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UNSOLVABLE PROBLEMS IN GROUPS WITH SOLVABLE WORD PROBLEM

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1. Introduction. Let G be a finitely presented group with solvable word problem. It is of some interest to ask which other decision problems must necessarily be solvable for such a group. Thus it is easy to see that there exist effective procedures to determine whether or not such a group is trivial, or nilpotent of a given class. On the other hand, the conjugacy problem need not be solvable for such a group, for Fridman [**5**] has shown that the word problem is solvable for the group with unsolvable conjugacy problem given by Novikov [**9**].

The order problem is said to be solvable for a given presentation of a group G if there exists an effective procedure to determine, given any word in the generators of the presentation, the order of the group element represented by the word; the power problem is said to be solvable for a given presentation of G if there exists an effective procedure to determine, given any words x and y in the generators of the presentation, whether or not the group element corresponding to x belongs to the cyclic subgroup of G generated by the group element corresponding to y. In this paper we show that there exists a finitely presented group with solvable word problem and unsolvable order and power problems. While these problems are known to be unsolvable in general for finitely presented groups, from the work of Baumslag, Boone, and Neumann [**1**], it is the case that the groups studied by these authors have unsolvable word problem. This is clear from the way in which these groups are constructed from a group with unsolvable word problem.

The results we obtain concerning decision problems are proved using an embedding theorem for recursively presented groups (Theorem 1 below). This theorem combines [**8**, Theorem 2] with a particular case of a result due to Clapham [**4**].

2. Results. A given presentation $(S|D)$ of a group G is said to be recursive if the set S of generators is effectively enumerable and the set D of defining relations is recursively enumerable; that is, D is given as $u_1, u_2, \ldots$, where, in some effective enumeration of the elements of the free group F on the elements of S as free generators, u_n is the $f(n)$th element of F, and $f(n)$ is a recursive function of n. This definition was given by Higman [**6**]. We have the following result.

THEOREM 1. *Let $(S|D)$ be a recursive presentation of a group H such that*

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the word problem is solvable for $(S|D)$. *Then* H *can be effectively embedded in a finitely presented group* L *with solvable word problem, so that there exists an effective procedure to determine, given any word in a finite set of generators of* L, *whether or not this word represents an element of the copy of* H *embedded in* L.

Proof. It was shown in [**8**, Theorem 2] that H can be embedded in a finitely generated, recursively presented group G with solvable word problem, in such a way that an effective procedure existed to determine, given any word in the generators of G, whether or not this word represents an element of H. It is clear that the embedding process described in the proof of that theorem is effective.

Clapham [**4**] has shown that a finitely generated, recursively presented group G, with word problem of degree $\mathbf{a}$, can be embedded (necessarily effectively) in a finitely presented group with word problem of degree $\mathbf{a}$. Taking $\mathbf{a} = 0$ in this result, we see that the group G can be embedded effectively in a finitely presented group L with solvable word problem. Thus H can be effectively embedded in such a group.

It remains to show that L can be chosen so that there exists an effective procedure to decide, given a word in a finite set of generators of L, whether or not this word represents an element of G; for if such a procedure exists, then since such a procedure exists for the copy of H embedded in G, it would follow that such a procedure exists for this copy of H considered as embedded in L.

In the notation of the proof of [**4**, Theorem 6] (reading recursively solvable and recursive for $\mathscr{A}$-solvable and $\mathscr{A}$-recursive, respectively), Clapham proved that $L = \{K \times G, t\}$ has recursively solvable word problem by showing that $\{(f_1, 1), (f_2, 1)\}$ and $\{(f_1, fR), (f_2, 1)\}$ in $K \times G$ are recursive. It is clear that the isomorphism he exhibits between these two groups is recursive; now applying [**3**, Lemma 3.6] shows that $K \times G$ is recursive in L. Since G is clearly recursive in $K \times G$, this proves the result.

We now prove our result concerning decision problems.

THEOREM 2. *There exists a finitely presented group* L, *with solvable word problem, such that the order problem and the power problem are unsolvable for* L. *Moreover, there is no effective procedure to determine, given a finite set of elements of* L, *whether or not the subgroup* H *of* L *generated by these elements is*

(i) *finite;*
(ii) *free abelian;*
(iii) *free.*

Proof. It is well known that there exists a recursive function ϕ whose range is not recursive. We pick one such ϕ, taking it to be one-to-one (this is easily done; cf. [**2**, Lemma 2.31]). Thus there exists no effective procedure to determine whether or not a given positive integer is in the range of ϕ. We construct two recursive presentations Π_1 and Π_2 using this function. We define

$$\Pi_1 = \{x_n \ (n = 1, 2, \ldots) | \ x_{\phi(n)}{}^{n!} = 1 \ (n = 1, 2, \ldots)\}$$

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and

$$\Pi_2 = \{y_n, z_n\ (n = 1, 2, \ldots) \mid y_n z_n = z_n y_n,\ y_{\phi(n)} = z_{\phi(n)}{}^n\ (n = 1, 2, \ldots)\}.$$

It was shown in [**7**] that the power problem is solvable for Π_1 and that the order problem is solvable for Π_2. Now the existence of a solution to either of these problems for an arbitrary recursive presentation is easily seen to imply the existence of a solution to the word problem for the presentation; hence the word problem is solvable for both Π_1 and Π_2.

It follows easily from our choice of ϕ that there is no effective procedure to determine whether or not

(2.1) any given $x_r \in \Pi_1$ has finite order;

(2.2) the subgroup generated by any given $x_r \in \Pi_1$ is infinite cyclic (and therefore free abelian and free);

(2.3) any given $y_r \in \Pi_2$ belongs to the cyclic subgroup of Π_2 generated by z_r.

This is so because (2.1) and (2.3) hold if and only if there exists an s such that $\phi(s) = r$, while (2.2) holds if and only if such an s does not exist.

Thus the order problem is unsolvable for Π_1 and the power problem is unsolvable for Π_2. Also, there is no effective procedure to determine, given any finite set of elements of Π_1, whether or not the subgroup generated by these elements is (1) finite, (2) free abelian, (3) free; for such a procedure does not exist for sets consisting of a single x_r.

Now let L_i be a finitely presented group with solvable word problem which contains a copy of the group represented by Π_i $(i = 1, 2)$, where L_i is obtained as in Theorem 1. Take L to be the direct product $L_1 \times L_2$. Then it is clear that L has the properties listed in the statement of the theorem.

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