any* $\mathbf{M}$*-measure ψ' for which $\psi' \leqslant \psi$.*

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This result can be extended to any μ-finite (resp., Radon) **M**-measure ψ. In fact, for $M \in \mathbf{M}$,

$$\psi(M) = \psi_s(M) + \int_M f(x)\, d\mu,$$

where f represents the Radon-Nikodym integrand of ψ. We suppose that ψ' is any **M**-measure, $\psi' \leqslant \psi$. Then

$$\psi'(M) = \psi'_s(M) + \int_M f'(x)\, d\mu$$

is the corresponding decomposition for ψ'.

If N_0 denotes an **N**-set for which $\psi_s(R - N_0) = 0$, then ψ is μ-absolutely continuous on the **M**-subsets of $R - N_0$, consequently, so is ψ'; therefore $\psi'_s(R - N_0) = 0$,

$$\psi'_\tau(M) = \psi'(M \cdot (R - N_0)) \leqslant \psi(M \cdot (R - N_0)) = \psi_\tau(M),$$
$$\psi'_s(M) = \psi'(M \cdot N_0) \leqslant \psi(M \cdot N_0) = \psi_s(M).$$

Thus ψ'_τ and ψ'_s are dominated by ψ_τ and ψ_s, respectively. The assumption that $\tilde{B}$ differentiates ψ means that $D\psi = f$ (mod $\mathbf{N}^*$) on E, hence $D\psi_\tau = f$ (mod $\mathbf{N}^*$) on E, and $D\psi_s = 0$ (mod $\mathbf{N}^*$) on E. Since $D^*\psi'_s \leqslant D^*\psi_s$, then $D\psi'_s$ exists and equals zero (mod $\mathbf{N}^*$) on E. Thus, we have the following general result.

THEOREM 2.23. *If a weak derivation basis $\tilde{B}$ differentiates the μ-finite (resp., Radon) **M**-measure ψ, then $\tilde{B}$ differentiates any μ-finite (resp., Radon) **M**-measure dominated by ψ.*

COROLLARY 2.24. *If a weak derivation basis differentiates the total variation τ of a signed μ-finite (Radon) **M**-measure ψ, it differentiates ψ itself.*

As a special case, if the weak basis $\tilde{B}$ differentiates the integral $\int |f(x)| d\mu$, where f is a σ-bounded measurable function, then $\tilde{B}$ differentiates $\int f(x) d\mu$.

LEMMA 2.25. *If the weak derivation basis $\tilde{B}$ differentiates the μ-finite Radon **M**-measure ψ, $M \in \mathbf{M}$, and $\tau = \psi + \mu$, then the τ-density*

$$\lim_\iota \frac{\tau(M \cdot M_\iota(x))}{\tau(M_\iota(x))}$$

exists almost everywhere on E and equals C_M (characteristic function of M).

Proof. We let f denote the Radon-Nikodym integrand of ψ. Then, for $M' \in \mathbf{M}$,

$$\begin{aligned}\tau_M(M') = \tau(M' \cdot M) &= \psi(M' \cdot M) + \mu(M' \cdot M)\\ &= \int_{M' \cdot M} f(x)\, d\mu + \psi_s(M' \cdot M) + \mu(M' \cdot M)\\ &= \int_{M' \cdot M} (f(x) + 1)\, d\mu + \psi_s(M' \cdot M)\\ &= \int_{M'} C_M(x)(f(x) + 1)\, d\mu + \psi_s(M' \cdot M),\end{aligned}$$

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where ψ_s is the μ-singular part of ψ. Since $\tilde{B}$ differentiates τ, $\lim \tau(M_\iota(x))/\mu(M_\iota(x))$ exists and equals $f(x) + 1$ for μ^*-almost all $x \in E$. But $\tilde{B}$ also differentiates τ_M, so that

$$\lim \tau(M_\iota(x) \cdot M)/\mu(M_\iota(x))$$

exists and, by above, is equal to $C_M(x)(f(x) + 1)$ for μ^*-almost all $x \in E$. Hence, by division, $\lim \tau(M_\iota(x) \cdot M)/\tau(M_\iota(x))$ exists and equals C_M for μ^*-almost all $x \in E$.

LEMMA 2.26. *If $\tilde{B}$, ψ, and τ are as in the preceding lemma, X is a subset of E of finite outer measure, M_1 is a measure-cover of X, $\mathbf{V}$ is a $\tilde{B}$-fine covering of X, ϵ is a positive number, and $0 < \alpha < 1$, then there exists a finite or infinite sequence of $\mathbf{V}$-sets V_n for which*

2.261 $$\bigcup_n V_n \supset X (\text{mod } \mathbf{N}^*), \quad \sum_n \tau(V_n) < \tau\Big(M \cdot \bigcup_n V_n\Big)/\alpha.$$

Proof. For any $\mathbf{M}$-set such that $\bar{\mu}(M \cdot E) > 0$, we define $\mathbf{V}(\tau, M, \alpha)$, as the family of $\mathbf{V}$-sets V for which

2.262 $$\tau(M \cdot V) > \alpha\tau(V),$$

and $\mu(\tau, M)$ as the supremum of the numbers $\mu(V)$ for $V \in \mathbf{V}$ (τ, M, α). From Lemma 2.25, it follows that there is at least one point $x \in M \cdot E$ at which the τ-density of M equals 1, hence $\mathbf{V}(\tau, M, \alpha)$ is non-vacuous, and $\mu(\tau, M) > 0$. In case $M \in \mathbf{M}$ and $\bar{\mu}(M \cdot E) = 0$, we define $\mu(\tau, M) = 0$.

From this point on the proof follows closely that of Theorem 2.12, with τ replacing μ and the measure covers having to be specially selected, since τ need not be μ-absolutely continuous. By a process similar to that of Theorem 2.12, for fixed κ, $0 < \kappa < 1$, we determine inductively a finite or infinite sequence $V_1, V_2, \ldots$ of $\mathbf{V}$-sets with properties as follows. We put $X_1 = X$, and for any positive integer $n \geqslant 1$,

$$X_{n+1} = X_1 - X_1 \cdot \bigcup_{i=1}^{n} V_i,$$

M_{n+1} denotes a measure cover of X_{n+1} contained in

$$M_n - M_n \cdot \bigcup_{i=1}^{n} V_i.$$

If $\bar{\mu}(X_{n+1}) > 0$, then V_{n+1} is so chosen from $\mathbf{V}$ that

2.263 $$\tau(M_{n+1} V_{n+1}) > \alpha\tau(V_{n+1}), \quad \mu(V_{n+1}) > \kappa\mu(\tau, M_{n+1}).$$

If $\bar{\mu}(X_{n+1}) = 0$, the process stops.

Our choice of the sets M_n ensures that the sets $M_n \cdot V_n$ are strictly disjoint; hence, using 2.263, we have

2.264 $$\tau(M_1) \geqslant \tau\Big(M_1 \cdot \bigcup_n V_n\Big) \geqslant \tau\Big(\bigcup_n M_n \cdot V_n\Big) = \sum_n \tau(M_n \cdot V_n) > \alpha \sum_n \tau(V_n),$$

which is the second relation of 2.261.

If V_n is a finite sequence, then $\bar{\mu}(X_n) = 0$ holds for some integer N, and the first relation of 2.261 clearly holds. If V_n is infinite, we let $X_\infty = X - X \cdot \bigcup V_n$; we may, and do, choose a μ-cover M_∞ of X_∞, contained in $\bigcap M_n$. From 2.263 and 2.264 we have $\lim \mu(\tau, M_n) = 0$. Since $M_\infty \subset M_n$, we have $\mathbf{V}(\tau, M_\infty, \alpha) \subset \mathbf{V}(\tau, M_n, \alpha)$, and $\mu(\tau, M_\infty) \leqslant \mu(\tau, M_n)$ for $n = 1, 2, \ldots$; thus $\mu(\tau, M_\infty) = 0$, and $\bar{\mu}(M_\infty \cdot E) = 0$. But $X_\infty \subset M_\infty \cdot E$; hence $\bar{\mu}(X_\infty) = 0$, and the first condition of 2.261 holds.

THEOREM 2.27. *If a weak derivation basis $\tilde{B}$ differentiates the μ-finite (Radon) $\mathbf{M}$-measure ψ, then $\tilde{B}$ possesses the Vitali ψ-property.*

Proof. Taking X, M_1, $\mathbf{V}$, and ϵ as in the statement of Lemma 2.26, we select α so that

2.271 $$0 < (\alpha^{-1} - 1)\, \tau(M_1) < \epsilon,$$

and choose an M-family $\mathbf{E}$ in accordance with Lemma 2.26, satisfying 2.261. For $S = \sigma\mathbf{E}$, the (τ, M_1)-redundancy of covering is given by

$$\left\{\sum_n \tau(V_n) - \tau(S)\right\} + \tau(S - S \cdot M_1) = \sum_n \tau(V_n) - \tau(S \cdot M_1),$$

which, by 2.261 and 2.271, is less than ϵ. Thus the τ-overlap of $\mathbf{E}$ and the (τ, M_1)-overflow of $\mathbf{E}$ are less than ϵ, so that the Vitali τ-property holds. Since $\psi \leqslant \tau$, the Vitali ψ-property also holds.

Remark. If desired, the sets V_n may be "anchored" to points of X_n, as in the de Possel theorem.

Combining Theorem 2.27 and Theorem 1.52, we obtain the following criterion of differentiability of an individual $\mathbf{M}$-measure.

THEOREM 2.28. *A necessary and sufficient condition for a weak derivation basis $\tilde{B}$ to differentiate the μ-finite (Radon) $\mathbf{M}$-measure ψ is the validity of the Vitali ψ-property.*

THEOREM 2.29. *The Vitali property for μ-finite (resp., Radon) μ-integrals is equivalent to the $\tilde{B}$-differentiability of every μ-finite (resp., Radon) μ-integral; the Vitali property for μ-finite (resp., Radon) $\mathbf{M}$-measures is equivalent to the $\tilde{B}$-differentiability of every μ-finite (resp., Radon) $\mathbf{M}$-measure.*

Proof. This follows from Theorem 1.52, Theorem 2.12, and Theorem 2.27.

2.3. Relation to Younovitch's equivalence theorem. We return to the setting of 1.7 in the following discussion.

Younovitch formulates a Vitali μ-property (Y2) exhibiting the features (i), (ii), and (iii) in the Remarks under 1.71, and asserts its equivalence with the density property. That is, under his assumptions, Younovitch proves Theorem 2.12 with a weakened Vitali μ-property in which X is required to belong to $\mathbf{M}$. We have been unable to prove this; otherwise, Younovitch's theory would be a special instance of ours.

Younovitch also formulates a criterion for the differentiation of μ-integrals, which are necessarily finite since $\mu(R) < \infty$ for his space R. This will be stated after some preliminary definitions are given.

If $\mathfrak{D}$ is a decomposition of the space R into a sequence of disjoint μ-measurable sets $R_1, R_2, \ldots$, so that $R = \bigcup R_\nu$, then $\mathfrak{D}$ is *Y-summable* if and only if

$$\sum_\nu \nu\mu(R_\nu)$$

is finite. For any positive integer k and any positive number ϵ, $U_\mu(\mathfrak{D}, k, \epsilon)$ denotes the set of points x in R for which there exists a sequence

$$A_{n_i}(x)$$

satisfying the relation

$$\sum_{\nu=k}^{\infty} \nu\mu(R_\nu \cdot A_{n_i}(x)) > \epsilon\mu(A_{n_i}(x)).$$

Younovitch's basis $\tilde{B}$ is said to have the property (Y3) if and only if for each Y-summable decomposition $\mathfrak{D}$ of R and each positive number ϵ,

$$\lim_k \bar{\mu}\{U_\mu(\mathfrak{D}, k, \epsilon)\} = 0.$$

(Younovitch does not place a bar over μ, evidently regarding the bracketed set as μ-measurable, which seems to confirm the conjecture of 1.71 that he establishes the $\mu(=\mu^*)$-measurability of the derivatives of μ-integrals).

YOUNOVITCH'S CRITERION 2.31. (Y2) and (Y3) together are equivalent to the $\tilde{B}$-differentiability of every (finite) μ-integral.

Assuming (Y2) to be equivalent to the Vitali μ-property, this result can be proved from the theorems of 2.1, as will now be indicated. Corresponding to the decomposition occurring in the formulation of (Y3), we define a function ϕ by $\phi(x) = \nu$ for $x \in R_\nu$, $\nu = 1, 2, \ldots$; ϕ is the frequency function of the M-family $\mathbf{C}$ consisting of the sets

$$S_\nu = \bigcup_{i=\nu}^{\infty} R_i.$$

$\mathbf{C}$ is a covering of R. The Y-summability of the decomposition means the finite integrability of $\phi = \phi_{\mathbf{C}}$ over R; conversely, any μ-measurable function on R taking only positive integral values may be regarded as the frequency function of a measurable M-family covering R.

We apply the considerations of 2.2 to $f = \phi$. We have

$$\sum_{\nu=k}^{\infty} \nu\mu(R_\nu \cdot A_{n_i}(x)) = \rho_k(A_{n_i}(x)).$$

Younovitch's set $U_\mu(\mathfrak{D}, k, \epsilon)$ satisfies the relations

$$[D^*\rho_k \geqslant \epsilon] \supset U_\mu(\mathfrak{D}, k, \epsilon) \supset [D^*\rho_k > \epsilon].$$

From Lemma 2.21, assuming the Vitali μ-property, it follows that condition (Y3) is equivalent to $D^*\psi = D_*\psi = f \pmod{\mathbf{N}^*}$ for the integrals of measurable frequency (or multiplicity) functions f.

By Theorem 2.12, the Vitali μ-property is equivalent to the differentiability property when the measurable function f takes only a finite number of values, or even only the values 0 and 1. These latter functions are the characteristic functions of μ-measurable sets.

Combining, we see that Younovitch's theorem amounts to the assertion that a necessary and sufficient condition for the validity of the differentiability of μ-integrals is its validity for the integrals of μ-measurable functions taking only positive integral values.

It remains to be shown only that a weak derivation basis differentiating the integrals of μ-measurable frequency functions differentiates all μ-integrals $\psi(M) = \int_M f(x)\, d\mu$. This follows using the decomposition $\psi(M) = \psi^+(M) + \psi^-(M)$, and for non-negative f, the representation $f(x) = [n](x) + e(x)$, where $[n](x)$ takes only positive integral values and $-1 \leqslant e(x) < 0$. Any weak derivation basis differentiates the integrals of the functions e.

2.4. A converse theorem for bases differentiating the $\mu^{(q)}$-functions. In what follows, we assume that the basis $\tilde{B}$ is a general derivation basis and that R has the property (G_σ).

$\mathbf{E}$ denoting an M-family of $\tilde{B}$-constituents, r and α positive numbers, we let $\mathbf{E}(\alpha, r)$ denote the family of $\mathbf{E}$-sets V for which

$$\int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu > \alpha\mu(V),$$

and we further let $\sigma_{\alpha,r}(\mathbf{E})$ denote the union of the sets $\mathbf{E}(\alpha, r)$. Clearly, if $r' > r''$, then $\mathbf{E}(\alpha, r') \supset \mathbf{E}(\alpha, r'')$, and $\sigma_{\alpha,r'}(\mathbf{E}) \supset \sigma_{\alpha,r''}(\mathbf{E})$.

LEMMA 2.41. *If* $\mathbf{H}$ *represents the* M*-family of the* $\mathbf{E}$*-constituents* V *for which*

$$\int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu \leqslant \alpha\mu(V);$$

that is, if $\mathbf{H} = \mathbf{E} - \mathbf{E}(\alpha, r)$, *then*

$$\omega^{(r+1)}(\mathbf{H}) \leqslant \alpha \sum_{V \epsilon \mathbf{H}} \mu(V),$$

where $\omega^{(r+1)}(\mathbf{H})$ *denotes the* $\mathfrak{L}^{(r+1)}$*-overlap of* $\mathbf{H}$.

Proof.

$$\begin{aligned} \omega^{(r+1)}(\mathbf{H}) &= \int_{\sigma\mathbf{H}} \{\epsilon_{\mathbf{H}}(x)\}^{r+1} d\mu = \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{H}}(x) - 1\}^{r+1} d\mu \\ &\leqslant \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{E}}(x) - 1\}^r \phi_{\mathbf{H}}(x)\, d\mu \\ &= \sum_{V\epsilon\mathbf{H}} \int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu \leqslant \alpha \sum_{V\epsilon\mathbf{H}} \mu(V). \end{aligned}$$

In the preceding considerations, if $r = 0$, we shall interpret $\{\phi_{\mathbf{E}}(x) - 1\}^r$ as the function defined on $\sigma\mathbf{E}$, taking the value 0 if $\epsilon_{\mathbf{E}}(x) = 0$, or the value 1 if $\epsilon_{\mathbf{E}}(x) \geqslant 1$; that is, the restriction to $\sigma\mathbf{E}$ of the characteristic function

$C_{\theta \mathbf{E}}$ of the $\mathbf{E}$-overlap set $\theta\mathbf{E}$ (see Definitions 1.32). $\mathbf{E}(\alpha, 0)$ is the family of the $\mathbf{E}$-sets V for which $\mu(V \cdot \theta\mathbf{E}) > \alpha\mu(V)$.

The above lemma remains true when $r = 0$, since

$$\omega^{(1)}(\mathbf{H}) = \omega(\mathbf{H}, \mu) = \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{H}}(x) - 1\}\, d\mu$$
$$\leqslant \int_{\sigma\mathbf{H}} C_{\theta\mathbf{H}}(x)\phi_{\mathbf{H}}(x)\, d\mu \leqslant \sum_{V \in \mathbf{H}} \int_V C_{\theta\mathbf{E}}(x)\, d\mu \leqslant \alpha \sum_{V \in \mathbf{H}} \mu(V).$$

DEFINITION 2.42. We say that the basis $\tilde{B}$ has the property (H_p), for $p > 1$, if and only if for any bounded set $X \subset E$, any $\tilde{B}$-fine covering $\mathbf{V}$ of X, any $z^* > \bar{\mu}(X)$, any $\epsilon^* > 0$, and any $\alpha^* > 0$, there exists a finite M-family $\mathbf{G}$ of $\mathbf{V}$-sets such that

2.421 $$\bar{\mu}(X - X \cdot \sigma\mathbf{G}) < \epsilon^*; \ \sum_{V \in \mathbf{G}} \mu(V) < z^*; \ \mu(\sigma_{\alpha^*,\, p-1}(\mathbf{G})) < \epsilon^*.$$

Remarks. Without the third condition, we have the Vitali μ-property in the ϵ-version, and for bounded subsets of E, which, as noted earlier, is equivalent to the original definition of the Vitali μ-property.

We see that if $p' > p''$, then $(H_{p'})$ implies $(H_{p''})$.

LEMMA 2.43. *If $\tilde{B}$ is an $S^{(z)}$-basis, $z \geqslant 1$, and if $\tilde{B}$ does not possess the property $(H_{p'})$, where $p' > 1$, then there exists a bounded set $X_0 \subset E$, a bounded $\mathbf{G}$-set $G_0 \supset X_0$, a $\tilde{B}$-fine covering $\mathbf{V}_0$ of X_0, and positive numbers α_0, ϵ_0 such that for every M-family $\mathbf{F}$ of $\mathbf{V}_0$-sets satisfying the relations*

2.431 $$\bar{\mu}(X_0 - X_0 \cdot \sigma\mathbf{F}) < \epsilon_0, \quad \mu(\sigma\mathbf{F} - \bar{X}_0 \cdot \sigma\mathbf{F}) < \epsilon_0, \quad \omega^{(z)}(\mathbf{F}) < \epsilon, \sigma\mathbf{F} \subset G_0,$$

we have

$$\mu\{\sigma_{\alpha_0, p'-1}(\mathbf{F})\} > 2\epsilon_0.$$

Proof. If G is a bounded $\mathbf{G}$-set, $X \subset E \cdot G$, $\mathbf{V}$ is a $\tilde{B}$-fine covering of X, α and ϵ are both positive numbers, then we call the entity $(X, G, \mathbf{V}, \alpha, \epsilon)$ an *admissible quintuple.* Since $\tilde{B}$ is an $S^{(z)}$-basis, for any such quintuple there exist M-familes $\mathbf{F}$ of $\mathbf{V}$-sets for which $\bar{\mu}(X - X \cdot \sigma\mathbf{F}) < \epsilon$, $\mu(\sigma\mathbf{F} - \bar{X} \cdot \sigma\mathbf{F}) < \epsilon$, $\omega^{(z)}(\mathbf{F}) < \epsilon$ and $\sigma\mathbf{F} \subset G$. For such families $\mathbf{F}$ we thus have

$$\sum \mu(V) = \mu(\sigma\mathbf{F}) + \omega(\mathbf{F}, \mu) \leqslant \mu(\sigma\mathbf{F}) + \omega^{(z)}(\mathbf{F}) < \bar{\mu}(X) + 2\epsilon.$$

For any fixed admissible quintuple, we let η denote the infimum, among all such families $\mathbf{F}$, of the numbers $\mu(\sigma_{\alpha, p'-1}(\mathbf{F}))$. It follows that if, for each admissible quintuple, the corresponding η were zero, then $\tilde{B}$ would have the property $(H_{p'})$, contrary to hypothesis. Thus, for some admissible quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$, the corresponding η_0 is a positive number, and for each finite M-family $\mathbf{F}$ of $\mathbf{V}_0$-sets satisfying the relations 2.431, we have

2.532 $$\mu\{\sigma_{\alpha, p'-1}(\mathbf{F})\} \geqslant \eta_0 > 0.$$

Now if $\mathbf{F}$ is any M-family which satisfies the relations obtained from 2.431 merely by replacing ϵ_0 by any smaller positive number, then $\mathbf{F}$ necessarily

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satisfies the unchanged relation 2.431. Thus, we may assume that ϵ_0 has been chosen so small that $0 < \epsilon_0 < \frac{1}{2}\eta_0$, which, in the light of 2.432, completes the proof.

Henceforth, any quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$ satisfying the conditions of Lemma 2.43 will be called a *privileged quintuple.*

LEMMA 2.44. *If the basis $\tilde{B}$ possesses the property (H_p), where $p > 1$, then $\tilde{B}$ is an $S^{(p)}$-basis.*

Proof. By virtue of the Remarks following Theorem 1.67, it suffices to show that the $S^{(p)}$-properties hold when X is any bounded subset of E. Thus, we take a bounded set $X \subset E$, a $\tilde{B}$-fine covering $\mathbf{V}$ of X, choose $\epsilon > 0$, and select z^* so that $\bar{\mu}(X) < z^* < \bar{\mu}(X) + \frac{1}{2}\epsilon$. We let $\epsilon^* = \frac{1}{4}\epsilon$ and choose any positive α^* with $\alpha^* z^* < \epsilon$.

We use the property (H_p) to find a finite M-family $\mathbf{G}$ of $\mathbf{V}$-constituents such that 2.421 holds. We define $\mathbf{F}$ as the family of $\mathbf{G}$-sets V for which

$$\int_V \{\epsilon_{\mathbf{G}}(x)\}^{p-1} d\mu \leqslant \alpha^* \mu(V).$$

If $Y = X \cdot \sigma\mathbf{G}$, then $\mathbf{F}$ covers $Y - Y \cdot \sigma_{\alpha^*, p-1}(\mathbf{G})$. Hence, by 2.421, and the definition of ϵ^*, we have

2.441 $$\bar{\mu}(X - X \cdot \sigma\mathbf{F}) \leqslant \bar{\mu}(X - X \cdot \sigma\mathbf{G}) + \mu(\sigma_{\alpha^*, p-1}(\mathbf{G})) < 2\epsilon^* = \epsilon/2;$$

that is, $\mathbf{F}$ is an ϵ-covering of X.

Using Lemma 2.41, with $r = p - 1$, and taking account of the second relation in 2.421 and the choice of α^*, we obtain

$$\omega^{(p)}(\mathbf{F}) \leqslant \alpha^* \sum_{V \in \mathbf{F}} \mu(V) \leqslant \alpha^* \sum_{V \in \mathbf{G}} \mu(V) < \alpha^* z^* < \epsilon.$$

Finally, from conditions 2.421 we have

$$\mu(\sigma\mathbf{F} - \bar{X} \cdot \sigma\mathbf{F}) \leqslant \mu(\sigma\mathbf{G} - \bar{X} \cdot \sigma\mathbf{G}) = \mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{G}) + \mu(\sigma\mathbf{G}) - \mu(\bar{X})$$
$$\leqslant \mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{G}) + \Big\{\sum_{V \in \mathbf{G}} \mu(V) - \mu(\bar{X})\Big\} < \epsilon,$$

which completes the proof that $\tilde{B}$ is an $S^{(p)}$-basis.

DEFINITION 2.45. We assume R to be a measure space as described in 1.1. We suppose that $\mathbf{U}$ is a given family of $\mathbf{M}$-sets of finite positive measure, δ a positive finite function defined on $\mathbf{U}$. We define E as the set of points x for which there exists at least one ordinary sequence of sets $V_n \in \mathbf{U}$ with $\lim \delta(V_n) = 0$ and $x \in V_n$ for $n = 1, 2, \ldots$. We define the $\mathfrak{D}$-*basis* $[\mathbf{U}, \delta]$ by associating with each $x \in E$ the totality of ordinary sequences of sets $M_\iota(x)$ $(\iota = 1, 2, \ldots)$ for which $x \in M_\iota(x)$, $M_\iota(x) \in \mathbf{U}$, and $\lim \delta(M_\iota(x)) = 0$. From our assumptions it follows that the domain of $[\mathbf{U}, \delta]$ is E and the spread is a subfamily of $\mathbf{U}$. The function δ is called the *index of uniform contraction.*

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For any fixed $\eta > 0$, we denote by $\mathbf{V}_\eta$ the subfamily of $\mathbf{V}$ consisting of the $\mathbf{V}$-sets V for which $\delta(V) < \eta$.

For a family (or an M-family) $\mathbf{F}$ of $\mathbf{V}$-sets V, we define the δ-fineness or δ-norm $\nu(\mathbf{F})$ as sup $\delta(V)$, for $V \in \mathbf{V}$.

Remarks. Denjoy **(2)** considered bases more general than those just defined, insofar as the requirement "$x \in M_\iota(x)$ for each ι" is replaced by "$x \in E(M_\iota)$ for each ι," where $E(V)$ is defined for each $\tilde{B}$-constituent V as a subset of R, not necessarily μ-measurable, containing V. However, the contraction requirement is lim $\mu(M_\iota) = 0$ **(19)**. Nevertheless, we name the bases introduced here after Denjoy since his memoir points to their specific properties. Following Haupt, they are called "**U**-Basen" in **(21**, p. 71**)**, for reasons there explained.

Once δ is fixed, a $\mathfrak{D}$-subbasis of $[\mathbf{U}, \delta]$ is uniquely defined by its spread $\mathbf{T} \subset \mathbf{U}$; its domain $D[\mathbf{T}, \delta]$ (abbreviated $D[\mathbf{T}]$) is no longer an arbitrary subset of D[**U**] as is the case with a general subbasis. For instance, if $\tilde{B}$ is an $S^{(1)}$-basis, the domain of any $\mathfrak{D}$-subbasis of $\tilde{B}$ is a μ^*-measurable set.

A $\tilde{B}$-fine covering $\mathbf{V}$ of X is characterized as a subfamily of $\mathbf{U}$ with $D[\mathbf{V}] \supset X$ (mod $\mathbf{N}^*$).

LEMMA 2.46. *If $\tilde{B}$ is a $\mathfrak{D}$-basis $[\mathbf{U}, \delta]$, $p' > z \geqslant 1$, and $\tilde{B}$ is an $S^{(z)}$-basis but not an $S^{(p')}$-basis, then there exists a bounded* **G**-*set G_0 and a set X_0, with $G_0 \cdot E \supset X_0$, a $\tilde{B}$-fine covering $\mathbf{V}_0$ of X_0, positive numbers α_0 and ϵ_0, and a sequence,* $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n, \ldots$ *of finite* M-*families of* $\mathbf{V}_0$-*sets for which*

2.461 $$\lim \nu(\mathbf{F}_n) = 0,\ S_n = \sigma F_n \subset G_0;$$

2.462 $$\bar{\mu}(X_0 - X_0 \cdot S_n) < \epsilon_0, \quad \omega^{(z)}(\mathbf{F}_n) < \epsilon_0/2^{n+1}, \quad \mu(\sigma_{\alpha_0, p'-1}(\mathbf{F}_n) > 2\epsilon_0.$$

Proof. By Lemma 2.44, $\tilde{B}$ does not possess the property (H_p'), and we may apply Lemma 2.43 t find a privileged quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$. Since $\tilde{B}$ is an $S^{(z)}$-basis, we define $\mathbf{F}_n$ as a finite M-family of $\mathbf{V}_{1/n}$-sets included in G_0, satisfying the first two relations in 2.462. The last relation in 2.462 holds by our choice of a privileged quintuple, and 2.461 is clearly valid.

LEMMA 2.47. *We let $\tilde{B}$ denote an $S^{(1)}$-basis which is also a $\mathfrak{D}$-basis* $[\mathbf{U}, \delta]$. *We define p_0 as the supremum of numbers p such that $\tilde{B}$ is an $S^{(p)}$-basis; we assume that $p_0 < \infty$. We so define q_0 that $p_0^{-1} + q_0^{-1} = 1$ if $p_0 > 1$; otherwise $q_0 = \infty$ if $p_0 = 1$. Then for any number q, $1 < q < q_0$, there exists a $\mu^{(q)}$-function ψ_0, a positive number α_0, and a subset C_0 of E of positive outer measure such that:*

2.471 $$\psi_0(M) = \int_M f_0(x)\, d\mu \text{ for } M \in \mathbf{M}, \text{ and } \int_R |f_0(x)|^q d\mu < \infty;$$

2.472 $$f_0(x) = 0 \text{ for each } x \in C_0;$$

2.473 $$D^* \psi_0 \geqslant \alpha_0 > 0 \text{ for each } x \in C_0.$$

Proof. We so define p that $p^{-1} + q^{-1} = 1$. Our hypotheses on q ensure that $p_0 < p < \infty$. In case $p_0 > 1$, we clearly have

$$0 < q(p_0 - 1) < q_0(p_0 - 1) = p_0;$$

hence we can choose a number p' so that

$$0 < q(p' - 1) < p_0 < p' < p.$$

Even in case $p_0 = 1$, this last inequality may be satisfied for a suitable choice of p'.

In either case, we so define q' that $(p')^{-1} + (q')^{-1} = 1$; clearly, then, $q < q' < q_0$. We let z denote the larger of the two numbers $q(p' - 1)$ and 1.

From our assumptions, it follows that $\tilde{B}$ is an $S^{(z)}$-basis but not an $S^{(p')}$-basis. Lemma 2.46 asserts the existence of a privileged quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$ and a sequence $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n, \ldots$ of finite M-families of $\mathbf{V}_0$-sets satisfying 2.461 and 2.462. We let

$$S_n = \sigma\mathbf{F}_n, O_n = \theta\mathbf{F}_n(\mathbf{F}_n\text{-overlap set}), \quad D = \bigcup_n O_n,$$

$$H_n = \sigma_{\alpha_0, p'-1}(\mathbf{F}_n), Q_n = H_n - H_n \cdot D, \quad C_0 = \limsup Q_n,$$

$$\epsilon_n(x) = \epsilon_{\mathbf{F}n}(x) \text{ if } x \in O_n, \epsilon_n(x) = 0, x \notin O_n.$$

We have

$$\mu(D) \leqslant \sum_n \mu(O_n) \leqslant \sum_n \omega(\mathbf{F}_n) \leqslant \sum_n \omega^{(z)}(\mathbf{F}_n) < \epsilon_0.$$

Since $\mu(H_n) > 2\,\epsilon_0$ for each n, then $\mu(Q_n) > \epsilon_0$ for all n. Since the sets Q_n are subsets of the set G_0 of finite outer measure, then $\mu(C_0) \geqslant \epsilon_0$.

We define

2.474 $$f_0(x) = \sum_n \{\epsilon_n(x)\}^{p'-1}, \quad \psi_0(M) = \int_M f_0(x)\, d\mu, \qquad M \in \mathbf{M}.$$

Now f_0 is non-negative and vanishes on $R - D$, hence on $C_0 \subset (R - D)$; this verifies 2.472.

From the definition of f_0 and Minkowski's inequality, we have

$$\left(\int_R |f_0(x)|^q d\mu\right)^{1/q} \leqslant \sum_n \left(\int_R \{\epsilon_n(x)\}^{(p'-1)q} d\mu\right)^{1/q} \leqslant \sum_n \left(\int_R \{\epsilon_n(x)\}^z d\mu\right)^{1/q}$$
$$= \sum_n \{\omega^{(z)}(\mathbf{F}_n)\}^{1/q} \leqslant \sum_n \left(\frac{\epsilon_0}{2^{n+1}}\right)^{1/q} = \epsilon_0^{1/q} \cdot \sum_n \rho^{n+1},$$

where $\rho = 2^{-1/q}$. Since $q > 1$, then $\rho < 1$, and the sum of the geometrical series is finite; thus 2.471 holds.

We denote by $\mathbf{F}_n(\alpha_0, p' - 1)$ the family of those sets $V \in \mathbf{F}_n$ for which

2.475 $$\int_V \{\epsilon_n(x)\}^{p'-1} d\mu > \alpha_0 \mu(V);$$

then $H_n = \sigma\mathbf{F}_n(\alpha_0, p' - 1)$.

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To each point $x \in C_0 \subset \limsup H_n$ there corresponds a sequence of natural numbers n_j such that

$$x \in V(n_j),\ V(n_j) \in \mathbf{F}_{n_j}(\alpha_0, p' - 1).$$

The sequence of the sets $V(n_j)$ is an x-contracting sequence of $\tilde{B}$, thus $x \in E$, and $C_0 \subset E$.

From 2.474 and 2.475 it follows that

$$\psi_0(V(n_j))/\mu(V(n_j)) \geqslant \left(\int_{V(n_j)} \{\epsilon_{n_j}(x)\}^{p'-1} d\mu\right) \Big/ \mu(V(n_j)) > \alpha_0,$$

from which we obtain 2.473.

THEOREM 2.48. *If $\tilde{B}$ is a $\mathfrak{D}$-basis which differentiates the $\mu^{(q)}$-functions, where $1 < q < \infty$, and if p is so defined that $p^{-1} + q^{-1} = 1$, then $\tilde{B}$ is an $S^{(p')}$-basis for each number p' such that $1 \leqslant p' < p$.*

Proof. Since $q > 1$, and $\tilde{B}$ differentiates the $\mu^{(q)}$-functions, then $\tilde{B}$ must differentiate the $\mu^{(\infty)}$-functions, that is, the integrals of μ-measurable functions which are bounded on each set G°_n. By Theorem 2.12, $\tilde{B}$ is an $S^{(1)}$-basis. Next, we define p_0 and q_0 as in Lemma 2.47. In case $p_0 = \infty$, it is clear that $\tilde{B}$ is an $S^{(p')}$-basis and the theorem holds. In case $1 \leqslant p_0 < \infty$, Lemma 2.47 tells us that for each number q' such that $1 < q' < q_0$, there exists at least one $\mu^{(q')}$-function which $\tilde{B}$ fails to differentiate. Our hypotheses thus compel us to conclude that $q \geqslant q_0$, hence $p \leqslant p_0$, from which the statement of the theorem is seen to be true.

Remarks. Theorem 2.48 is not a clear-cut converse theorem because it does not say that $\tilde{B}$ is an $S^{(p)}$-basis. We conjecture that a $\mathfrak{D}$-basis can be constructed, which is an $S^{(p')}$-basis for each $p' < p$, yet fails to be an $S^{(p)}$-basis.

According to Zygmund (**29**, pp. 143–144), the interval basis in the plane differentiates the $\mu^{(q)}$-functions for $q > 1$. Therefore, by Theorem 2.48, it is an $S^{(p)}$-basis for all finite $p \geqslant 1$. There exists a blanket (**10**, pp. 294–295) which is an $S^{(1)}$-basis but is not an $S^{(p)}$-basis for any $p > 1$. The two bases just mentioned are extreme cases in the continuous chain of $S^{(p)}$-bases.

2.5 An individual differentiability criterion of Busemann-Feller type. Throughout this section we assume that $\tilde{B}$ is a $\mathfrak{D}$-basis and that (G_σ) holds.

DEFINITION 2.51. For $M \in \mathbf{M}$, we denote by $\dot{M}$ the element (soma) of $\mathfrak{M} = \mathbf{M}/\mathbf{N}$ corresponding to M; that is, the M-coset of $\mathbf{N}$. If $\mathbf{Z}$ is any subfamily of $\mathbf{M}$, and $\mathfrak{Z}$ is its image by mapping $M \rightarrow \dot{M}$, then we define the union (mod $\mathbf{N}$) of the $\mathbf{Z}$-sets, namely $\sigma^*(\mathbf{Z})$, as any set corresponding to the union or join of the $\mathfrak{Z}$-elements in $\mathbf{M}/\mathbf{N}$, regarded as a complete lattice (**26**, pp. 378–380). The lattice relation is so defined that $\dot{Z}' > \dot{Z}''$ holds if and only if for

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every $Z' \in \dot{Z}$, and $Z'' \in \dot{Z}''$, the relation $Z' \supset Z''$ (mod $\mathbf{N}$) holds. The union σ^* ($\mathbf{Z}$), defined modulo $\mathbf{N}$, is thus characterized by the property that every $\mathbf{M}$-set which includes each $\mathbf{Z}$-set (mod $\mathbf{N}$) includes $\sigma^*(\mathbf{Z})$ (mod $\mathbf{N}$). There exist enumerably many $\mathbf{Z}$-sets $Z_1, Z_2, \ldots, Z_n, \ldots$ such that

$$\bigcup_n Z_n = \sigma^*(\mathbf{Z}) \text{ (mod } \mathbf{N}).$$

If all the $\mathbf{Z}$-sets are included (mod $\mathbf{N}$) in an $\mathbf{M}$-set of finite measure, then $\sigma^*(\mathbf{Z})$ is any $\mathbf{M}$-set of minimal measure including each $\mathbf{Z}$-set (mod $\mathbf{N}$).

DEFINITION 2.52. If M denotes a bounded $\mathbf{M}$-set (nucleus), η a positive number, α a number such that $0 < \alpha < 1$, then, similarly to Busemann and Feller (**1**, p. 230), we define the *halo* $\sigma_{\alpha,\eta}(M)$ as the union (mod $\mathbf{N}$) of the $\tilde{B}$-constituents V for which $\mu(M\cdot V) > \alpha\mu(V)$ and $\delta(V) \leqslant \eta$.

DEFINITION 2.53. If ψ is an $\mathbf{M}$-measure, $M \in \mathbf{M}$, $\alpha > 0$, and $\eta > 0$, then we denote by $\sigma_{\alpha,\psi,\eta}(M)$ the union (mod $\mathbf{N}$) of the constituents V for which $\psi(M\cdot V) > \alpha\mu(V)$ and $\delta(V) \leqslant \eta$. The set $\sigma_{\alpha,\psi,\eta}(M)$ is called a *ψ-halo*.

Remarks. The modification of the Busemann-Feller definition involving the strict union is due to the possible non-measurability of the strict union in our more general sense.

The term "halo" was first used by K. O. Househam in his talks in Capetown, 1950, on A. P. Morse's differentiation theory, to denote Morse's set $\Delta:\beta$ (**14**, p. 208). We diverge from the colloquial use of the term by permitting our halos to have points in common with the nucleus or even to include the nucleus. However, all our halo conditions control the proper halo; that is, the part of the halo outside the nucleus, thus retaining the basic meaning of the term.

The relation between $\sigma_{\alpha,r}(\mathbf{E})$, defined in 2.4, and the notion just defined may be written $\sigma_{\alpha,r}(\mathbf{E}) = \sigma_{\alpha,\psi,\eta}(M)$, provided $M = \theta(\mathbf{E})$, ψ is the indefinite integral of that function coinciding with $\{\epsilon_{\mathbf{E}}\}^r$ on $\theta\mathbf{E}$, zero elsewhere, and $\eta = \nu(\mathbf{E})$.

THEOREM 2.53. *If $\tilde{B}$ is a $\mathfrak{D}$-basis and an $S^{(1)}$-basis, and ψ is a non-negative μ-integral, then a necessary and sufficient condition that $\tilde{B}$ differentiate ψ is the following* halo evanescence condition*: For any bounded non-increasing sequence of* $\mathbf{M}$*-sets M_n, with* $\lim \mu(M_n) = 0$, *any non-increasing sequence η_n of positive numbers with* $\lim \eta_n = 0$, *and any $\alpha > 0$, we have*

$$\mu\{\lim_n (\sigma_{\alpha,\psi,\eta_n}(M_n))\} = 0.$$

Proof. We first establish the sufficiency. We show that the halo evanescence property implies the Vitali ψ-property in the ϵ-covering version. We let X denote a bounded subset of E, $\mathbf{V}$ a $\tilde{B}$-fine covering of X, ϵ a positive number. We shall prove the existence of an M-family $\mathbf{E}$ of $\mathbf{V}$-sets which is an ϵ-covering of X, with ψ-overflow with respect to X, and ψ-overlap, both less than ϵ.

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For a suitable N, we have $X \subset G^{\circ}_{\mathrm{N}}$ (recall Definitions 1.31). Pruning **V** if necessary, we may assume that all the **V**-sets lie in G°_{N}. Since $\tilde{B}$ is an $S^{(1)}$-basis and a $\mathfrak{D}$-basis, then corresponding to each positive integer n, there exists an M-family $\mathbf{E}_n$ of **V**-sets such that if $S_n = \sigma\mathbf{E}_n$ then

2.531 $S_n \supset X (\text{mod } \mathbf{N}^*)$, $\mu(S_n - S_n \cdot \bar{X}) < 2^{-n-1}$, $\omega(\mathbf{E}_n, \mu) < 2^{-n-1}$, $\nu(\mathbf{E}_n) \leqslant 1/n$.

Putting

$$O_n = \theta\mathbf{E}_n, \; D_n = \bigcup_{k=n}^{\infty} O_k,$$

we have $\mu(O_n) < 1/2^{n+1}$, $\mu(D_n) < 1/2^n$.

Next we define $\alpha = \epsilon/(\bar{\mu}(X) + 1)$. We denote by $\mathbf{H}_n$ the family of those $\mathbf{E}_n$-sets V for which

2.532 $$\psi(V \cdot O_n) \leqslant \alpha\mu(V).$$

We have

2.533 $$\omega(\mathbf{H}_n, \psi) = \int_{\sigma\mathbf{H}_n} \epsilon_{\mathbf{H}_n}(x)\, d\psi \leqslant \int_{O_n} \phi_{\mathbf{H}_n}(x)\, d\psi$$
$$= \sum_{V \epsilon \mathbf{H}_n} \psi(V \cdot O_n) \leqslant \alpha \sum_{V \epsilon \mathbf{H}_n} \mu(V) \leqslant \alpha(\mu(S_n) + \omega(\mathbf{E}_n, \mu))$$
$$< \alpha(\bar{\mu}(X) + 2^{-n-1} + 2^{-n-1}) \leqslant \alpha(\bar{\mu}(X) + 1) = \epsilon.$$

Now $\mathbf{E}_n$ is an 0-covering of X. The constituents in $\mathbf{E}_n - \mathbf{H}_n$ are included in $\sigma_{\alpha,\psi,1/n}(O_n)$, (mod $\mathbf{N}^*$); therefore H_n covers

$$X - X \cdot \sigma_{\alpha,\psi,1/n}(O_n) (\text{mod } \mathbf{N}^*),$$

and consequently

2.534 $$\sigma\mathbf{H}_n \supset \{X - X \cdot \sigma_{\alpha,\psi,1/n}(D_n)\} (\text{mod } \mathbf{N}^*).$$

Since the sets D_n are all included in the set G°_{N} of finite outer measure, and form a non-increasing sequence with $\lim \mu(D_n) = 0$, we may invoke the halo evanescence property to conclude that

$$\lim_n \mu(\sigma_{\alpha,\psi,1/n}(D_n)) = 0.$$

There exists a positive number η such that for any **M**-set $M \subset G^{\circ}_N$, for which $\mu(M) < \eta$, we have $\psi(M) < \epsilon$. We fix n so that

2.535 $$\mu(\sigma_{\alpha,\psi,1/n}(D_n)) < \eta, \; 2^{-n-1} < \eta.$$

We may and do further assume that $\eta < \epsilon$. Then the family $\mathbf{E} = \mathbf{H}_n$ corresponding to this index n satisfies the ϵ-covering condition due to 2.534 and 2.535. Using the second relation in 2.531 and the fact that $\sigma\mathbf{E} \subset S_n \subset G^{\circ}_N$, we have $\psi(\sigma\mathbf{E} - \sigma\mathbf{E} \cdot \bar{X}) < \epsilon$; that is, the ψ-overflow of **E** with respect to X is less than ϵ. Finally, the ψ-overlap of **E** is less than ϵ by virtue of 2.533.

We now attend to the proof of the necessity. We consider an arbitrary non-increasing sequence of bounded **M**-sets $M_1, M_2, \ldots, M_n \ldots$ with lim

$\mu(M_n) = 0$, an arbitrary non-increasing positive sequence $\delta_1, \delta_2, \ldots, \delta_n \ldots$, with $\lim \delta_n = 0$, and an arbitrary $\alpha > 0$. We put

$$H_n = \sigma_{\alpha,\psi,\delta_n}(M_n),\ H = \bigcap_n H_n.$$

Since the halo $\sigma_{\alpha,\psi,\delta}(M)$ is a non-decreasing function of δ, it follows readily that for any pair of positive integers n, ν, we have

$$H \subset \sigma_{\alpha,\psi,\delta_n}(M_\nu);$$

hence, for each such ν,

2.536 $$H \subset \bigcap_n \sigma_{\alpha,\psi,\delta_n}(M_\nu).$$

For each such pair of positive integers n, ν, there exist enumerably many constituents

$$V^1_{n,\nu}, V^2_{n,\nu}, \ldots, V^j_{n,\nu}, \ldots$$

such that

2.537 $$\delta(V^j_{n,\nu}) \leqslant \delta_n, \quad \psi(V^j_{n,\nu} \cdot M_\nu) > \alpha\mu(V^j_{n,\nu}),$$

$$S_{n,\nu} = \bigcup_j V^j_{n,\nu} = \sigma_{\alpha,\psi,\delta_n}, (M_\nu) \text{ (mod } \mathbf{N}).$$

Corresponding to each point x of the set

$$H^*_\nu = \bigcap_n S_{n,\nu},$$

there exists a sequence of $\tilde{B}$-constituents W_n satisfying the relations

2.538 $$x \in W_n, \quad \delta(W_n) \leqslant \delta_n, \quad \psi(W_n \cdot M_\nu) > \alpha\mu(W_n).$$

We let f denote the integrand of ψ, so define r that

$$r_\nu(x) = f(x) \text{ if } x \in M_\nu, \quad r_\nu(x) = 0 \text{ if } x \notin M_\nu,$$

and let

$$\rho_\nu(M) = \int_M r_\nu(x)\, d\mu,$$

for $M \in \mathbf{M}$. We deduce from 2.538 that

2.539 $$D^*\rho_\nu(x) \geqslant \alpha$$

for each $x \in H^*_\nu$.

Since $\tilde{B}$ is an $S^{(1)}$-basis and differentiates ψ, then by Theorem 2.23, $\tilde{B}$ differentiates the non-negative Radon integrals dominated by ψ for almost all $x \in E$. Hence, from 2.539, $r_\nu(x) \geqslant \alpha$ almost everywhere in H^*_ν. But $r_\nu(x) = 0$ for each $x \in M_\nu$, thus $H^*_\nu \subset M_\nu$ (mod $\mathbf{N}^*$), which, by 2.537, means that

$$\bigcap_n \sigma_{\alpha,\psi,\delta_n}(M_\nu) \subset M_\nu (\text{mod } \mathbf{N}^*);$$

using 2.536, we observe that $H \subset M_\nu$ (mod $\mathbf{N}^*$). Since ν is arbitrary and $\lim \mu(M_\nu) = 0$, we conclude that $\mu(H) = 0$, which completes the proof.

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§3. Examples of Bases Possessing the Vitali Property for Radon Measures

Throughout this section the setting of 1.1 is adopted.

3.1 Preliminary definitions.

Definition 3.11. By *external* $\mathfrak{D}$*-closed* set we shall mean the R-complement of a **G**-set; **A** will denote the family of all such sets.

Definition 3.12. We say that Δ is a (*Morse*) *disentanglement function* if and only if Δ is a positive finite function defined on the spread (family of the constituents) of the basis $\tilde{B}$ (**14**, p. 207).

Definition 3.13. If α is a fixed number greater than 1, Δ is a disentanglement function, and V_0 is a $\tilde{B}$-constituent, then the *Morse halo* $H(\Delta, \alpha, V_0)$ is the union of those constituents V which intersect V_0, and for which $\Delta(V) \leqslant \alpha\Delta(V_0)$. The *halo dilatation* $\rho(\Delta, \alpha, V_0)$ is defined as the ratio $\bar{\mu}(H(\Delta, \alpha, V_0))/\mu(V_0)$.

Definition 3.14. We shall say that the basis $\tilde{B}$ has the *strong Vitali property* (abbreviated (S.V.)) if and only if for each $\epsilon > 0$, each set $X \subset E$ of finite outer measure, and each $\tilde{B}$-fine covering **V** of X there exists an (enumerable) M-family **E** of **V**-constituents such that, for $S = \sigma\mathbf{E}$:

(S.V.1) $X - X{\cdot}S \in \mathbf{N}^*$;
(S.V.2) $\mu(S - S{\cdot}\bar{X}) < \epsilon$;
(S.V.3 str.) the **E**-constituents are pairwise disjointed.

If, in (S.V.3 str.) we replace the strict disjunction by 0-disjunction, that is, disjunction mod $\mathbf{N}^*$, we obtain the *strong Vitali property mod* $\mathbf{N}^*$; if we discard (S.V.2), we have the *reduced strong Vitali property* (abbreviated R.S.V.). Recalling the Definitions 1.65, we find it convenient to designate as an $S^{(\infty)}$-basis, any basis having the property (S.V.) mod $\mathbf{N}^*$.

The straightforward proofs of the following are omitted.

Proposition 3.15. (S.V.) mod $\mathbf{N}^*$ *implies the Vitali property for μ-finite μ-integrals.*

Proposition 3.16. (R.S.V.) *and Haupt's adaptation property together imply the Vitali property for Radon measures.*

Definition 3.17. We say that $\tilde{B}$ has the *generalized Morse halo property* (**14**, p. 213, Def. 6.4) if and only if there exists $\alpha > 1$ and a disentanglement function Δ for which

$$\sup\{\lim\sup[\Delta(M_\iota(x)) + \rho(\Delta, \alpha, M_\iota(x))]\} < \infty$$

for μ-almost all points $x \in E$. Here, as in 1.1, the limit superior is taken for a sequence $M_\iota(x)$ and the supremum is taken over all x-converging sequences.

Remarks. The strong Vitali properties lack the flexibility of the Vitali properties of §1. In their formulation, one cannot replace the 0-covering condition by an ϵ-covering one, nor in the (G_σ) case, replace the phrase "of finite outer measure" by "bounded." However, such alterations are permissible if the constituents are **A**-sets.

3.2 Generalized Morse bases.

FUNDAMENTAL THEOREM 3.21. *We suppose that $\tilde{B}$ is such a basis that for each $x \in E$, the sets of every x-converging sequence contain x, and each $\tilde{B}$-constituent is a member of* **A**. *We assume, further, that each* **M**-*set of finite measure is a subset of some* **G**-*set of finite outer measure, and that Morse's halo property holds. Then $\tilde{B}$ possesses the property* (R.S.V.).

The proof of this theorem occurs in **(21**, p. 80**)**.

Remarks. In Morse's version of the fundamental theorem, R is a metric space, μ a classical Radon measure, $\tilde{B}$ is a blanket F. Since the contraction process is defined by means of the metric, the (metrically) open sets belong to **G**, the (metrically) closed to **A**. Thus a Radon measure in the classical sense is a Radon measure in the sense of 1.31, the reference sequence $G^\circ{}_1, G^\circ{}_2, \ldots$ consisting of concentric open spheres, whose radii tend to infinity. Morse assumes that $V \in F(x)$ implies $x \in V$, and that the $\tilde{B}$-constituents are closed. Without the closeness assumption for the constituents, $\tilde{B}$ need not be strong as the following examples confirm.

EXAMPLE 3.22. R is a plane Euclidean space, μ is plane Borel measure, and E is the open unit square with principal vertices at $(0, 0)$ and $(1, 1)$. To avoid repetition, throughout this discussion, t will denote an arbitrary point of E, n an arbitrary positive integer. We let $\mathbf{T}_n$ denote the set of points in R of the form $(r/2^n, s/2^n)$, where r and s are arbitrary integers. $\mathbf{K}_n$ denotes the family of closed squares whose four vertices are points of $\mathbf{T}_n$, with sides of length 2^{-n}. Each point t lies in or on the boundary of at least one square in $\mathbf{K}_n$; we associate, with each such t, exactly one square $I_{n,t}$ in $\mathbf{K}_n$ such that $t \in I_{n,t}$. We define $I'_{n,t}$ as the square concentric with $I_{n,t}$, with sides parallel to the axes and three times as long as those of $I_{n,t}$.

At each point $z \in \mathbf{T}_n$, we construct a square centered at z, with sides parallel to the axes, and of length 2^{-2n}. We let $\mathbf{H}_n$ denote the family of all such squares, and we define

$$J_n = \bigcup_{m=n+1}^{\infty} \sigma \mathbf{H}_m.$$

We further define $I''_{n,t} = I_{n,t} + I'_{n,t} \cdot J_n$. Finally, we so define the blanket F with domain E that $F(t)$ is the family consisting of the sets $I''_{1,t}, I''_{2,t}, \ldots, I''_{n,t}, \ldots$.

For each integer $m \geqslant n + 1$, there are not over $16 \cdot 2^{2m-2n}$ points of $\mathbf{T}_m$

lying on or in $I_{n,t}$, thus not over $16 \cdot 2^{2m-2n}$ members of $\mathbf{H}_m$, each of μ-measure 2^{-4m}, intersecting $I''_{n,t}$. Therefore

$$\mu(I'_{n,t} \cdot J_n) \leqslant 16 \sum_{m=n+1}^{\infty} 2^{-2m} \cdot 2^{-2n} < 2^{-2n+3}\mu(I_{n,t});$$

since $n \geqslant 1$, we have

3.221 $$\mu(I''_{n,t}) \leqslant \mu(I_{n,t}) + \mu(I'_{n,t} \cdot J_n) < 3\mu(I_{n,t}) < \tfrac{1}{2}\mu(I'_{n,t}).$$

We consider a subblanket of F, say F_1, with spread $\mathbf{F}_1$, having the property that each member of $\mathbf{F}_1$ is included in E. We let $\mathbf{G}$ be any countable subfamily of $\mathbf{F}_1$ whose μ-overlap is zero. Each set $\beta \in \mathbf{G}$ is a set $I''_{n,t}$, and we may associate with β the corresponding set $\beta' = I'_{n,t}$; we let $\mathbf{G}'$ denote the family of the corresponding sets β'. Due to our construction, it follows that $\mathbf{G}'$ has μ-overlap zero. Using 3.221 above and the fact that $\sigma\mathbf{G} \subset E$ we obtain

$$\mu(S - \sigma\mathbf{G}) \geqslant \mu(S) - \sum_{\beta \in \mathbf{G}} \mu(\beta) \geqslant 1 - \tfrac{1}{2}\sum_{\beta' \in \mathbf{G}'} \mu(\beta') = 1 - \tfrac{1}{2}\mu(\sigma\mathbf{G}') \geqslant 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Thus, no countable subfamily of $\mathbf{F}_1$ whose μ-overlap is zero can cover μ-almost all of E. At the same time, if we define $\Delta(\beta) = \operatorname{diam} \beta$ for $\beta \in \mathbf{F}$, then it is clear that Morse's halo property holds.

EXAMPLE 3.23. In this example, R and E are both the set of all real numbers, μ is linear Borel measure, V° denotes a fixed open subset of the open interval $I = (-1, 1)$, containing the point $x = 0$, everywhere dense in I, with $\mu(V^\circ) = 2\theta^\circ$, where $0 < \theta^\circ < 1$. We define $V(x, \zeta)$, for $\zeta > 0$, as the open set image of V° by the direct homothetic transformation carrying the interval $(x - \zeta, x + \zeta)$ onto I. $F(x)$ is defined as the family of all sets $V(x, \zeta)$, $\zeta > 0$. We define $\Delta(V) = \mu(V)$ for $V = V(x, \zeta)$; hence $\Delta(V) = 2\zeta\theta^\circ$. From this it follows that $H(\Delta, \alpha, V(\alpha, \zeta)) = (x - \zeta(2\alpha + 1),\ x + \zeta(2\alpha + 1))$; thus $\rho(\Delta, \alpha, V) = (2\alpha + 1)/\theta^\circ$. Since

$$\limsup_{F(x) \ni V \to x} \Delta(V) = 0$$

for each real number x, then Morse's halo property is valid.

We let F_1 denote a subblanket of F with domain I, with the property that the closure of every member of the spread of F_1 is included in I. We consider any countable subfamily $\mathbf{G}$ of the spread of F_1, whose μ-overlap is zero. If $\beta = V(x, \zeta) \in \mathbf{G}$, then the closure β' of β is the closed interval $[x - \zeta, x + \zeta]$, and $\mu(\beta) = \theta^\circ \mu(\beta')$. If $\mathbf{G}'$ denotes the family of the corresponding sets β', it follows from the density of β in β' that the μ-overlap of $\mathbf{G}'$ is zero. Thus, since $\sigma\mathbf{G} \subset \sigma\mathbf{G}' \subset I$, we have

$$\mu(I - \sigma\mathbf{G}) \geqslant \mu(I) - \sum_{\beta \in \mathbf{G}} \mu(\beta) = 2 - \theta^\circ \cdot \sum_{\beta' \in \mathbf{G}'} \mu(\beta') = 2 - \theta^\circ \mu(\sigma\mathbf{G}') \geqslant 2 - 2\theta^\circ > 0.$$

This shows that F has not the property (S.V.), although F is regular.

THEOREM 3.24. *If $\tilde{B}$ is such a basis that each $\tilde{B}$-constituent of any x-convergent sequence at any point x of E includes an* $\mathbf{A}$*-set containing x, and if Haupt's*

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adaptation property and Morse's halo property both hold, then $\tilde{B}$ possesses the Vitali property for Radon measures.

Proof. We let ψ denote an arbitrary Radon measure, X an arbitrary bounded subset of E, say $X \subset G^\circ_N$, $\mathbf{V}$ any $\tilde{B}$-fine covering of X, and ϵ any positive number. Using Proposition 1.38 we have only to show the existence of an M-family $\mathbf{E}$ of $\mathbf{V}$-sets such that

$$\bar{\mu}(X - X\cdot\sigma\mathbf{E}) < \epsilon, \quad \omega(\mathbf{E}, \psi) < \epsilon.$$

For $n = 1, 2, \ldots$ we denote by $\tilde{B}^*_n$ the set of the $\tilde{B}$-sequences $M_\iota(x)$ such that $x \in X$, whose constituents V belong to $\mathbf{V}$, are included in G°_N, satisfying

3.241 $$\Delta(V) + \rho(\Delta, \alpha, V) < n,$$

by η a number for which $0 < \eta < 1$, and by X_n the domain of $\tilde{B}^*_n$. We have $X_1 \subset X_2 \subset \ldots \subset X_n \ldots$. Since G°_N is a $\mathbf{G}$-set, $\mathbf{V}$ a $\tilde{B}$-fine covering of X, and Morse's halo property holds, it follows that $\lim X_n = X$ (mod $\mathbf{N}^*$), hence $\lim \mu(\bar{X}_n) = \mu(\bar{X})$. Since $\mu(\bar{X})$ is finite, we may and do choose n so that

3.242 $$\mu(\bar{X}) - \mu(\bar{X}_n) < \epsilon.$$

We put $\psi^\circ = \sup \psi(M)$ for $\mathbf{M} \ni M \subset G^\circ_N$, and, since $\psi^\circ < \infty$, select δ so that

3.243 $$0 < (\delta^{-1} - 1)\,\psi^\circ < \epsilon,\ 0 < \delta < 1.$$

For each $\tilde{B}^*_n$-sequence $M_\iota(x)$, we determine all possible sequences $A_\iota(x)$ of $\mathbf{A}$-sets, for which the properties

$$x \in A_\iota(x),\, \mu(A_\iota(x))/\mu(M_\iota(x)) > \eta,\ \ \psi(A_\iota(x)) \geqslant \delta\psi(M_\iota(x))$$

all hold. The corresponding sequences $A_\iota(x)$ may all be regarded as converging to x. The fact that for *each* $\tilde{B}^*_n$-sequence such associated sequences exist follows from the first assumption of the theorem and the universal lower approximation property of the $\mathbf{A}$-sets, applied to $\psi + \mu$, which is implied by Haupt's adaptation property (recall Proposition 1.37). Thus if $\tilde{A}^*$ denotes the family of the sequences $A_\iota(x)$, then $\tilde{A}^*$ is a basis with domain X_n.

Following Morse, we associate with each $\tilde{A}^*$-constituent V^* a $\tilde{B}^*_n$-constituent $V = \mathrm{D}(V^*)$ (the *dilatation* of V^*) satisfying the conditions

$$V^* \subset V,\, \mu(V^*) > \eta\mu(V),\ \ \psi(V^*) \geqslant \delta\psi(V),$$

and we define on the spread of $\tilde{A}^*$ the disentanglement function Δ^* by

3.244 $$\Delta^*(V^*) = \Delta(V),$$

where $V = \mathrm{D}(V^*)$.

We observe that the halo $H^*(\Delta^*, \alpha, V^*)$, which is a union of $\tilde{A}^*$-constituents, is included in $H(\Delta, \alpha, V)$, hence

3.245 $$\bar{\mu}(H^*(\Delta^*, \alpha, V^*))/\mu(V^*) = \rho^*(\Delta^*, \alpha, V^*) < \rho(\Delta, \alpha, V)/\eta.$$

Combining 3.241, 3.244, and 3.245, we deduce

3.246 $$\Delta^*(V^*) + \rho^*(\Delta^*, \alpha, V^*) < (\Delta(V) + \rho(\Delta, \alpha, V))/\eta < n/\eta,$$

where $V = D(V^*)$. Thus $\tilde{A}^*$ possesses the Morse halo property, in fact, uniformly. The assumptions of Theorem 3.21 hold, hence $\tilde{A}^*$ has the property R.S.V. Thus there exists a disjointed M-family $\mathbf{E}^*$ of $\tilde{A}^*$-constituents, such that for $S^* = \sigma\mathbf{E}^*$, $S^* \supset X_n$ (mod $\mathbf{N}^*$).

We define $\mathbf{E}$ as the M-family obtained from $\mathbf{E}^*$ by the correspondence $V = D(V^*)$. The $\mathbf{E}$-constituents belong to $\mathbf{V}$ and lie in $G^\circ{}_N$. Since $D(V^*) \supset V^*$, then

$$S = \sigma\mathbf{E} \supset S^* \supset X_n \text{ (mod } \mathbf{N}^*), \quad S \supset \bar{X}_n \text{ (mod } \mathbf{N}).$$

Using 3.242, we have

$$\mu(\bar{X} - \bar{X}\cdot S) \leqslant \mu(\bar{X}) - \mu(\bar{X}_n) < \epsilon.$$

Also, since $S \subset G^\circ{}_N$, we have $\psi(S) \leqslant \psi^\circ < \infty$; hence, from 3.243,

$$\omega(\mathbf{E}, \psi) = \sum_{V\epsilon\mathbf{E}} \psi(V) - \psi(S)$$
$$\leqslant \delta^{-1}\Big(\sum_{V^*\epsilon\mathbf{E}^*} \psi(V^*)\Big) - \psi(S^*) = (\delta^{-1} - 1)\,\psi(S^*) < \epsilon.$$

The M-family $\mathbf{E}$ fulfils the required conditions, which means that the basis $\tilde{B}$ possesses the Vitali property for Radon measures. The following is the immediate consequence of Theorem 1.64.

COROLLARY 3.25. *Under the assumptions of Theorem* 3.24, $\tilde{B}$ *differentiates every Radon measure.*

Remarks. The essential steps in the foregoing proof are (i) the contraction of the $\tilde{B}^*{}_n$-sets into $\mathbf{A}$-sets of nearly equal ψ-measure, with μ-exhaustion power greater than η, (ii) the transfer, expressed by $\Delta^*(V^*) = \Delta(V)$, of the function Δ from the original sets to the new ones. The second step shows the power of Morse's methods residing here in the choice of the new disentanglement function.

In Morse's paper **(14)** it is remarked that the metric axiom

$$\delta(p', p'') = 0 \text{ implies } p' = p''$$

is never used. Discarding this axiom, the (metric) closure P of the set $\{p\}$, consisting of the point p only, is the set P of points x with $\delta(p, x) = 0$. To the various points x of P may be attached different families $F(x)$, but any Borel set containing p must contain P. This is why the first assumption in Theorem 3.24 is satisfied under Morse's relaxed hypotheses.

Morse's halo property in the general case involves the contracting process; this is not so, however, in the special case of uniformity. If, for a blanket F, there exists $\alpha > 1$ and Δ such that $\Delta(V) + \rho(\Delta, \alpha, V)$ is bounded on the spread of F, then the same is true of any blanket with the same spread, in particular for the blanket F^* so defined that $F^*(x)$ is the family of F-constituents containing x, which is a $\mathfrak{D}$-basis. Not only does F differentiate any Radon measure but F^* does, too.

Referring to Theorem 2.29 and Theorem 3.24, the Examples 3.22 and 3.23 give a negative answer to a question raised by O. Nikodym in 1947: "Is the property (SV) mod $\mathbf{N}^*$ equivalent to the validity of the differentiation theorem for μ-finite μ-integrals?"

FUNDAMENTAL THEOREM 3.26. *We suppose that $\tilde{B}$ is a basis whose constituents are* $\mathbf{A}$*-sets, and that there exists a disentanglement function* Δ *such that*

$$\sup\{\limsup \rho(\Delta, 1, V)\} < \infty$$

almost everywhere on E. *We assume that $\tilde{B}$ has the* property (L), *namely, corresponding to any subset X of E of finite outer measure, any $\tilde{B}$-fine covering* $\mathbf{V}$ *of* X, *and any* $\epsilon > 0$, *there exists an* M*-family* $\mathbf{E}$ *of* $\mathbf{V}$*-constituents such that for* $S = \sigma\mathbf{E}$ *we have*

(V1) $S \supset X \pmod{\mathbf{N}^*}$,

(V2) $\mu(S - S\cdot\bar{X}) < \epsilon$.

Then $\tilde{B}$ has the property (S.V.).

Proof. This can be found in (**21**, p. 83). See also the Remarks after Theorem 3.27.

Remarks. Property (L) asserts the existence of an M-family covering $X \pmod{\mathbf{N}^*}$ without any overlap condition. The essential step in the proof of the theorem is called *disentanglement*, and rests upon the halo property. For this reason (L) may be regarded as a rough or pre-Vitali property. In the formulation of (L), (V1) may be replaced by $\mu(\bar{X} - S\cdot\bar{X}) < \epsilon$, and in case (UG) holds, (V2) may be dropped.

Property (L) is named after Lindelöf. In fact, if R is a metric space with a countable basis (*separable*, in Fréchet's terminology), μ a Radon measure, F a blanket such that the constituents of $F(x)$ are open sets containing x, then Lindelöf's classical topological property implies (even expresses) the property (L).

In Theorem 3.26, it is not assumed that the constituents of an x-converging sequence contain x, as is the case in Theorem 3.24.

THEOREM 3.27. *If there exists a disentanglement function* Δ *such that*

$$\sup\{\limsup \rho(\Delta, 1, V)\} < \infty$$

almost everywhere on E, *and if Haupt's adaptation property and* (L) *holds, then $\tilde{B}$ possesses the Vitali property for Radon measures.*

Proof. We let ψ denote a Radon measure, X a bounded subset of E (we assume $X \subset G^{\circ}{}_N$), $\mathbf{V}$ a $\tilde{B}$-fine covering of X, ϵ a positive number. Due to

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Proposition 1.38, we need to prove only that there exists an M-family **E** of **V**-sets such that $\bar{\mu}(X - X \cdot \sigma \mathbf{E}) < \epsilon$ and $\omega(\mathbf{E}, \psi) < \epsilon$.

For $n = 1, 2, \ldots$ we denote by $\tilde{B}^*_n$ the set of $\tilde{B}$-sequences $M_\iota(x)$ with $x \in X$, whose constituents belong to **V**, are included in G°_N, and satisfy

3.271 $$\rho(\Delta, 1, V) < n;$$

we denote the domain of $\tilde{B}^*_n$ by X_n. Since the sequence $X_1, X_2, \ldots, X_n, \ldots$ is increasing with $\lim X_n = X \pmod{\mathbf{N}^*}$, we may and do choose ν so that

3.272 $$\mu(\bar{X}) - \mu(\bar{X}_\nu) < \epsilon;$$

we let $Z = X_\nu$.

We denote by $\mathbf{V}^1$ the spread of $\tilde{B}^*_\nu$, by η a fixed number, $0 < \eta < 1$, by δ a positive number such that

3.273 $$0 < (\delta^{-1} - 1)\,\psi^\circ < \epsilon,$$

where $\psi^\circ = \sup \psi(M)$ for $\mathbf{M} \ni M \subset G_\nu$. Finally we fix an auxiliary sequence of positive numbers ϵ_i whose sum is less than ϵ.

Property (L) allows us to select an M-family $M_1, M_2, \ldots M_j, \ldots$ of **V**-sets such that

3.274 $$T = \bigcup_{j=1}^{\infty} M_j \supset Z (\text{mod } \mathbf{N}^*), \quad \mu(T - T \cdot \bar{Z}) < \epsilon^1$$

where $\epsilon^1 = \min\{\epsilon_1, \eta\, \bar{\mu}(Z)/2(\eta + \nu)\}$.

Since $\lim \mu(T_q) = \mu(T) < \infty$, where

$$T_q = \bigcup_{j=1}^{q} M_j,$$

we can choose Q so that

3.275 $$\mu(T - T_Q) < \epsilon^1.$$

Now, using 3.271, we can extract from the finite family $M_1, M_2, \ldots, M_Q$ a disjoint subfamily $M^1{}_1, M^1{}_2, \ldots, M^1{}_{q_1}$, such that for

$$T^1 = \bigcup_{k=1}^{q_1} M^1{}_k,$$

we have

3.276 $$\mu(T^1) > \mu(T_Q)/\nu.$$

This is the disentanglement step.

With each set $M^1{}_k$ we associate an **A**-set $A^1{}_k \subset M^1{}_k$ in such a way that

3.277 $$\mu(A^1{}_k) > \eta\, \mu(M^1{}_k), \quad \psi(A^1{}_k) \geqslant \delta \psi(M^1{}_k).$$

We evaluate the Z-exhaustion of

$$S^1 = \bigcup_{k=1}^{q_1} A^1{}_k$$

by which we mean the value $\mu(S^1\cdot\bar{Z}) = \bar{\mu}(S^1\cdot Z)$. Since $S^1 \subset T$, $Z \subset T$ (mod $\mathbf{N}^*$), then, using 3.274, 3.275, 3.276, and 3.277, we derive

$$\begin{aligned}\mu(S^1\cdot\bar{Z}) &\geqslant \mu(\bar{Z}) + \mu(S^1) - \mu(T)\\ &> \bar{\mu}(Z) + \eta(\bar{\mu}(Z) - \epsilon^1)/\nu - (\bar{\mu}(Z) + \epsilon^1)\\ &= \eta\bar{\mu}(Z)/\nu - \epsilon^1(\eta+\nu)/\nu \geqslant \eta\bar{\mu}(Z)/2\nu.\end{aligned}$$

So far we have been able to select a finite disjoint subfamily $M^1{}_1, \ldots M^1{}_{q_1}$ of $\mathbf{V}^1$-sets (hence $\mathbf{V}$-sets), with a Z-overflow less than ϵ_1, and to contract each $M^1{}_k$ into an $\mathbf{A}$-set $A^1{}_k$ such that the new family has a Z-power of exhaustion greater than $\eta/2\nu$; that is, the *ratio of Z-exhaustion* is greater than $\zeta = \eta/2\nu$.

We repeat the process with $Y = Z - Z\cdot S^1$ and the family $\mathbf{V}^2$ consisting of those $\mathbf{V}^1$-constituents which do not intersect any $A^1{}_k$, $k = 1, 2, \ldots, q_1$, to produce two finite disjoint subfamilies $M^2{}_1, M^2{}_2, \ldots, M^2{}_{q_2}$ and $A^2{}_1, A^2{}_2, \ldots, A^2{}_{q_2}$ satisfying, for $l = 1, 2, \ldots, q_2$, $k = 1, 2, \ldots, q_1$, the relations

$$M^2{}_l \in \mathbf{V}^2,\ A^2{}_l \subset M^2{}_l,\ A^2{}_l \in \mathbf{A},\ A^1{}_k\cdot M^2{}_l = 0,\ \mu(S^2\cdot\bar{Y}) > \zeta\bar{\mu}(Y)$$

and

$$\mu(S^2 - S^2\cdot\bar{Y}) < \epsilon_2,$$

where

$$S^2 = \bigcup_{l=1}^{q_2} A^2{}_l.$$

The iteration of this exhaustion process yields two M-families, namely, $\mathbf{E}$ consisting of the sets $M^1{}_1, \ldots, M^1{}_{q_1}, M^2{}_1, \ldots, M^2{}_{q_2}, M^3{}_1, \ldots, M^3{}_{q_3}, \ldots,$ and $\mathbf{C}$ consisting of the sets $A^1{}_1, \ldots, A^1{}_{q_1}, A^2{}_1, \ldots, A^2{}_{q_2}, A^3{}_1, \ldots, A^3{}_{q_3}, \ldots$ such that (i) the $\mathbf{E}$-constituents belong to $\mathbf{V}$, (ii) $S = \sigma\mathbf{E} \supset Z$ (mod $\mathbf{N}^*$), (iii) the overflow $\bar{\mu}(S - S\cdot\bar{Z}) < \epsilon$, (iv) to each $\mathbf{C}$-set A there corresponds an $\mathbf{E}$-constituent $V = \mathrm{D}(A)$ (the dilatation of A) satisfying $A \subset V$, $\psi(A) \geqslant \delta\psi(V)$. In view of these facts it is easily seen that the remainder of the proof involves merely a repetition of a portion of the proof of Theorem 3.24, hence $\mathbf{E}$ has met the two necessary requirements, and the theorem is proved.

Remarks. The proof of Theorem 3.26 is obtained from the preceding proof by discarding the contraction process.

Haupt's adaptation property is used only in the contraction process. If the weaker condition (UG) is substituted, then the assertion of the theorem remains true for Radon integrals.

In the proof of Theorem 3.24, Morse's function Δ is used to disentangle the infinite family of constituents to produce the desired countable family, hence the necessity of a choice condition based on the boundedness of Δ. In the foregoing proof we disentangle a finite family and repeat the process, producing the desired family by juxtaposition of sections.

In the proof of Theorem 3.27, we do not treat of Morse's halo $H(\Delta, \alpha, V_0)$ itself, but only with a finite family of halo constituents, which may be aptly

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called a *partial halo*. For this reason we define $H'(\Delta, \alpha, V_0)$ as the union mod **N** (Definition 2.51) of the constituents V intersecting V_0, with $\Delta(V) \leqslant \alpha\Delta(V_0)$. In the formulation of Theorems 3.26 and 3.27 we can replace the halo dilatation $\rho(\Delta, \alpha, V_0)$ by

$$\rho'(\Delta, \alpha, V_0) = \mu(H'(\Delta, \alpha, V_0))/\mu(V_0)$$

clearly $\rho' \leqslant \rho$.

In defining H or H' we accept all constituents intersecting V_0 and satisfying $\Delta(V) \leqslant \alpha\Delta(V_0)$. The *incidence* requirement is $V \cdot V_0 \neq 0$. Correspondingly, disentanglement requires the determination of a *strictly disjointed* family of constituents. The incidence requirement may be altered to *essential intersection*, that is, $\mu(V \cdot V_0) > 0$, and in turn the disentanglement changed to require the production of a family of pairwise mod **N** disjointed constituents. This point of view can be adopted in Theorem 3.26 if we wish to achieve the strong Vitali property mod **N**, and in Theorem 3.27, if we restrict the assertion to Radon integrals. The stronger we make the incidence requirement, the weaker the halo condition becomes. Busemann and Feller gave a necessary and sufficient halo condition for the validity of the density theorem, equivalent, by virtue of Theorem 2.12, to the Vitali μ-property, for Euclidean derivation bases of the $\mathfrak{D}$-type, the constituents being open sets, and the contraction being defined metrically. We have already encountered a halo of Busemann-Feller type in 2.4, namely $\sigma_{\alpha,r}(\mathbf{E})$.

3.3. Half-regular and regular branches of a derivation basis.

DEFINITIONS 3.31. To some of the sequences $M_\iota(x)$ of the basis $\tilde{B}$ we correlate Moore-Smith sequences $M^*_\iota(x)$ of **M**-sets of positive finite measure, with the same indices and the same convergence point, such that for each sequence

(R1) $$\liminf \mu(M^*_\iota)/\mu(M_\iota) > 0$$

(R2) $$\limsup \mu(M^*_\iota - M_\iota \cdot M^*_\iota)/\mu(M_\iota) = 0.$$

The set $\tilde{B}^*$ of the sequences $M^*_\iota(x)$ is called a *half-regular branch* of $\tilde{B}$, and $\tilde{B}^* \cup \tilde{B}$ is called a *half-regular extension* of $\tilde{B}$. If (R2) is replaced by the strengthened requirement $M^*_\iota \subset M_\iota$, then $\tilde{B}^*$ is called a *regular branch*, and $\tilde{B}^* \cup \tilde{B}$ is called a *regular extension* of $\tilde{B}$. (**6**, 9.7).

De Possel has shown that the Vitali μ-property is preserved by half-regular extension. An example exists in which the sequences $M_\iota(x)$ are ordinary sequences of concentric closed squares in the plane (**10**, pp. 292–295). The corresponding set $M^*_\iota(x)$ consists of $M_\iota(x)$ augmented by small satellite squares, in such a manner that $\lim \mu(M^*_\iota)/\mu(M_\iota) = 1$, where μ denotes plane Lebesgue measure.

Example 3.23 deals with a regular branch of the basis of closed concentric intervals on the line. It shows that the strong Vitali property is not preserved under regular extension.

We shall now investigate the behavior of the Vitali properties under regular extension.

LEMMA 3.32. *We let ψ denote a Radon measure. We assume that there exists $\zeta > 0$ such that corresponding to any bounded set $X \subset E$ of positive outer measure, any $\tilde{B}$-fine covering* **V** *of X, any μ-cover M of X, and any $\epsilon' > 0$, there exists an* M*-family* **E**′ *of* **V***-constituents with union $S' = \sigma\mathbf{E}'$ for which*

$$\bar{\mu}(X \cdot S') > \zeta\bar{\mu}(X), \quad \psi(S' - S' \cdot M) < \epsilon', \quad \omega(\mathbf{E}', \psi) < \epsilon'.$$

Then $\tilde{B}$ has the Vitali ψ-property.

Proof. We assume that $\epsilon > 0$ and attempt to find an M-family satisfying (V1), (V2), (V3) of Definitions 1.33. We introduce a sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers whose sum is less than $\epsilon/2$. Since X is bounded and **V** is a B-fine covering of X, we may and do assume that both X and all the **V**-sets lie in some G°_N.

By hypothesis, there exists an M-family $\mathbf{E}_1$ of **V**-sets for which

$$\mu(X_1 \cdot S_1) > \zeta\bar{\mu}(X_1), \quad \psi(S_1 - S_1 \cdot M_1) < \epsilon_1, \quad \omega(\mathbf{E}, \psi) < \epsilon_1,$$

where $S_1 = \sigma\mathbf{E}_1$, $X_1 = X$, $M_1 = M$. We repeat this process on the sets $X_2 = X_1 - X_1 \cdot S_1$, $M_2 = M_1 - M_1 \cdot S_1$; by iteration in this way, we obtain a sequence of M-families $\mathbf{E}_1$, a nested sequence of sets $X_i \subset X$, with a nested sequence of μ-covers M_i, for $i = 1, 2, \ldots$, such that, for each such i, putting $S_i = \sigma\mathbf{E}_i$,

$$\mu(X_i \cdot S_i) > \zeta\bar{\mu}(X_i), \quad \psi(S_i - S_i \cdot M_i) < \epsilon_i, \quad \omega(\mathbf{E}_i, \psi) < \epsilon_i.$$

Finally, we let **E** denote the union of the families $\mathbf{E}_i$.

Since the rate of exhaustion at each step exceeds ζ, then **E** exhausts X; that is $S = \sigma\mathbf{E} \supset X$ (mod $\mathbf{N}^*$). We complete the proof by evaluating the ψ-overflow and ψ-overlap. We have

$$\psi(S - S \cdot M) \leqslant \sum_j \psi(S_j - S_j \cdot M_j) < \epsilon;$$

$$\begin{aligned}
\omega(\mathbf{E}, \psi) &= \sum_{V \in \mathbf{E}} \psi(V) - \psi(S) \\
&\leqslant \sum_j \Big(\sum_{V \in \mathbf{E}_j} \psi(V)\Big) - \psi(S \cdot M) = \sum_j \Big(\sum_{V \in \mathbf{E}_j} \psi(V) - \psi(S_j \cdot M_j)\Big) \\
&= \sum_j \Big(\sum_{V \in \mathbf{E}_j} (\psi(V) - \psi(S_j))\Big) + \sum_j \psi(S_j - M_j \cdot S_j) \\
&= \sum_j \omega(\mathbf{E}_j, \psi) + \sum_j \psi(S_j - M_j \cdot S_j) < \epsilon.
\end{aligned}$$

Remark. If Haupt's adaptation property holds, then the overflow requirement in the above may be omitted. The same is true if the weaker (UG) holds and ψ is a Radon μ-integral.

LEMMA 3.33. *If ψ is a Radon measure, $\tau = \psi + \mu$, $\tilde{B}$ possesses the Vitali τ-property, and $\tilde{B}^*$ is a regular branch of $\tilde{B}$, then $\tilde{B}^*$ possesses the Vitali ψ-property.*

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Proof. We let X denote a subset of E of positive outer measure included in some G°_N, M a μ-cover of X, $\mathbf{V}^*$ a $\tilde{B}^*$-fine covering of X, and ϵ a positive number.

For $n = 1, 2, \ldots$ we define $\tilde{B}^*_n$ as the set of all $\tilde{B}^*$-sequences S^* consisting of the sets $V^*_\iota(x)$ for which (i) $x \in X$, (ii) $V^*_\iota(x) \in \mathbf{V}^*$ for all ι, and (iii) there corresponds to each S^* a $\tilde{B}$-sequence $\mathrm{D}(S^*) = S$, consisting of those sets $V_\iota(x)$ for which $V^*_\iota \subset V_\iota$ and $\mu(V^*_\iota)/\mu(V_\iota) > 1/n$ for all ι.

We denote by $\tilde{B}_n$ the set of the $\tilde{B}$-sequences associated by D to the $\tilde{B}^*_n$-sequences, $\tilde{B}_n = \mathrm{D}(\tilde{B}^*_n)$, and by X_n the domain of $\tilde{B}_n$, which is also the domain of $\tilde{B}^*_n$. Since $\mathbf{V}^*$ is a $\tilde{B}^*$-fine covering of X and $\tilde{B}^*$ is a regular branch of $\tilde{B}$, then X_n increases with n and $\lim \mu(\bar{X}_n) = \mu(\bar{X})$. As in the proof of Theorem 3.24 we may and do choose k so that

3.331 $$\mu(\bar{X}) - \mu(\bar{X}_k) < \epsilon.$$

We shall show by Lemma 3.32 that the subbasis $\tilde{B}^*_k$ of $\tilde{B}^*$ possesses the Vitali ψ-property. We consider a subset Y of X_k of positive outer measure, P a μ-cover of Y, $\tilde{T}^*$ a subbasis of $\tilde{B}^*_k$ with domain Y (mod $\mathbf{N}^*$), ϵ' a positive number. We define $\zeta = 1/2k$, and $\epsilon'' = \min(\epsilon', \bar{\mu}(\mathrm{Y})/4k)$. $\tilde{T} = \mathrm{D}(\tilde{T}^*)$ is a subbasis of $\tilde{B}$ with the same domain as $\tilde{T}^*$. Since $\tilde{B}$ has the Vitali τ-property, there exists an M-family $\mathbf{E}$ of $\tilde{T}$-sets such that

3.332 $$S = \sigma\mathbf{E} \supset Y (\mathrm{mod}\ \mathbf{N}^*), \quad \tau(S - S\cdot P) < \epsilon'', \quad \omega(\mathbf{E}, \tau) < \epsilon''.$$

With each $\mathbf{E}$-set V we associate a $\mathbf{V}^*$-set $V' = \mathrm{C}(V)$ (the *contraction* of V), with

3.333 $$\mathrm{C}(V) \subset V, \ \mu(V')/\mu(V) > 1/k.$$

We define the M-family $\mathbf{E}'$, demanded by Lemma 3.32, as $\mathrm{C}(\mathbf{E})$. Clearly

3.334 $$\omega(\mathbf{E}', \psi) \leqslant \omega(\mathbf{E}, \psi) \leqslant \omega(\mathbf{E}, \tau) < \epsilon'' \leqslant \epsilon';$$

the ψ-overlap condition is satisfied.

Putting $S' = \sigma\mathbf{E}'$,

3.335 $$\psi(S' - S'\cdot P) \leqslant \psi(S - S\cdot P) \leqslant \tau(S - S\cdot P) < \epsilon'' \leqslant \epsilon';$$

thus the ψ-overflow condition is satisfied.

We turn to the evaluation of the Y-exhaustion of ϵ', namely,

$$\mu(S'\cdot P) = \mu(S') - \mu(S' - S'\cdot P).$$

Since $\mu(S' - S'\cdot P) \leqslant \mu(S - S\cdot P) \leqslant \tau(S - S\cdot P) < \epsilon''$, we have $\mu(S'\cdot P) > \mu(S') - \epsilon''$. From the last relation in 3.332, $\omega(\mathbf{E}', \mu) \leqslant \omega(\mathbf{E}, \mu) < \epsilon''$, hence

$$\mu(S' \cdot P) > \sum_{V' \epsilon \mathbf{E}'} \mu(V') - 2\epsilon'' > (1/k) \sum_{V \epsilon \mathbf{E}} \mu(V) - 2\epsilon''$$
$$\geqslant (1/k)\, \bar{\mu}(Y) - 2\epsilon'' \geqslant \bar{\mu}(Y)/2k = \zeta\bar{\mu}(Y).$$

The basis satisfies the requirements of Lemma 3.32. Consequently, there exists an M-family $\mathbf{E}^*$ of $\tilde{B}^*_n$-constituents, hence $\mathbf{V}^*$-constituents for which

$$S^* = \sigma \mathbf{E}^* \supset X_k (\text{mod } \mathbf{N}^*), \ \psi(S^* - S^* \cdot M) < \epsilon, \ \omega(\mathbf{E}^*, \psi) < \epsilon.$$

Due to the choice of k, from 3.331 we have obtained the Vitali ψ-property in the ϵ-version.

The following is an immediate consequence of Lemma 3.33.

THEOREM 3.34. *If $\tilde{B}^*$ is a regular branch of $\tilde{B}$, and $\tilde{B}$ possesses the Vitali property for Radon measures (resp., integrals), then $\tilde{B}^*$ possesses the Vitali property for Radon measures (resp., integrals).*

Examples 3.35. $\tilde{B}$ is the cube basis in Euclidean space E_3, μ the Borel measure, $\mathbf{M}$ the family of Borel sets, $\tilde{B}^*$ consists of all sequences of Borel sets of positive measure converging regularly to a point. According to Lemma 3.33, $\tilde{B}^*$ possesses the Vitali property for Radon measures, hence it differentiates them.

The subbasis of $\tilde{B}^*$ consisting of all sequences of closed sets of positive measure, converging regularly to a point, is the classical Lebesgue basis which enjoys the strong Vitali property. We may notice that any sequence of $\mathbf{M}$-sets of positive measure converging homothetically to a point belongs to $\tilde{B}^*$.

If we denote by $\tilde{B}^{**}$ the superbasis of $\tilde{B}^*$ consisting of all sequences of Borel sets of positive measure converging to a point x (without regularity condition), it is easy to see that for a Radon integral $\psi(M) = \int_M f(M)\, d\mu$ we have $D^*\psi(x)$ = essential maximum of f at x, $D_*\psi(x)$ = essential minimum of f at x. It follows that the only Radon integrals differentiated by $\tilde{B}^{**}$ are the integrals of functions summable at finite range and essentially continuous almost everywhere. This shows clearly the loss of differentiation power when discarding the regularity requirement for converging sequences. Finally, we notice that $\tilde{B}^{**}$ is a blanket, different from both $\tilde{B}^*$ and the Lebesgue basis.

3.4 Star blankets. To conclude, we give another example where the surrendering of the closeness assumption for the constituents of a basis means the replacement of the strong Vitali property by a Vitali property for Radon measures, with no loss of differentiation power.

DEFINITIONS 3.41. R denotes Euclidean n-dimensional space. The *hub* of a set $X \subset R$ is the set of these points $x \in X$ such that the segment joining x and x' lies in X whenever $x' \in X$; the *hub radius* of X at a point $x \in X$ is the supremum of those numbers ρ for which the solid sphere with x as center and ρ as radius is included in the hub of X. A *star blanket* (**15**, p. 432) according to Morse is a blanket F in R, whose constituents are closed sets, and such that for each x of its domain,

$$\limsup_{F(x) \ni V \to x} (\text{diam } V)/(\text{hub radius of } V \text{ at } x) < \infty.$$

We define *Borelian* star blankets by discarding in Morse's definition the closeness requirement, and replacing it by the weaker demand that the constituents be Borel sets.

Remarks. Morse proved that a star blanket in his sense possesses the strong Vitali property *whenever μ is a Radon measure.* From his differentiation theorems he deduces the existence μ–almost everywhere of a finite derivative for every Radon measure ψ.

THEOREM 3.42. *Borelian star blankets possess the Vitali property for Radon measures, whenever the basic measure μ is itself a Radon measure.*

Sketch of the proof. Referring to **(15)**, all properties of star blankets exhibited in §5 up to the application of Morse's Theorem 3.4 in 5.11 are valid without using the fact of closeness of the constituents. The statement of this basic theorem, but for minor changes, is as follows: R is a metric space, μ a Radon measure, $0 < \zeta < \infty$, $X \subset R$, and $\mathbf{F}$ is a family of closed sets. Corresponding to each bounded open set G there exists a countable disjointed subfamily $\mathbf{K}$ of $\mathbf{F}$ for which $\sigma\mathbf{K} \subset G$, $\bar{\mu}(X \cdot G) \leqslant \zeta\mu(\sigma\mathbf{K})$. Then every disjointed subfamily of $\mathbf{F}$ can be extended to a countable subfamily of $\mathbf{F}$ covering X (mod $\mathbf{N}^*$).

If the $\mathbf{F}$-sets are required merely to be Borel sets, instead of closed, then we have to change the conclusion as follows: For any Radon measure ψ and any $\epsilon > 0$, every finite subfamily of $\mathbf{F}$ whose ψ-overlap is less than ϵ can be extended to a countable subfamily of $\mathbf{F}$ covering X (mod $\mathbf{N}^*$), and with ψ-overlap less than ϵ.

THEOREM 3.43. *Borelian star blankets differentiate every Radon measure, the basic measure μ being itself any Radon measure.*

Proof. This follows from Theorem 3.42.

Remark. This differentiation theorem, like Morse's theorems, implies that the set of points provided with μ-nullsequences is a μ-nullset.

§4. AN APPROACH TO A THEORY OF DIFFERENTIATION OF ABSTRACT INTERVAL FUNCTIONS

We take our setting as in 1.1. (G_σ) and the reduced strong Vitali property are assumed to hold. λ denotes a finite numerical function defined on the spread $\mathbf{D}$.

4.1 Preliminary definitions.

DEFINITIONS 4.11. Any enumerable disjointed family $\mathbf{P}$ of $\tilde{B}$-constituents is called a *Vitali partition.* A partition $\mathbf{P}$ is called $\mathbf{V}$-*fine* if $\mathbf{V}$ is a family of $\tilde{B}$-constituents and the sets in $\mathbf{P}$ belong to $\mathbf{V}$. $\mathbf{P}$ is said to be *bounded* if the $\mathbf{P}$-constituents are included in some $G^\circ{}_N$. If the sum

$$\sum_{V \in \mathbf{P}} \lambda(V)$$

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represents a real number (including $+\infty$ and $-\infty$), we denote it by $\psi(\lambda, \mathbf{P})$ and we say that $\mathbf{P}$ is λ-*integrable.*

This last condition is always fulfilled when $\lambda \geqslant 0$. In general, however, λ may be of variable sign, and we shall henceforth assume that $\mathbf{P}$ is λ-integrable whenever $\mathbf{P}$ is a bounded Vitali partition.

DEFINITIONS 4.12. A *Vitali partition over* $X \subset R$ is defined as a Vitali partition covering X (mod $\mathbf{N}^*$).

A Vitali partition over X is thus a Vitali partition over any set Y having a μ-cover in common with X. From the reduced strong Vitali property it follows that for any bounded set $X \subset E$ and any $\tilde{B}$-fine covering $\mathbf{V}$ of X, there exists a $\mathbf{V}$-fine Vitali partition over X.

DEFINITION 4.13. A set X is called λ-*admissible* if any full $\tilde{B}$-fine covering of X includes a λ-integrable Vitali partition over X. According to our assumption above, the bounded subsets of E are λ-admissible.

DEFINITION 4.14. For any λ-admissible set $X \subset E$ we define the *upper Vitali integral* $\psi^\circ(X)$ or $\psi^\circ(\lambda, X)$ and the *lower Vitali integral* $\psi_0(X)$ or $\psi_0(\lambda, X)$ as lim sup $\psi(\lambda, \mathbf{P})$ and lim inf $\psi(\lambda, \mathbf{P})$, respectively, the limits being taken in the family of the λ-integrable partitions $\mathbf{P}$ over X with the full $\tilde{B}$-fine coverings of X serving as a scale of fineness. We have

$$\psi^\circ(X) = \psi^\circ(\bar{X}\cdot E),\ \psi_0(X) = \psi_0(\bar{X}\cdot E),\ \psi^\circ(N^*) = 0, \qquad N^* \in E\cdot\mathbf{N}^*.$$

λ is said to be *Vitali integrable* over X if $\psi^\circ(X) = \psi_0(X)$, *Vitali summable* if $\psi^\circ(X)$ and $\psi_0(X)$ are equal and finite. In either case the common value is denoted by $\psi(X)$.

Explicitly, λ is Vitali summable to $\psi(X)$ if $\psi(X)$ is finite, and corresponding to any $\epsilon > 0$, there exists a full $\tilde{B}$-fine covering $\mathbf{W}(\epsilon)$ of X such that for any λ-integrable Vitali partition $\mathbf{P}$ over X whose constituents belong to $\mathbf{W}(\epsilon)$, we have $|\psi(X) - \psi(\lambda, \mathbf{P})| < \epsilon$.

The Vitali integrals are discussed in **(18)**, wherein $E = R$, the strong Vitali property holds, and the $\tilde{B}$-constituents are assumed to be $\mathbf{G}$-sets. Thus the Vitali partitions over a set X form a directed system which is used as the scale of fineness. The definition for the Vitali integrals is subsumed by the more general one adopted here.

4.2 Differentiation theorems for Vitali summable functions.

LEMMA 4.21. *If $\tilde{B}$ has the strong Vitali property, X is a bounded subset of E, and α and β are any finite numbers, then:*

(a) *$\psi^\circ(X) \geqslant \alpha\bar{\mu}(X)$ whenever $D^*\lambda > \alpha$ almost everywhere on X;*

(b) *$\psi_0(X) \leqslant \beta\bar{\mu}(X)$ whenever $D_*\lambda < \beta$ almost everywhere on X.*

Proof. We first establish (a). If $\psi^\circ(X) = \infty$, there is nothing to prove. Accordingly, we assume that $\psi^\circ(X) < \infty$, $\epsilon > 0$, and $D^*\lambda > \alpha$ almost every-

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where on X. In accordance with the definition of $\psi^\circ(X)$, there exists a full $\tilde{B}$-fine covering $\mathbf{W}'$ of X such that for any bounded $\mathbf{W}'$-fine Vitali partition $\mathbf{P}$ over X, we have $\psi(\lambda, \mathbf{P}) < \psi^\circ(X) + \epsilon$. The family $\mathbf{V}'$ of the $\mathbf{W}'$ constituents satisfying the relation $\lambda(V) > \alpha\mu(V)$ is a $\tilde{B}$-fine covering of X, hence, on account of the strong Vitali property and the boundedness of X, there exists a disjointed enumerable bounded subfamily $\mathbf{P}$ of $\mathbf{V}$ covering X (mod $\mathbf{N}^*$).

We have

$$\psi(\lambda, P) \geqslant \alpha \sum_{V \epsilon \mathbf{P}} \mu(V) \geqslant \alpha\bar{\mu}(X).$$

Since $\mathbf{P}$ is $\mathbf{W}'$-fine, $\psi^\circ(X) > \psi(\lambda, \mathbf{P}) - \epsilon$. Combination of the two relations yields (a), since ϵ is arbitrary.

We turn to (b). If $\psi_0(X) = -\infty$, there is nothing to prove; we thus assume that $\psi_0(X) > -\infty$, $\epsilon > 0$, and $D_*\lambda < \beta$ almost everywhere on X. There exists a full $\tilde{B}$-fine covering $\mathbf{W}''$ of X such that for any bounded $\mathbf{W}''$-fine Vitali partition $\mathbf{P}$, we have

$$\psi(\lambda, \mathbf{P}) > \psi_0(X) - \epsilon.$$

The family $\mathbf{V}''$ of the $\mathbf{W}''$-constituents satisfying $\lambda(V) < \beta\mu(V)$ is a $\tilde{B}$-fine covering of X. Due to the strong Vitali property and the boundedness of X, $\mathbf{V}''$ includes a disjointed enumerable bounded subfamily $\mathbf{P}$ with

$$S = \sigma\mathbf{P} \supset X (\text{mod } \mathbf{N}^*), \ \mu(S - S\cdot\bar{X}) < \epsilon.$$

We have

$$\psi(\lambda, \mathbf{P}) \leqslant \beta \sum_{V \epsilon \mathbf{P}} \mu(V) = \beta\mu(S) < \beta(\mu(\bar{X}) + \epsilon),$$

and since $\mathbf{P}$ is $\mathbf{W}''$ fine, $\psi(\lambda, \mathbf{P}) > \psi_0(X) - \epsilon$. Combining, we obtain $\psi_0(X) < \beta\bar{\mu}(X) + \epsilon(1 + \beta)$. From the arbitrary nature of ϵ, (b) follows.

THEOREM 4.22. *Assuming the strong Vitali property, if the function λ is Vitali summable over every bounded subset X of E, then the $\tilde{B}$-derivative $D\lambda$ exists and is finite almost everywhere on E and $D\lambda$ is $\mu_E{}^*$-measurable. In particular, if on the bounded subsets X of E, $\psi(X)$ can be represented as*

$$\int_{\bar{X}\cdot E} f(x)\, d\mu_E$$

then $D\lambda = f$ (mod $\mathbf{N}^*$).

Proof. It follows readily from the preceding lemma that $N' = [D^*\lambda = \infty]$ and $N'' = [D_*\lambda = -\infty]$ are $\mathbf{N}^*$-sets.

We regard $D^*\lambda$ and $D_*\lambda$, restricted to the set $E^* = E - (N' \cup N'')$, as the functions f and g occurring in Lemma 1.23. Recalling the Remarks following Definitions 1.31, it clearly suffices to derive a contradiction from the assumed existence of two bounded μ^*-entangled sets A and B and two finite numbers α and β with $\beta < \alpha$, such that $D^*\lambda > \alpha$ on A and $D_*\lambda < \beta$ on B. However, since $\psi^\circ(A) = \psi^\circ(B) = \psi_0(B) = \psi_0(A)$, Lemma 4.21 yields the desired contradiction at once.

To prove the second part of the theorem, we let A and B denote two bounded subsets of E of positive measure, $A = B$ (mod $\mathbf{N}^*$), such that the convex closure of $f(A)$ and the convex closure of $D\lambda$ (B) are at a positive distance apart. This means that there exists two finite numbers α and β, with $\alpha > \beta$, and either

4.221 $$f(x) < \beta \text{ on } A, \quad D\lambda(x) > \alpha \text{ on } B$$

or

4.222 $$f(x) > \alpha \text{ on } A, \quad D\lambda(x) < \beta \text{ on } B.$$

Since the integrand f is $E{\cdot}\mathbf{M}$-measurable, then $f(x) < \beta$ almost everywhere on $\bar{A}{\cdot}E$ in case $f(x) < \beta$ on A, whence

$$\int_{\bar{A}.E} f(x)\, d\mu_E < \beta\mu_E(\bar{A}\,.\,E) = \beta\,\bar{\mu}(A);$$

while, according to the preceding lemma, $\psi(B) \geqslant \alpha\bar{\mu}(B)$ if $D\lambda(x) > \alpha$ on B. Since $\bar{\mu}(A) = \bar{\mu}(B) > 0$, and $\psi(A) = \psi(B)$, the inequalities in 4.221 are incompatible. Similarly, it follows that 4.222 cannot hold. Referring to Lemma 1.23 and the Remarks following Corollary 1.24, we obtain $D\lambda = f$ (mod $\mathbf{N}^*$).

4.3 An example of Vitali summable functions: The non-negative upper semi-additive functions.

DEFINITION 4.31. The non-negative function λ is called *upper semi-additive* on E (with respect to $\tilde{B}$) if, corresponding to any $\tilde{B}$-constituent V and any $\epsilon > 0$, there exists a full $\tilde{B}$-fine covering $\mathbf{W}_\epsilon$ of $V{\cdot}E$ such that for any $\mathbf{W}_\epsilon$-fine Vitali partition $\mathbf{P}$, $\psi(\lambda, \mathbf{P}) < \lambda(V) + \epsilon$.

THEOREM 4.32. *If λ is a non-negative upper semi-additive function, then λ is Vitali integrable over the subset X of E and*

$$\psi(X) = \inf \sum_{V \epsilon \mathbf{E}} \lambda(V)$$

for all M-*families* $\mathbf{E}$ *of $\tilde{B}$-constituents*[5] *covering X* (mod $\mathbf{N}^*$). *In particular, if for any bounded Vitali partition* $\mathbf{P}$, $\psi(\lambda, \mathbf{P})$ *is finite, then* $\psi(X)$ *is finite on bounded* X.

Proof. We regard X as fixed and let

$$\gamma_0 = \inf \sum_{V \epsilon \mathbf{E}} \lambda(V),$$

let ϵ denote an arbitrary positive number, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ a sequence of positive numbers whose sum is less than $\frac{1}{2}\epsilon$.

If $\gamma_0 = \infty$, then $\psi(X) = \infty$ and there is nothing to prove. We assume, then, that $\gamma_0 < \infty$. From the definition of γ_0, we are able to find an M-family $V_1, V_2, \ldots, V_n, \ldots$ of $\tilde{B}$-constituents covering X (mod $\mathbf{N}^*$), such that

$$\sum_n \lambda(V_n) < \gamma_0 + \tfrac{1}{2}\epsilon.$$

[5]While the Vitali integral $\Psi(X)$ may be defined for vector valued functions, the right hand side of the equation just given presupposes a complete lattice structure.

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For each V_n we determine a full $\tilde{B}$-fine covering $\mathbf{W}_n$ in such a way that for any $\mathbf{W}_n$-fine Vitali partition $\mathbf{P}_n$ over $V_n \cdot E$,

$$\psi(\lambda, \mathbf{P}_n) < \lambda(V_n) + \epsilon_n,$$

and we define $\mathbf{W}^*$ as the union of the $\mathbf{W}_n$. $\mathbf{W}^*$ is a full $\tilde{B}$-fine covering of X.

The theorem will be established if we prove that for any $\mathbf{W}^*$-fine Vitali partition $\mathbf{P}$ over X

$$\psi(\lambda, \mathbf{P}) < \gamma_0 + \epsilon,$$

since $\psi(\lambda, \mathbf{P}) \geqslant \gamma_0$. Accordingly, we let $\mathbf{P}$ be such a partition, and subdivide $\mathbf{P}$ into disjointed sub-partitions $\mathbf{P}_n$, $\mathbf{P}_n$ being $\mathbf{W}_n$-fine. This decomposition is certainly possible but not necessarily unique. Due to the upper semi-additivity of λ,

$$\psi(\lambda, \mathbf{P}_n) < \lambda(V_n) + \epsilon_n,$$

hence by addition

$$\sum_n \psi(\lambda, \mathbf{P}_n) = \psi(\lambda, \mathbf{P}) < \sum_n \lambda(V_n) + \tfrac{1}{2}\epsilon < \gamma_0 + \epsilon.$$

As for the second part of the theorem, if X is bounded, then X is covered (mod $\mathbf{N}^*$) by a bounded Vitali partition $\mathbf{P}$ for which $\psi(\lambda, \mathbf{P})$ is finite, therefore γ_0 is also finite.

Remark. Under the hypotheses of the theorem the function ψ defined on all subsets of E is a Carathéodory outer measure, meaning that it satisfies (C1) and (C2) of **(25**, p. 43**)**. Besides, ψ vanishes on the $\mathbf{N}^*$-subsets of E. With the Carathéodory restriction process of an outer measure to a measure in mind, we may expect the restriction of ψ to the $E \cdot \mathbf{M}$-sets to be a Radon μ_E-integral, which would enable us to apply Theorem 4.22. This proves true under the assumptions in **(18)**; more on this occurs in 4.4 below.

4.4 Morse's addivelous functions. Morse's definition of addivelous functions given when $\tilde{B}$ is a Borelian blanket can be readily transposed to a general basis.

DEFINITION 4.41. We say that the function λ defined on the family $\mathbf{D}^*$ of subsets of R is *addivelous* if:

(i) λ is non-negative;

(ii) $\mathbf{D}^*$ includes the spread $\mathbf{D}$ of $\tilde{B}$;

(iii) Whenever the $\tilde{B}$-constituents $V_1, V_2, \ldots$ are disjoint and included in the $\mathbf{D}^*$-set D^*, then $\Sigma\lambda(V_n) \leqslant \lambda(D^*)$;

(iv) if $V \in \mathbf{D}$ and $\epsilon > 0$, then there exists a $\mathbf{D}^*$-set $D = D(V)$ for which $\lambda(D) \leqslant \lambda(V) + \epsilon$, $I(D) \supset V \cdot E$ (mod $\mathbf{N}^*$).[6]

Defining $\mathbf{W}_\epsilon$ as the family of the $\tilde{B}$-constituents included in D, we see that an addivelous function is upper semi-additive. The strong Vitali property secures the existence of a Vitali partition $V^\circ_1, V^\circ_2, \ldots,$ over E. We define (recall the final Remark after Definitions 1.31)

[6]For the definition of $I(D)$ refer to 1.1.

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$$R^{\circ}_n = D(V_1) \cup D(V_2) \cup \ldots \cup D(V_n),$$

and regard a Vitali partition **P** as bounded if, for some N, the **P**-sets are included in $I\ (R^{\circ}_N)$. Thus for any bounded Vitali partition **P**, $\psi(\lambda, \mathbf{P})$ is finite.

Theorem 4.32 is applicable, and expresses the equivalence between Vitali integration and Morse's regularization, for the subsets of E.

In Morse's case, $\tilde{B}$ is a Borelian blanket. Carathéodory's condition (C3) is clearly fulfilled. The restriction of ψ to the $E \cdot \mathbf{M}$-sets is a μ_E-integral, more precisely a Radon μ_E-integral with respect to the expanding reference sequence R°_n. Theorem 4.22 can be applied; $D\lambda$ is equal to a Radon-Nikodym μ_E-integrand of $\psi | E \cdot \mathbf{M}$. In an unpublished lecture delivered before the American Mathematical Society in 1948, Morse gave an interpretation, when $E = R$, of the indefinite integral of $D\lambda$ as $\psi|\mathbf{M}$, where $\psi(X)$ is defined as the infimum of numbers of the form

$$\sum_{V \in \mathbf{E}} \lambda(V),$$

where **E** is such a countable subfamily of **D** that **E** covers almost all of X.

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