European Journal of Applied Mathematics (2023), **34**, pp. 838–869
doi:10.1017/S0956792523000025

PAPER

Dimension reduction analysis of a three-dimensional thin elastic plate reinforced with fractal ribbons

Mustapha El Jarroudi*, Mhamed El Merzguioui, Mustapha Er-Riani, Aadil Lahrouz and Jamal El Amrani

Abdelmalek Essaâdi University, LMA, FST Tanger, B.P. 416, Tangier, Morocco
*Correspondence author. Email: eljarroudi@hotmail.com

Received: 11 June 2022; **Revised:** 21 December 2022; **Accepted:** 17 January 2023; **First published online:** 2 March 2023

Keywords: Thin elastic plates, fractal ribbons, dimension reduction analysis, microscopic interactions, effective material

2020 Mathematics Subject Classification: 35B40 (Primary); 28A80, 35J20 (Secondary)

Abstract
The aim of this paper is to study the dimension reduction analysis of an elastic plate with small thickness reinforced with increasing number of thin ribbons developing fractal geometry. We prove the Γ-convergence of the energy functionals to a two-dimensional effective energy including singular terms supported within the Sierpinski carpet.

1 Introduction

Starting from the pioneering work by Adkins and Rivlin [1] who studied the deformation of a structure reinforced with thin parallel, flexible and inextensible cords, strong research efforts have been devoted to the study of reinforced structures in order to describe their constitutive parameters. A lot of earlier works have focused on the homogenisation of elastic materials reinforced with fibres or ribbons composed of highly contrasting elastic materials (see for instance [5, 12, 15], and the references therein). The obtained homogenised composites are generally characterised by high strength and improved stiffness.

In this paper, we consider the deformation of a three-dimensional elastic plate with vertical small varying thickness reinforced with highly contrasted thin vertical ribbons following fractal paths. More specifically, we assume that the ribbons are thin vertical elastic strips of height $2r_h$ which are built on a pre-fractal curve obtained after h-iterations of the contractive similarities of the Sierpinski carpet Σ. We suppose that the plate occupies the domain $\omega \times (-\varepsilon_h, \varepsilon_h)$ of thickness $2\varepsilon_h$; $h \in \mathbb{N}$, where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\omega$.

Our main purpose is to describe, under suitable scaling regimes of the Lamé constants of the plate and that of the ribbons, the state of equilibrium of a such structure as the thickness of the plate and the height of the ribbons tend to zero, and the sequence of pre-fractal curves converges in the Hausdorff metric to the Sierpinski carpet. Using Γ-convergence methods (see for instance [11]), we obtain the following effective potential energy of the composite:

$$F_\infty(u,v) = \begin{cases} \int_\omega \eta_{\alpha\beta}(\overline{u})\, e_{\alpha\beta}(\overline{u})\, dx' + \int_\omega \varpi_{\alpha\beta}(u_3) \dfrac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} dx' \\ +\mu^* \int_\Sigma d\mathcal{L}_\Sigma(v) \\ +\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \sum_{\alpha=1,2} \int_\Sigma A_{\alpha\alpha}(s)\,(u_\alpha - v_\alpha)^2\, d\mathcal{H}^d(s) \\ +\dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A_{33} u_3^2 d\mathcal{H}^d(s) \\ \qquad\qquad \text{if } (u,v) \in H\left(\omega, \mathbb{R}^3\right), \\ +\infty \qquad\qquad \text{otherwise,} \end{cases} \tag{1.1}$$

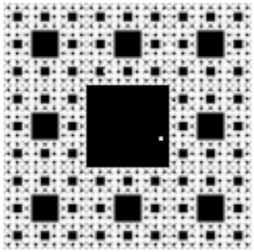

Figure 1. *The network $\{C_l\}_{l\in\mathbb{N}}$ is represented by black squares.*

where $x' = (x_1, x_2)$, $\overline{u} = (u_1, u_2)$, $v = (v_1, v_2)$,

$$\begin{cases} \eta_{\alpha\beta}\left(\overline{u}\right) = 4\mu\left(e_{\alpha\beta}\left(\overline{u}\right) + \dfrac{\lambda}{2\mu+\lambda} e_u\left(\overline{u}\right)\delta_{\alpha\beta}\right), \\ e_{\alpha\beta}\left(\overline{u}\right) = \dfrac{1}{2}\left(\dfrac{\partial u_\alpha}{\partial x_\beta} + \dfrac{\partial u_\beta}{\partial x_\alpha}\right); \alpha, \beta = 1, 2, \\ \varpi_{\alpha\beta}\left(u_3\right) = \dfrac{4}{3}\mu\left(\dfrac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} + \dfrac{\lambda}{2\mu+\lambda}\left(\Delta_{x'} u_3\right)\delta_{\alpha\beta}\right); \alpha, \beta = 1, 2, \\ \Delta_{x'} u_3 = \dfrac{\partial^2 u_3}{\partial x_1^2} + \dfrac{\partial^2 u_3}{\partial x_2^2}, \end{cases} \tag{1.2}$$

where the summation convention with respect to repeated indices has been used and will be used in the sequel, $\lambda > 0$ and $\mu > 0$ are the Lamé constants of the material in ω, μ^* is the effective shear modulus of the material occupying the fractal Σ, and where δ_{ij} denotes Kronecker's symbol, the parameter $\gamma \in (0, +\infty)$ is given by

$$\gamma = \lim_{h\to\infty} \left(\frac{8}{3}\right)^h \frac{a}{\varepsilon_h \ln r_h}, \tag{1.3}$$

a being a positive constant which will be specified in the next Section, $\mathcal{H}^d$ is the d-dimensional Hausdorff measure where d is the fractal dimension of Σ with

$$d = \ln 8/\ln 3, \tag{1.4}$$

$H\left(\omega, \mathbb{R}^3\right)$ is the space of admissible displacements defined by

$$H\left(\omega, \mathbb{R}^3\right) = \left\{ \begin{array}{l} (u, v) \in L^2\left(\Sigma, \mathbb{R}^3\right) \times L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right); \overline{u} \in H_0^1\left(\omega, \mathbb{R}^2\right) \\ v \in \mathcal{D}_{\Sigma,\mathcal{E}}, u_3 \in H_0^2\left(\omega\right) \end{array} \right\}, \tag{1.5}$$

$\mathcal{D}_{\Sigma,\mathcal{E}}$ is the domain of the energy supported on the fractal Σ (see (3.7), Section 3), $\kappa = \dfrac{3\mu+\lambda}{\mu+\lambda}$,

$$A\left(s\right) = \begin{cases} \text{Diag}\left(1, \dfrac{2}{(1+\kappa)}, \dfrac{2}{(1+\kappa)}\right) \text{ if } \nu\left(s\right) = \pm\left(0, 1\right), \\ \text{Diag}\left(\dfrac{2}{(1+\kappa)}, 1, \dfrac{2}{(1+\kappa)}\right) \text{ if } \nu\left(s\right) = \pm\left(1, 0\right), \end{cases} \tag{1.6}$$

where $\nu(s)$ is the outward unit normal on $\Sigma \cap \partial C_l$ seen from C_l; $\{C_l\}_{l\in\mathbb{N}}$ being the network of the squares removed from $[0, 1]^2$ to obtain the Sierpinski carpet Σ (see Figure 1), and $\mathcal{L}_\Sigma$ is a measure-valued Lagrangian with $\mathcal{L}_\Sigma\left(v\right) = \mathcal{L}_\Sigma\left(v, v\right) \geq 0$ is a positive measure (see Section 3, Proposition 2 for more details). The Lagrangian $\mathcal{L}_\Sigma$ takes on the fractal Σ the role of the Euclidean Lagrangian $d\mathcal{L}\left(\mathfrak{u}, \mathfrak{v}\right) = \nabla\mathfrak{u}.\nabla\mathfrak{v}dx'$.

The effective energy (1.1) is composed of stretching and bending energies for an isotropic elastic plate occupying the domain ω, a singular fractal energy term supported on the Sierpinski carpet Σ, and a nonlocal term due to the microscopic interactions between the constituent materials. The equilibrium

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

of the fractal Σ is asymptotically described by a generalised Laplace equation which is related to the discontinuity of the effective stress on Σ through the following relation:

$$\begin{cases} \mu^* \Delta_{\alpha,\Sigma}(v) \dfrac{\mathcal{H}^d}{\mathcal{H}^d(\Sigma)} = \dfrac{-2\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} A_{\alpha\alpha}(s)(u_\alpha - v_\alpha)\mathcal{H}^d \text{ on } \Sigma, \\ \qquad = [\eta_{\alpha\beta}(\overline{u})\nu_\beta]_\Sigma ; \alpha = 1, 2 \qquad \text{in } \Sigma, \end{cases} \tag{1.7}$$

where (u, v) is the solution of the limit problem stated in Corollary 13 of Section 5, $\Delta_\Sigma = \begin{pmatrix} \Delta_{1,\Sigma} \\ \Delta_{2,\Sigma} \end{pmatrix}$ is a second-order operator in $L^2_{\mathcal{H}^d}(\Sigma, \mathbb{R}^2)$ defined by the form $\mathcal{E}_\Sigma$ in Lemma 3 Section 3, and

$$[\eta_{\alpha\beta}(\overline{u})\nu_\beta]_\Sigma = \eta^+_{\alpha\beta}(\overline{u})\nu_\beta - \eta^-_{\alpha\beta}(\overline{u})\nu_\beta; \alpha = 1, 2, \tag{1.8}$$

where $\eta^+_{\alpha\beta}(\overline{u})\nu_\beta$ is the outward normal stress on $\Sigma \cap \partial C_l$; $l \in \mathbb{N}$, and $\eta^-_{\alpha\beta}(\overline{u})\nu_\beta$ is the inward normal stress.

If $\gamma = +\infty$ then, for every $(u, v) \in H(\omega, \mathbb{R}^3)$, $F_\infty(u, v) < +\infty \Rightarrow \overline{u} = v$ and $u_3 = 0$ in ω. In this case, the energy supported by the structure is given by

$$F_\infty(\overline{u}) = \begin{cases} \int_\omega \eta_{\alpha\beta}(\overline{u}) e_{\alpha\beta}(\overline{u})\, dx' + \mu^* \int_\Sigma d\mathcal{L}_\Sigma(\overline{u}) \\ \qquad\qquad \text{if } \overline{u} \in H^1_0(\omega, \mathbb{R}^2) \cap \mathcal{D}_{\Sigma,\mathcal{E}}, \\ +\infty \qquad\qquad \text{otherwise,} \end{cases} \tag{1.9}$$

where we see the disappearance of the term corresponding to the bending energy.

If $\gamma = 0$, then the effective energy of the structure turns out to be

$$F_{0,\infty}(u, v) = \begin{cases} \int_\omega \eta_{\alpha\beta}(\overline{u}) e_{\alpha\beta}(\overline{u})\, dx' + \int_\omega \varpi_{\alpha\beta}(u_3) \dfrac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} dx' \\ +\mu^* \int_\Sigma d\mathcal{L}_\Sigma(v) \\ \qquad\qquad \text{if } (u, v) \in H(\omega, \mathbb{R}^3), \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{1.10}$$

In this case, there is no connection between the energy of the plate and the effective energy stored in the Sierpinski carpet.

The homogenisation of structures reinforced with thin inclusions developing a fractal geometry has attracted attention in recent years due to the geometrical and physical characteristics of the inclusions (see for instance [6–10, 13, 16, 26, 32–35]). The homogenised problems obtained at the limit generally consist of singular forms containing fractal terms. The asymptotic analysis of a three-dimensional elastic material reinforced with thin vertical strips constructed on horizontal iterated Sierpinski gasket curves was studied in [16]. The problem considered in this work is quite different as we deal here with a three-dimensional plate with varying thickness reinforced with vertical strips disposed on iterated Sierpinski carpet curves. So far, much analysis has been realised on a very small class of self-similar sets, called finitely ramified fractals, which are characterised by the property that they are disconnected by removing a finite set of points. The standard example of finitely ramified fractals is the Sierpinski gasket. The Sierpinski carpet is an infinitely ramified fractal for which a purely analytic local regular Dirichlet form was very recently constructed in [20]. Note that the asymptotic analysis of elastic materials containing microcracks located along the Sierpinski carpet and the Menger sponge fractal (three-dimensional Sierpinski carpet) has been carried out in [14].

The homogenisation of three-dimensional elastic materials reinforced by highly rigid fibres with variable cross-section, which may have fractal geometry, has been studied in [15]. The authors showed that the geometrical changes induced by the oscillations along the fibre-cross-section interfaces, which may include fractal ones, can provide jumps of displacement fields or stress fields.

This paper is organised as follows. The statement of the problem is presented in Section 2. In Section 3, we introduce the energy form and the notion of a measure-valued local energy on the

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

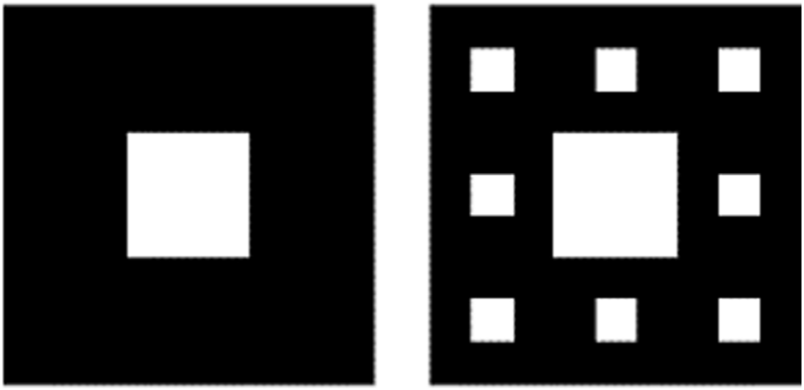

Figure 2. *The construction of Sierpinski carpet.*

Sierpinski carpet Σ. Section 4 is devoted to compactness results, which will be useful for the proof of the main results. In Section 5, we formulate the main results of this work. Section 6 is devoted to the proof of the main results.

2 Statement of the problem

Let us consider the unit square $E_0 = [0, 1]^2$. Let us divide E_0 into 9 equal subcubes of side 1/3. Let $\mathcal{SC}_1$ be the set of eight subsquares remaining after removing the interior of the central subsquare and let $E_1 = \bigcup \{C; C \in \mathcal{SC}_1\}$. Repeating the process, subdividing each element of $\mathcal{SC}_1$ into 9 equal subcubes of side 1/9, we obtain $E_2 = \bigcup \{C; C \in \mathcal{SC}_2\}$, where $\mathcal{SC}_2$ is the set of subcubes remaining after removing the interior of the central subsquare from each element of $\mathcal{SC}_1$. Continuing in this way (see Figure 2), we obtain a decreasing sequence of compact sets $(E_h)_{h\in\mathbb{N}}$. The set Σ defined by

$$\Sigma = \bigcap_{h=0}^{\infty} E_h, \tag{2.1}$$

is the standard Sierpinski carpet. The set Σ can be obtained as an iterated function system construction. Let $a_1 = (0, 0)$, $a_2 = (1/2, 0)$, $a_3 = (1, 0)$, $a_4 = (1, 1/2)$, $a_5 = (1, 1)$, $a_6 = (1/2, 1)$, $a_7 = (0, 1)$, $a_8 = (0, 1/2)$. We suppose that

$$E_0 \subset \overline{\omega} \text{ and } E_0 \cap \partial\omega = \{a_1, a_3, a_5, a_7\}, \tag{2.2}$$

where ω is the bounded domain of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\omega$, which was already set in the Introduction. Let us denote by $\{\psi_i\}_{i=1,\dots,8}$ the family of contractive similitudes defined on $\mathbb{R}^2$ by

$$\psi_i\left(x'\right) = \frac{x' + 2a_i}{3}, \forall x' = (x_1, x_2) \in \mathbb{R}^2. \tag{2.3}$$

Then, Σ is the unique non-empty compact set of $\mathbb{R}^2$ satisfying

$$\Sigma = \bigcup_{i=1}^{8} \psi_i(\Sigma). \tag{2.4}$$

Let us set $\mathcal{V}_0 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. Let $h \in \mathbb{N}^*$. We consider the set of vertices $\mathcal{V}_{i_1\dots i_h}$; $i_1, \dots, i_h \in \{1, \dots, 8\}$, defined by

$$\mathcal{V}_{i_1\dots i_h} = \psi_{i_1} \circ \dots \circ \psi_{i_h}(\mathcal{V}_0). \tag{2.5}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

We then set

$$\mathcal{V}_h = \begin{cases} \mathcal{V}_0 & \text{for } h = 0, \\ \bigcup_{i_1,\dots,i_h \in \{1,\dots,8\}} \mathcal{V}_{i_1\dots i_h} & \text{for } h \in \mathbb{N}^* \end{cases} \tag{2.6}$$

and

$$\mathcal{V}_\infty = \bigcup_{h \in \mathbb{N}} \mathcal{V}_h. \tag{2.7}$$

We consider the connected graph $\Sigma_{i_1\dots i_h} = \left(\mathcal{V}_{i_1\dots i_h}, S_{i_1\dots i_h}\right)$, where $S_{i_1\dots i_h}$ is the set of edges $\left[p, q\right]$ with $p, q \in \mathcal{V}_{i_1\dots i_h}$, such that $|p - q| = 3^{-h}/2$; $|p - q|$ being the Euclidian distance between p and q. We denote by S_0 the set of edges $\left[p, q\right]$ with $p, q \in \mathcal{V}_0$, such that $|p - q| = 1/2$, $\Sigma_0 = (\mathcal{V}_0, S_0)$, and set

$$\begin{cases} S_h = \bigcup_{i_1,\dots,i_h \in \{1,\dots,8\}} S_{i_1\dots i_h}, \ \forall h \in \mathbb{N}^*, \\ \Sigma_h = \bigcup_{i_1,\dots,i_h \in \{1,\dots,8\}} \Sigma_{i_1\dots i_h}, \ \forall h \in \mathbb{N}^*. \end{cases} \tag{2.8}$$

We also define

$$\begin{cases} S_h^1 = \bigcup_{[p,q] \subset S_h:\ [p,q] \perp (0,1)} \left[p, q\right], \\ S_h^2 = \bigcup_{[p,q] \subset S_h:\ [p,q] \perp (1,0)} \left[p, q\right], \end{cases} \tag{2.9}$$

where $\left[p, q\right] \perp (0, 1)$ (resp. $\left[p, q\right] \perp (1, 0)$) means that the line segment $\left[p, q\right]$ is perpendicular to the unit vector $(0, 1)$ (resp. $(1, 0)$).

Let N_h^v be the number of vertices in $\mathcal{V}_h$ and let N_h^e be the number of edges in S_h. These numbers can be computed by using the proof of [30, Lemma 2.1.2]. Indeed, N_h^v can be obtained by adding the number of midpoints of the edges of the graph approximation of the Sierpinski carpet of [30, Paragraph 2.1] to the number of vertices obtained in [30, Lemma 2.1.2], then, using the proof of [30, Lemma 2.1.2], N_h^e can be obtained by induction. We have, for $h \geq 2$,

$$\begin{aligned} N_h^v &= \left(3^{h+1} + 1\right)\left(3^h + 1\right) - \sum_{k=0}^{h-2} 8^k \left(3^{h-1-k} - 1\right)\left(3^{h-k} - 1\right), \\ N_h^e &= 4\left(3^h\left(3^h + 1\right) - \sum_{k=0}^{h-2} 8^k 3^{h-1-k}\left(3^{h-1-k} - 1\right)\right), \end{aligned} \tag{2.10}$$

from which we deduce, by a straightforward computation, that

$$\begin{aligned} N_h^v &\underset{h\to\infty}{\sim} a8^h, \\ N_h^e &\underset{h\to\infty}{\sim} b8^h, \end{aligned} \tag{2.11}$$

where a and b are positive constants with $a \approx 3.657$ and $b \approx 4.8$. The edges belonging to S_h can be rearranged as S_h^k; $k \in I_h = \left\{1, 2, \dots, N_h^e\right\}$. We suppose that the sequences $(\varepsilon_h)_{h\in\mathbb{N}}$ and $(r_h)_h$ of positive numbers verify

$$\begin{cases} \lim_{h\to\infty} \varepsilon_h = 0, & \lim_{h\to\infty} r_h = 0, \\ \lim_{h\to\infty} r_h/\varepsilon_h = 0, & \lim_{h\to\infty} 3^h r_h = 0. \end{cases} \tag{2.12}$$

Let $p_h^k = \left(p_{h1}^k, p_{h2}^k\right)$, $q_h^k = \left(q_{h1}^k, q_{h2}^k\right)$ be the extremities of the line segment S_h^k; $k = 1, 2, \dots, N_h^e$. We define the ribbon T_h^k by

$$T_h^k = \left(\omega \cap S_h^k\right) \times (-r_h, r_h), \tag{2.13}$$

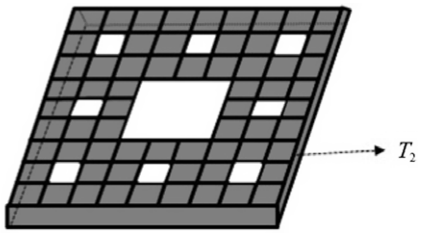

Figure 3. *An example of the union T_h of ribbons for $h = 2$.*

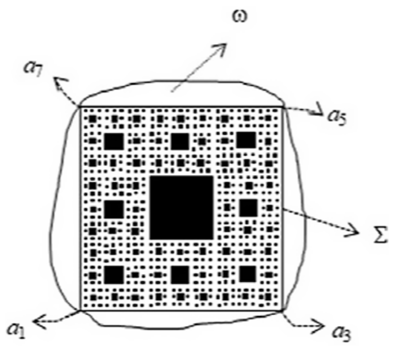

Figure 4. *The fractal Σ embedded in $\overline{\omega}$ such that $\Sigma \cap \partial\omega = \{a_1, a_3, a_5, a_7\}$.*

and their union (see Figure 3) by

$$T_h = \bigcup_{k=1}^{N_h^e} T_h^{,k}. \tag{2.14}$$

Denoting $|T_h|$ the 2-dimensional measure of T_h, we see that

$$|T_h| = \frac{4r_h N_h^e}{3^h}. \tag{2.15}$$

Let us recall that ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\omega$. We suppose that $\Sigma \subset \overline{\omega}$ and, according to (2.2), that (see Figure 4)

$$\Sigma \cap \partial\omega = E_0 \cap \partial\omega = \{a_1, a_3, a_5, a_7\} = \partial\Sigma. \tag{2.16}$$

We define

$$\begin{aligned} \Omega_h &= \omega \times (-\varepsilon_h, \varepsilon_h), \\ \Gamma_h &= \partial\omega \times (-\varepsilon_h, \varepsilon_h). \end{aligned} \tag{2.17}$$

We suppose that $\Omega_h \backslash T_h$ is the reference configuration of a linear, homogeneous, and isotropic elastic material with Lamé coefficients $\mu_h > 0$ and $\lambda_h > 0$. This means that the deformation tensor $e(u) = \left(e_{ij}(u)\right)_{i,j=1,2,3}$, with $e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ for some displacement u, is linked to the stress tensor $\sigma^h(u) = \left(\sigma_{ij}^h(u)\right)_{i,j=1,2,3}$, through Hooke's law

$$\sigma_{ij}^h(u) = \lambda_h e_{mm}(u)\,\delta_{ij} + 2\mu_h e_{ij}(u)\,;\, i,j = 1,2,3, \tag{2.18}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

where $\lambda_h > 0$ and $\mu_h > 0$ are the Lamé constants of the material. We suppose that T_h is the reference configuration of a linear, homogeneous and isotropic elastic material with Lamé coefficients $\mu_h^*, \lambda_h^* > 0$, and stress tensor $\sigma^{*h}(u)$ with components

$$\sigma_{ij}^{*h}(u) = \lambda_h^* e_{mm}(u)\,\delta_{ij} + 2\mu_h^* e_{ij}(u)\,;\ i,j = 1,2,3, \tag{2.19}$$

with

$$\lambda_h^* = c_h \lambda^* \text{ and } \mu_h^* = c_h \mu^*, \tag{2.20}$$

where λ^*, μ^* are positive constants and

$$c_h = \frac{\rho^h}{r_h 3^h}, \tag{2.21}$$

is a scaling parameter which is related to the geometry of the fractal inclusion T_h, where $\rho > 1$ is a structural constant which, according to [3, 4], is related to the spectral dimension d_s of the Sierpinski carpet Σ by the following relation:

$$\rho = 8^{2/d_s - 1}. \tag{2.22}$$

The exact value of ρ remains still unknown, and only some bounds for ρ are given in [3, 4]:

$$\begin{cases} \rho \in \left[7/6, 3/2\right] \text{ based on shorting and cuttingarguments,} \\ \rho \in [1.25147, 1.25149] \text{ based on computer calculation.} \end{cases} \tag{2.23}$$

We suppose that a perfect adhesion occurs between Ω_h and T_h along their common interfaces. We suppose that the material in Ω_h is held fixed on Γ_h, remains free on $\partial\Omega_h \backslash \Gamma_h$ and submitted to volumic forces $f^h \in L^2\left(\Omega_h, \mathbb{R}^3\right)$. We assume that the applied forces f^h have the following form:

$$\begin{cases} f_\alpha^h(x) = f_\alpha\left(x_1, x_2, x_3/\varepsilon_h\right)/\varepsilon_h;\ \alpha = 1, 2, \\ f_3^h(x) = f_3\left(x_1, x_2, x_3/\varepsilon_h\right), \end{cases} \tag{2.24}$$

with $f = (f_1, f_2, f_3) \in L^2\left(\omega \times (-1, 1), \mathbb{R}^3\right)$ and that

$$\lim_{h\to\infty} \varepsilon_h \mu_h = \mu > 0 \text{ and } \lim_{h\to\infty} \varepsilon_h \lambda_h = \lambda > 0. \tag{2.25}$$

We define the energy functional F_h on $L^2\left(\Omega_h, \mathbb{R}^3\right)$ by

$$F_h(u) = \begin{cases} \int_{\Omega_h \backslash T_h} \sigma_{ij}^h(u)\, e_{ij}(u)\, dx + \int_{T_h} \sigma_{ij}^{*h}(u)\, e_{ij}(u)\, ds dx_3 \\ \qquad\qquad \text{if } u \in H\left(\Omega_h, \mathbb{R}^3\right), \\ +\infty \qquad\qquad \text{otherwise,} \end{cases} \tag{2.26}$$

where ds is the measure on S_h defined by

$$ds = \begin{cases} dx_1 \text{ on } S_h^1, \\ dx_2 \text{ on } S_h^2 \end{cases}$$

and

$$H\left(\Omega_h, \mathbb{R}^3\right) = H_{\Gamma_h}^1\left(\Omega_h, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right), \tag{2.27}$$

with

$$H_{\Gamma_h}^1\left(\Omega_h, \mathbb{R}^3\right) = \left\{u \in H^1\left(\Omega_h, \mathbb{R}^3\right);\ u = 0 \text{ on } \Gamma_h\right\}.$$

The equilibrium of the elastic material occupying Ω_h is described by the minimisation problem

$$\min_{u \in L^2\left(\Omega_h, \mathbb{R}^3\right)} \left\{F_h(u) - 2\int_{\Omega_h} f^h.u dx\right\}. \tag{2.28}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

3 Energy forms on the Sierpinski carpet

In this section, we introduce the energy form and the notion of a measure-valued local energy (or Lagrangian) on the Sierpinski carpet. For any function $w : \mathcal{V}_\infty \longrightarrow \mathbb{R}^{2,}$ we define

$$\mathcal{E}^h_\Sigma (w) = \rho^h \sum_{\substack{p,q \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} |w(p) - w(q)|^2 , \tag{3.1}$$

where ρ is given in (2.22). We then define the energy

$$\mathcal{E}_\Sigma (z) = \lim_{h\to\infty} \mathcal{E}^h_\Sigma (z) , \tag{3.2}$$

with domain $\mathcal{D}_\infty = \left\{ z : \mathcal{V}_\infty \longrightarrow \mathbb{R}^2 : \mathcal{E}_\Sigma (z) < \infty \right\}$. This energy has been constructed in [20]. Every function $z \in \mathcal{D}_\infty$ can be uniquely extended to be an element of $C\left(\Sigma, \mathbb{R}^2\right)$, still denoted as z. Let us set

$$\mathcal{D} = \left\{ z \in C\left(\Sigma, \mathbb{R}^2\right) : \mathcal{E}_\Sigma (z) < \infty \right\} , \tag{3.3}$$

where $\mathcal{E}_\Sigma (z) = \mathcal{E}_\Sigma \left(z\,|_{\mathcal{V}_\infty} \right)$. We define the space $\mathcal{D}_\mathcal{E}$ as

$$\mathcal{D}_\mathcal{E} = \overline{\mathcal{D}}^{\|.\|_{\mathcal{D}_\mathcal{E}}} , \tag{3.4}$$

where $\|.\|_{\mathcal{D}_\mathcal{E}}$ is the intrinsic norm

$$\|z\|_{\mathcal{D}_\mathcal{E}} = \left\{ \mathcal{E}_\Sigma (z) + \|z\|^2_{L^2_{\mathcal{H}^d}(\Sigma,\mathbb{R}^2)} \right\}^{1/2} , \tag{3.5}$$

where

$$L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right) = \left\{ u : \Sigma \longrightarrow \mathbb{R}^2 ; \int_\Sigma |u|^2 (s)\, d\mathcal{H}^d (s) < \infty \right\} . \tag{3.6}$$

Let us now define the space

$$\mathcal{D}_{\Sigma,\mathcal{E}} = \{ z \in \mathcal{D}_\mathcal{E} : z = 0 \text{ on } \partial\Sigma \} , \tag{3.7}$$

where $\partial\Sigma$ is defined in (2.16). We denote $\mathcal{E}_\Sigma (.,.)$ the bilinear form defined on $\mathcal{D}_{\Sigma,\mathcal{E}} \times \mathcal{D}_{\Sigma,\mathcal{E}}$ by

$$\mathcal{E}_\Sigma (w,z) = \frac{1}{2} \left(\mathcal{E}_\Sigma (w+z) - \mathcal{E}_\Sigma (w) - \mathcal{E}_\Sigma (z) \right) , \forall w, z \in \mathcal{D}_{\Sigma,\mathcal{E}} . \tag{3.8}$$

One can see that

$$\mathcal{E}_\Sigma (w,z) = \lim_{h\to\infty} \mathcal{E}^h_\Sigma (w,z) , \tag{3.9}$$

where

$$\mathcal{E}^h_\Sigma (w,z) = \rho^h \sum_{\substack{p,q \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} (w(p) - w(q)) . (z(p) - z(q)) . \tag{3.10}$$

According to [20, Theorem 2.5 and Theorem 10.4], the form $\mathcal{E}_\Sigma$ is a strongly local regular closed form on $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$. This means (see for instance [19]) that

1. (local property) $u, v \in \mathcal{D}_{\Sigma,\mathcal{E}}$ with compact supp[u] and supp[v], and v is constant on a neighbourhood of supp[u] implies that $\mathcal{E}_\Sigma (u, v) = 0$,
2. (regularity) $\mathcal{D}_{\Sigma,\mathcal{E}} \cap C_0\left(\Sigma, \mathbb{R}^2\right)$ ($C_0\left(\Sigma, \mathbb{R}^2\right)$ being the space of functions of $C\left(\Sigma, \mathbb{R}^2\right)$ with compact support) is dense both in $C_0\left(\Sigma, \mathbb{R}^2\right)$ with respect to the uniform norm and in $\mathcal{D}_{\Sigma,\mathcal{E}}$ with respect to the norm (3.5),
3. (closedness) Let $(u_n)_n \subset \mathcal{D}_{\Sigma,\mathcal{E}}$ such that $\|u_n - u_m\|_{\mathcal{D}_\mathcal{E}} \longrightarrow 0$, $n, m \longrightarrow \infty$, there exists $u \in \mathcal{D}_{\Sigma,\mathcal{E}}$ such that $\|u_n - u\|_{\mathcal{D}_\mathcal{E}} \longrightarrow 0$, $n \longrightarrow \infty$.

The space $\mathcal{D}_{\Sigma,\mathcal{E}}$ is injected in $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$ and is complete with respect to the norm (3.5); thus, $\mathcal{D}_{\Sigma,\mathcal{E}}$ is an Hilbert space with the scalar product associated with the norm (3.5). Moreover, every function of

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

$\mathcal{D}_{\Sigma,\mathcal{E}}$ possesses a continuous representative. Indeed, according to [20, Theorem 2.7 and Remark 11.3], the space $\mathcal{D}_{\mathcal{E}}$ is continuously embedded in the space $C^{\beta}\left(\Sigma, \mathbb{R}^2\right)$ of Hölder continuous functions with $\beta = \ln \rho / \ln 9$.

Let us now consider the sequence $(m_h)_h$ of measures defined by

$$m_h = \sum_{\substack{p \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} \frac{\delta_p}{N_h^v}, \tag{3.11}$$

where δ_p is the Dirac measure at the point p. We have the following:

Lemma 1 *The sequence $(m_h)_h$ weakly converges in $C(\Sigma)^*$ to the measure*

$$m = \mathbf{1}_\Sigma \frac{d\mathcal{H}^d}{\mathcal{H}^d(\Sigma)},$$

where $C(\Sigma)^*$ is the topological dual of the space $C(\Sigma)$ and $\mathbf{1}_\Sigma$ is the indicator function of the set Σ.

Proof. Let $\varphi \in C(\Sigma)$. Then, according to the ergodicity result of [17, Theorem 6.1],

$$\begin{aligned}\lim_{h\to\infty} \int_\Sigma \varphi(x)\, dm_h &= \lim_{h\to\infty} \sum_{p\in\mathcal{V}_h} \frac{\varphi(p)}{N_h^v} \\ &= \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma \varphi(s)\, d\mathcal{H}^d(s).\end{aligned}$$

□

According to [31, Section 3], the approximating form $\mathcal{E}_\Sigma^h(.,.)$ can be written as

$$\mathcal{E}_\Sigma^h(w,z) = \int_\Sigma \nabla_h w . \nabla_h z\, dm_h, \tag{3.12}$$

with

$$\nabla_h w . \nabla_h z(p) = \frac{1}{2^{2\varkappa}} \sum_{\substack{q \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} \frac{(w(p)-w(q))}{|p-q|^{\varkappa/2}} . \frac{(z(p)-z(q))}{|p-q|^{\varkappa/2}},$$

where $\varkappa$ is the unique positive number for which the sequence $\left(\mathcal{E}_\Sigma^h(.,.)\right)_h$ has a non-trivial limit. We note that, using (3.1),

$$\varkappa = \frac{\ln 8\rho}{\ln 3}, \tag{3.13}$$

where ρ is given in (2.22). The following result holds true:

Proposition 2 *For every $w, z \in \mathcal{D}_{\Sigma,\mathcal{E}}$, the sequence of measures $\left(\mathcal{L}_\Sigma^h(w,z)\right)_h$ defined, for every $\forall A \subset \Sigma$, by*

$$\mathcal{L}_\Sigma^h(w,z)(A) = \int_{A\cap\Sigma} \nabla_h w . \nabla_h z\, dm_h$$

$$= \rho^h \sum_{\substack{p,q \in A\cap\mathcal{V}_h \\ |p-q|=3^{-h}/2}} (w(p)-w(q)) . (z(p)-z(q)),$$

weakly converges in $C\left(\Sigma, \mathbb{R}^2\right)^*$ to a signed finite Radon measure $\mathcal{L}_\Sigma(w,z)$ on Σ, called Lagrangian measure on Σ. Moreover,

$$\mathcal{E}_\Sigma(w,z) = \int_\Sigma d\mathcal{L}_\Sigma(w,z), \ \forall w, z \in \mathcal{D}_{\Sigma,\mathcal{E}}.$$

Proof. The proof follows the lines of the proof of [18, Proposition 2.3.]. Let us set, for $w \in \mathcal{D}_{\Sigma,\mathcal{E}}$, $\mathcal{L}_\Sigma^h(w) = \mathcal{L}_\Sigma^h(w,w)$. We deduce from (3.2), (3.9), (3.12), and (3.13) that the sequence $\left(\mathcal{L}_\Sigma^h(w)(\Sigma)\right)_h$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

is a uniformly bounded sequence. Then, observing that, for every $w \in \mathcal{D}_{\Sigma,\mathcal{E}}$ and every $\varphi e_1 \in \mathcal{D}_{\Sigma,\mathcal{E}} \cap C_0\left(\Sigma, \mathbb{R}^2\right)$; $e_1 = (1, 0)$,

$$\begin{aligned}\int_\Sigma \varphi d\mathcal{L}^h_\Sigma(w) &= \rho^h \sum_{\substack{p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}} \varphi(p)\,|w(p)-w(q)|^2\\ &= \rho^h \sum_{\substack{p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}} \frac{\varphi(p)+\varphi(q)}{2}\,|w(p)-w(q)|^2 \\ &= \mathcal{E}^h_\Sigma(\varphi w, w) - \frac{1}{2}\mathcal{E}^h_\Sigma\left(\varphi e_1, |w|^2 e_1\right),\end{aligned} \tag{3.14}$$

we deduce, taking into account the regularity of the form $\mathcal{E}_\Sigma(.,.)$, that

$$\lim_{h\to\infty} \int_\Sigma \varphi d\mathcal{L}^h_\Sigma(w) = \mathcal{E}_\Sigma(\varphi w, w) - \frac{1}{2}\mathcal{E}_\Sigma\left(\varphi e_1, |w|^2 e_1\right). \tag{3.15}$$

On the other hand, according to [28, Proposition 1.4.1], the energy form $\mathcal{E}_\Sigma(w)$, which is a Dirichlet form of diffusion type, admits the following integral representation:

$$\mathcal{E}_\Sigma(w) = \int_\Sigma d\mathcal{L}_\Sigma(w),$$

where $\mathcal{L}_\Sigma(w)$ is a positive Radon measure which is uniquely determined by the relation

$$\int_\Sigma \varphi d\mathcal{L}_\Sigma(w) = \mathcal{E}_\Sigma(\varphi w, w) - \frac{1}{2}\mathcal{E}_\Sigma\left(\varphi e_1, |w|^2 e_1\right), \forall \varphi \in C_0(\Sigma).$$

Thus, combining with (3.15), the sequence $\left(\mathcal{L}^h_\Sigma(w)\right)_h$ converges in the sense of measures to the measure $\mathcal{L}_\Sigma(w)$. Observing that, for every $w, z \in \mathcal{D}_{\Sigma,\mathcal{E}}$,

$$\mathcal{L}^h_\Sigma(w,z) = \frac{1}{2}\left(\mathcal{L}^h_\Sigma(w+z) - \mathcal{L}^h_\Sigma(w) - \mathcal{L}^h_\Sigma(z)\right),$$

we deduce that the sequence $\left(\mathcal{L}^h_\Sigma(w,z)\right)_h$ weakly converges in $C\left(\Sigma, \mathbb{R}^2\right)^*$ to the measure $\mathcal{L}_\Sigma(w,z)$. □

As $\mathcal{E}_\Sigma(.,.)$ is a closed Dirichlet form on $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$, we have, according to [25, Chap. 6, Theorem 2.1], the following result:

Lemma 3 *There exists a unique self-adjoint non-positive operator Δ_Σ on $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$ with domain*

$$\mathcal{D}_\Lambda = \left\{ \begin{array}{l} w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right); \\ \Delta_\Sigma(w) = \begin{pmatrix} \Delta_{1,\Sigma}(w) \\ \Delta_{2\Sigma}(w) \end{pmatrix} \in L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right) \end{array} \right\} \subset \mathcal{D}_{\Sigma,\mathcal{E}}$$

dense in $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$ such that, for every $w \in \mathcal{D}_\Lambda$ and $z \in \mathcal{D}_{\Sigma,\mathcal{E}}$,

$$\mathcal{E}_\Sigma(w,z) = -\int_\Sigma \Delta_\Sigma(w)\,.z\frac{d\mathcal{H}^d}{\mathcal{H}^d(\Sigma)}.$$

4 Compactness results

In this section, we establish some compactness results which will be useful for the proof of the main results.

4.1 A priori estimates

Lemma 4 *For every $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$ such that*

$$\sup_h F_h\left(u^h\right) < +\infty,$$

we have, under the hypothesis (2.25), the following estimates:

1. $\sup_h \sum_{\alpha,\beta=1,2} \frac{1}{\varepsilon_h} \int_{\Omega_h} \left(\left(\frac{\partial u_\alpha^h}{\partial x_\beta} \right)^2 + \left(\frac{\partial u_3^h}{\partial x_3} \right)^2 + \left(u_\alpha^h \right)^2 \right) dx < +\infty,$
2. $\sup_h \sum_{\alpha=1,2} \frac{1}{\varepsilon_h} \int_{\Omega_h} \left(\left(\varepsilon_h \frac{\partial u_\alpha^h}{\partial x_3} \right)^2 + \left(\varepsilon_h \frac{\partial u_3^h}{\partial x_\alpha} \right)^2 + \left(\varepsilon_h u_3^h \right)^2 \right) dx < +\infty.$

Proof. From the Korn inequality for clamped plates (see for instance [21, Subsection 2.1]), we deduce that

$$\begin{aligned} &\sum_{\alpha,\beta=1,2} \int_{\Omega_h} \left(\left(\frac{\partial u_\alpha^h}{\partial x_\beta} \right)^2 + \left(\frac{\partial u_3^h}{\partial x_3} \right)^2 + \left(u_\alpha^h \right)^2 \right) dx \\ &+ \sum_{\alpha=1,2} \int_{\Omega_h} \varepsilon_h^2 \left(\left(\frac{\partial u_\alpha^h}{\partial x_3} \right)^2 + \left(\frac{\partial u_3^h}{\partial x_\alpha} \right)^2 + \left(u_3^h \right)^2 \right) dx \\ &\leq C \sum_{i,j=1,2,3} \int_{\Omega_h} \left(e_{ij} \left(u^h \right) \right)^2 dx. \end{aligned} \tag{4.1}$$

Since

$$F_h \left(u^h \right) \geq \mu_h \varepsilon_h \sum_{i,j=1,2,3} \frac{1}{\varepsilon_h} \int_{\Omega_h} \left(e_{ij} \left(u^h \right) \right)^2 dx,$$

we deduce, using the hypothesis (2.25), that

$$\sup_h \sum_{i,j=1,2,3} \frac{1}{\varepsilon_h} \int_{\Omega_h} \left(e_{ij} \left(u^h \right) \right)^2 dx \leq C \sup_h F_h \left(u^h \right), \tag{4.2}$$

which, in view of (4.1), proves the claim.

We have now the following estimates:

Lemma 5 *For every sequence $\left(u^h\right)_h$ such that $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$ and*

$$\sup_h F_h \left(u^h \right) < +\infty,$$

we have, under the hypothesis (2.20), the following estimates:

1. $\sup_h \sum_{\substack{\alpha=1,2 \\ p,q \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} \rho^h \left(\frac{1}{2r_h} \int_{-r_h}^{r_h} \left(u_\alpha^h (p, x_3) - u_\alpha^h (q, x_3) \right) dx_3 \right)^2 < +\infty,$
2. $\sup_h \frac{1}{2r_h} \sum_{\alpha=1,2} \int_{-r_h}^{r_h} \int_\Sigma \left| u_\alpha^h \right|^2 dm_h dx_3 < +\infty$*; m_h being the measure defined in (3.11),*
3. $\frac{1}{2r_h} \int_{-r_h}^{r_h} \int_\Sigma \left| \varepsilon_h u_3^h \right|^2 dm_h dx_3 \leq C \left(\frac{3}{8} \right)^h$*; C being a positive constant independent of h.*

Proof. 1. Observing that

$$\begin{aligned} &\left(e_{11} \left(u^h \right) \right)^2 + 2 \left(e_{12} \left(u^h \right) \right)^2 + \left(e_{22} \left(u^h \right) \right)^2 \\ &= \left(\frac{\partial u_1^h}{\partial x_1} \right)^2 + \frac{1}{2} \left(\frac{\partial u_2^h}{\partial x_1} \right)^2 \quad \text{on } S_h^1 \end{aligned}$$

and

$$\begin{aligned} &\left(e_{11} \left(u^h \right) \right)^2 + 2 \left(e_{12} \left(u^h \right) \right)^2 + \left(e_{22} \left(u^h \right) \right)^2 \\ &= \left(\frac{\partial u_2^h}{\partial x_2} \right)^2 + \frac{1}{2} \left(\frac{\partial u_1^h}{\partial x_2} \right)^2 \quad \text{on } S_h^2, \end{aligned}$$

we deduce that

$$\begin{aligned}
&\int_{T_h} \sigma_{ij}^h\left(u^h\right) e_{ij}\left(u^h\right) dsdx_3 \\
&\geq 2\mu_h^*\left(\int_{T_h} \left(e_{11}\left(u_h\right)\right)^2 + 2\left(e_{12}\left(u_h\right)\right)^2 + \left(e_{22}\left(u_h\right)\right)^2 dsdx_3\right) \\
&\geq \mu_h^*\left(\int_{-r_h/2}^{r_h/2}\int_{S_h^1}\left(\frac{\partial u_1^h}{\partial x_1}\right)^2 + \left(\frac{\partial u_2^h}{\partial x_1}\right)^2 dsdx_3\right) \\
&+ \mu_h^*\left(\int_{-r_h/2}^{r_h/2}\int_{S_h^2}\left(\frac{\partial u_2^h}{\partial x_2}\right)^2 + \left(\frac{\partial u_1^h}{\partial x_2}\right)^2 dsdx_3\right).
\end{aligned} \tag{4.3}$$

Observing that, for $[p, q] \subset S_h^\beta$; $\beta = 1, 2$,

$$\begin{aligned}
\int_{[p,q]}\left(\frac{\partial u_\alpha^h}{\partial x_\beta}\right)^2 ds &\geq 3^h\left(\int_{[p,q]}\frac{\partial u_\alpha^h}{\partial x_\beta} ds\right)^2 \\
&= 3^h\left(u_\alpha^h\left(p, x_3\right) - u_\alpha^h\left(q, x_3\right)\right)^2,
\end{aligned}$$

we deduce from (4.3), using the hypothesis (2.20), that

$$\begin{aligned}
&\int_{T_h} \sigma_{ij}^h\left(u^h\right) e_{ij}\left(u^h\right) dsdx_3 \\
&\geq 3^h r_h \mu_h^* \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}} \frac{1}{2r_h}\int_{-r_h/2}^{r_h/2}\left(u_\alpha^h\left(p, x_3\right) - u_\alpha^h\left(q, x_3\right)\right)^2 dx_3 \\
&\geq 3^h r_h c_h \mu^* \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}} \frac{1}{2r_h}\int_{-r_h/2}^{r_h/2}\left(u_\alpha^h\left(p, x_3\right) - u_\alpha^h\left(q, x_3\right)\right)^2 dx_3 \\
&\geq \mu^* \rho^h \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}}\left(\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(u_\alpha^h\left(p, x_3\right) - u_\alpha^h\left(q, x_3\right)\right) dx_3\right)^2.
\end{aligned} \tag{4.4}$$

Hence,

$$\begin{aligned}
&\sup_h\ \mu^* \rho^h \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}}\left(\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(u_\alpha^h\left(p, x_3\right) - u_\alpha^h\left(q, x_3\right)\right) dx_3\right)^2 \\
&\leq \sup_h \int_{T_h} \sigma_{ij}^{*h}\left(u^h\right) e_{ij}\left(u^h\right) ds \\
&\leq \sup_h F_h\left(u^h\right) < +\infty.
\end{aligned} \tag{4.5}$$

2. Let p fixed in $\mathcal{V}_h$. Let us denote $(q_m)_{m=1,\ldots,N_h^v}$ the point of $\mathcal{V}_h$ such that $q_1 = p$, $q_{N_h^v} = a_1$, and $|q_m - q_{m+1}| = 3^{-h}/2$, for $m = 1, \ldots, N_h^v - 1$. As $u^h \in H^1_{\Gamma_h}\left(\Omega_h, \mathbb{R}^3\right) \cap H^1\left(T_h, \mathbb{R}^3\right)$, we have, in particular, $u^h\left(q_{N_h^v}\right) = u^h\left(a_1\right) = 0$. Then, using some convexity argument,

$$\begin{aligned}
&\sum_{\alpha=1,2}\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(u_\alpha^h\left(p, x_3\right)\right)^2 dx_3 \\
&\quad = \sum_{\alpha=1,2}\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(\sum_{m=1}^{N_h^v-1}\left(u_\alpha^h\left(q_m, x_3\right) - u_\alpha^h\left(q_{m+1}, x_3\right)\right)\right)^2 dx_3 \\
&\quad \leq C \sum_{\substack{\alpha=1,2\\ \theta,q\in\mathcal{V}_h\\ |\theta-q|=3^{-h}/2}}\frac{1}{2r_h}\int_{-r_h/2}^{r_h/2}\left(u_\alpha^h\left(\theta, x_3\right) - u_\alpha^h\left(q, x_3\right)\right)^2 dx_3,
\end{aligned}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

C being a positive constant independent of h. This implies, by summing over all $p \in \mathcal{V}_h$, that

$$\frac{1}{N_h^v} \sum_{\substack{\alpha=1,2 \\ p \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} \frac{1}{2r_h} \int_{-r_h}^{r_h} \left(u_\alpha^h (p, x_3)\right)^2 dx_3$$
$$\leq C\rho^h \sum_{\substack{\alpha=1,2 \\ \theta, q \in \mathcal{V}_h \\ |p-q|=3^{-h}/2}} \frac{1}{2r_h} \int_{-r_h/2}^{r_h/2} \left(u_\alpha^h (\theta, x_3) - u_\alpha^h (q, x_3)\right)^2 dx_3,$$

from which we deduce, using (4.5), that

$$\sup_h \frac{1}{2r_h} \sum_{\alpha=1,2} \int_{-r_h}^{r_h} \int_\Sigma \left|u_\alpha^h\right|^2 dm_h dx_3 < +\infty. \tag{4.6}$$

3. Observing that

$$u_3^h\left(x', z\right) - u_3^h\left(x', 0\right) = \int_0^z \frac{\partial u_3^h}{\partial x_3} dx_3, \tag{4.7}$$

we deduce the following inequality

$$\left(u_3^h\left(x', z\right)\right)^2 \leq C\left(\left(u_3^h\left(x', 0\right)\right)^2 + r_h \int_{-r_h}^{r_h} \left(\frac{\partial u_3^h}{\partial x_3}\right)^2 dx_3\right). \tag{4.8}$$

Then, integrating (4.8) over T_h, we obtain the inequality

$$\int_{T_h} \left(u_3^h\right)^2 ds dx_3 \leq C r_h \left(\int_{S_h} \left(u_3^h\left(x', 0\right)\right)^2 ds + r_h \int_{T_h} \left(e_{33}\left(u^h\right)\right)^2 ds dx_3\right),$$

from which we deduce that

$$\frac{1}{|T_h|} \int_{T_h} \left(u_3^h\right)^2 ds dx_3 \leq C\left(\frac{3}{8}\right)^h \int_{S_h} \left(u_3^h\left(x', 0\right)\right)^2 ds$$
$$+ C r_h \left(\frac{3}{8}\right)^h \int_{T_h} \left(e_{33}\left(u^h\right)\right)^2 ds dx_3. \tag{4.9}$$

On the other hand, using (4.7), we have that

$$\varepsilon_h \int_{S_h} \left(u_3^h\left(x', 0\right)\right)^2 ds \leq C \int_{-\varepsilon_h}^{\varepsilon_h} \int_{S_h} \left(u_3^h(x)\right)^2 ds dx_3$$
$$+ C\varepsilon_h^2 \int_{\Omega_h} \left(e_{33}\left(u^h\right)\right)^2 ds dx_3. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\frac{1}{|T_h|} \int_{T_h} \left(\varepsilon_h u_3^h\right)^2 ds dx_3 \leq C\left(\frac{3}{8}\right)^h \varepsilon_h \int_{\Omega_h} \left(u_3^h(x)\right)^2 dx$$
$$+ C\varepsilon_h^3 \left(\frac{3}{8}\right)^h \int_{T_h} \left(e_{33}\left(u^h\right)\right)^2 ds dx_3$$
$$+ C\varepsilon_h^2 r_h \left(\frac{3}{8}\right)^h \int_{T_h} \left(e_{33}\left(u^h\right)\right)^2 ds dx_3, \tag{4.11}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

from which we deduce, using Lemma 4, that

$$\frac{1}{|T_h|}\int_{T_h}\left(\varepsilon_h u_3^h\right)^2 dsdx_3 \le C\left(\frac{3}{8}\right)^h. \tag{4.12}$$

On the other hand, using the same arguments as in (4.7)–(4.11), we deduce that

$$\begin{aligned}\frac{1}{N_h^v}\sum_{\substack{\alpha=1,2\\ p\in\mathcal{V}_h}}\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(\varepsilon_h u_3^h(p,x_3)\right)^2 dx_3 \le \frac{C}{|T_h|}\int_{T_h}\left(\varepsilon_h u_3^h\right)^2 dsdx_3\\ + Cr_h\varepsilon_h^2\left(\frac{3}{8}\right)^h\int_{T_h}\left(e_{33}\left(u^h\right)\right)^2 dsdx_3,\end{aligned} \tag{4.13}$$

then, combining with (4.12), we obtain that

$$\frac{1}{2r_h}\int_{-r_h}^{r_h}\int_{\Sigma}\left(\varepsilon_h u_3^h\right)^2 dm_h dx_3 \le C\left(\frac{3}{8}\right)^h. \tag{4.14}$$

4.2 Convergence of displacements

Let $\varphi\in C_c^\infty(\omega\times(-1,1))$. Then,

$$\begin{aligned}\lim_{h\to\infty}\frac{1}{\varepsilon_h}\int_{\Omega_h}\varphi\left(x',x_3/\varepsilon_h\right)dx &= \lim_{h\to\infty}\frac{1}{\varepsilon_h}\int_\omega\int_{-\varepsilon_h}^{\varepsilon_h}\varphi\left(x',x_3/\varepsilon_h\right)dx\\ &= \int_\omega\int_{-1}^{1}\varphi\left(x',z\right)dx'dz.\end{aligned}$$

This suggests the following notion of convergence with respect to dimension reduction:

Definition 6 *Let $u_h\in L^2(\Omega_h)$. We say that the sequence $(u_h)_h$ converges to $u\in L^2(\omega\times(-1,1))$ with respect to dimension reduction and write*

$$u_h\overset{dr}{\rightharpoonup}u\quad L^2(\omega\times(-1,1)),$$

if

$$\lim_{h\to\infty}\frac{1}{\varepsilon_h}\int_{\Omega_h}u_h(x)\,\varphi\left(x',x_3/\varepsilon_h\right)dx = \int_\omega\int_{-1}^{1}u\left(x',z\right)\varphi\left(x',z\right)dx'dz,$$

for every $\varphi\in C_c^\infty(\omega\times(-1,1))$.

We have the following compactness result:

Lemma 7 *Let $u_h\in L^2(\Omega_h)$ such that $\sup_h\left(\frac{1}{\varepsilon_h}\int_{\Omega_h}u_h^2(x)\,dx\right)<+\infty$. Then, there exists a subsequence of $(u_h)_h$, still denoted $(u_h)_h$, and a function $u\in L^2(\omega\times(-1,1))$, such that*

$$u_h\overset{dr}{\rightharpoonup}u\;\; L^2(\omega\times(-1,1)).$$

Proof. Let us consider the sequence of measures $(\varsigma_h)_h$ defined on $\omega\times(-1,1)$ by

$$\varsigma_h=\frac{u_h(x)}{\varepsilon_h}\mathbf{1}_{\Omega_h}(x)\,\delta_{x_3/\varepsilon_h}(dx_3)\,dx.$$

As

$$\langle\varsigma_h,\varphi\rangle=\frac{1}{\varepsilon_h}\int_{\Omega_h}u_h(x)\,\varphi\left(x',x_3/\varepsilon_h\right)dx,$$

for every $\varphi \in C_c(\omega \times (-1,1))$, $|\Omega_h| = 2\varepsilon_h |\omega|$, and

$$\sup_h \left(\frac{1}{\varepsilon_h} \int_{\Omega_h} u_h^2(x)\, dx \right) < +\infty,$$

we deduce, using the Cauchy-Schwarz inequality, that, for every $h \in \mathbb{N}$,

$$\begin{aligned} |\varsigma_h(\omega \times (-1,1))| &= \frac{1}{\varepsilon_h} \left| \int_{\Omega_h} u_h(x)\, dx \right| \\ &\leq C \left(\frac{1}{\varepsilon_h} \int_{\Omega_h} u_h^2(x)\, dx \right)^{1/2} \leq C, \end{aligned}$$

where C is a positive constant independent of h. The sequence $(\varsigma_h)_h$ is thus of bounded variation, hence weakly converges, up to some subsequence, to a measure ς. Moreover, for every $\varphi \in C_c^\infty(\omega \times (-1,1))$,

$$\begin{aligned} \textstyle\int_\omega \int_{-1}^1 \varphi(x)\, d\varsigma_h &= \frac{1}{\varepsilon_h} \textstyle\int_{\Omega_h} \varphi(x', x_3/\varepsilon_h)\, u_h(x)\, dx \\ &\leq \left(\frac{1}{\varepsilon_h} \textstyle\int_{\Omega_h} u_h^2(x)\, dx \right)^{1/2} \left(\frac{1}{\varepsilon_h} \textstyle\int_{\Omega_h} \varphi^2(x', x_3/\varepsilon_h)\, dx \right)^{1/2} \\ &\leq C \left(\frac{1}{\varepsilon_h} \textstyle\int_{\Omega_h} \varphi^2(x', x_3/\varepsilon_h)\, dx \right)^{1/2}, \end{aligned}$$

from which we deduce, by passing to the limit as h tends to ∞, that

$$\int_\omega \int_{-1}^1 \varphi(x', z)\, d\varsigma \leq C \|\varphi\|_{L^2(\omega \times (-1,1))}.$$

It follows, according to Riesz' representation theorem, that there exists $u \in L^2(\omega \times (-1,1))$ such that $\varsigma = u(x', z)\, dx'dz$. This means that, up to some subsequence,

$$u_h \overset{dr}{\rightharpoonup} u \quad L^2(\omega \times (-1,1)).$$

Proposition 8 *Let $u^h \in H(\Omega_h, \mathbb{R}^3)$ such that $\sup_h F_h(u^h) < +\infty$. Then, under the assumption (2.25), there exists a subsequence of $(u^h)_h$, still denoted as $(u^h)_h$, such that*

1. $u_\alpha^h \overset{dr}{\rightharpoonup} U_\alpha \; L^2(\omega \times (-1,1), \mathbb{R}^3)$*;* $\alpha = 1, 2$*,*
   $\varepsilon_h u_3^h \overset{dr}{\rightharpoonup} U_3 \; L^2(\omega \times (-1,1), \mathbb{R}^3)$*,*
2. $U_3 = u_3(x')$ *is independent of* $z \in (-1,1)$*,* $\int_{-1}^1 U_\alpha(x', z) = u_\alpha(x')$*;* $\alpha = 1, 2$*, with* $U_\alpha(x', z) = -z\frac{\partial u_3}{\partial x_\alpha}(x') + u_\alpha(x')$*,* $\overline{u} = (u_1, u_2) \in H_0^1(\omega, \mathbb{R}^2)$*, and* $u_3 \in H_0^2(\omega)$*.*

Proof. 1. The two convergences follow from Lemmas 4 and 7.

2. Since

$$\sup_h \frac{1}{\varepsilon_h} \int_{\Omega_h} e_{ij}(u^h)\, e_{ij}(u^h)\, dx < +\infty,$$

it follows from Lemma 7 that there exists $\chi_{ij} \in L^2(\omega \times (-1,1))$; $i, j = 1, 2, 3$, such that, up to some subsequence,

$$e_{ij}(u^h) \overset{dr}{\rightharpoonup} \chi_{ij} \quad L^2(\omega \times (-1,1))\,;\, i, j = 1, 2, 3. \tag{4.15}$$

Let $\varphi \in C_c^\infty(\omega \times (-1,1))$. Then, for $\alpha = 1, 2$, we have

$$\frac{1}{\varepsilon_h} \int_{\Omega_h} e_{\alpha 3}(u^h)\, \varphi(x', x_3/\varepsilon_h)\, dx = -\frac{1}{\varepsilon_h} \int_{\Omega_h} \begin{pmatrix} \dfrac{u_\alpha^h}{2\varepsilon_h} \dfrac{\partial \varphi}{\partial z}(x', x_3/\varepsilon_h) \\ + \dfrac{u_3^h}{2} \dfrac{\partial \varphi}{\partial x_\alpha}(x', x_3/\varepsilon_h) \end{pmatrix} dx \tag{4.16}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

and

$$\frac{1}{\varepsilon_h}\int_{\Omega_h} e_{33}\left(u^h\right)\varphi\left(x',x_3/\varepsilon_h\right)dx=-\frac{1}{\varepsilon_h}\int_{\Omega_h}\frac{u_3^h}{\varepsilon_h}\frac{\partial\varphi}{\partial z}\left(x',x_3/\varepsilon_h\right)dx. \tag{4.17}$$

Multiplying by ε_h in (4.16) and passing to the limit, taking into account (5.15), we obtain

$$\begin{aligned}\lim_{h\to\infty}\int_{\Omega_h} e_{\alpha 3}\left(u^h\right)\varphi\left(x',x_3/\varepsilon_h\right)dx &= \lim_{h\to\infty}-\frac{1}{2\varepsilon_h}\int_{\Omega_h} u_\alpha^h\frac{\partial\varphi}{\partial z}\left(x',x_3/\varepsilon_h\right)dx\\ &\quad+\lim_{h\to\infty}-\frac{1}{2\varepsilon_h}\int_{\Omega_h}\varepsilon_h u_3^h\frac{\partial\varphi}{\partial x_\alpha}\left(x',x_3/\varepsilon_h\right)dx\\ &= -\frac{1}{2}\int_\omega\int_{-1}^1\left(U_\alpha\frac{\partial\varphi}{\partial z}\right)(x',z)\,dx'dz\\ &\quad-\frac{1}{2}\int_\omega\int_{-1}^1\left(U_3\frac{\partial\varphi}{\partial x_\alpha}\right)(x',z)\,dx'dz\\ &= 0,\end{aligned}$$

which implies that

$$\int_\omega\int_{-1}^1\left(\frac{\partial U_\alpha}{\partial z}+\frac{\partial U_3}{\partial x_\alpha}\right)\varphi\left(x',z\right)dx'dz=0;\ \alpha=1,2, \tag{4.18}$$

for every $\varphi\in C_c^\infty\left(\omega\times(-1,1)\right)$. Multiplying by ε_h^2 in (4.17) and passing to the limit, taking into account (4.15), we obtain that

$$\begin{aligned}\lim_{h\to\infty}\varepsilon_h\int_{\Omega_h} e_{33}\left(u^h\right)\varphi\left(x',x_3/\varepsilon_h\right)dx &= \lim_{h\to\infty}-\frac{1}{\varepsilon_h}\int_{\Omega_h}\varepsilon_h u_3^h\frac{\partial\varphi}{\partial z}\left(x',x_3/\varepsilon_h\right)dx\\ &= -\int_\omega\int_{-1}^1\left(U_3\frac{\partial\varphi}{\partial z}\right)(x',z)\,dx'dz\\ &= 0.\end{aligned}$$

This yields

$$\int_\omega\int_{-1}^1\frac{\partial U_3}{\partial z}\varphi\left(x',z\right)dx'dz=0, \tag{4.19}$$

for every $\varphi\in C_c^\infty\left(\omega\times(-1,1)\right)$. It follows from (4.19) that $\dfrac{\partial U_3}{\partial z}=0$ in $\mathcal{D}'\left(\omega\times(-1,1)\right)$, hence, according to [27, Lemma 4.1], $U_3\left(x',z\right)\equiv u_3\left(x'\right)$. In view of (4.18)–(4.19), it follows from Schwarz Lemma that there exists $u_\alpha\in L^2(\omega)$; $\alpha=1,2$, such that

$$U_\alpha\left(x',z\right)=-z\frac{\partial u_3}{\partial x_\alpha}\left(x'\right)+u_\alpha\left(x'\right);\ \alpha=1,2. \tag{4.20}$$

On the other hand, according to Lemma 4, we have

$$\sup_h\sum_{\alpha,\beta}\frac{1}{\varepsilon_h}\int_{\Omega_h}\left(\frac{\partial u_\alpha^h}{\partial x_\beta}\right)^2dx<+\infty,$$

from which we deduce, taking into account Lemma 7, that, up to some subsequence,

$$\frac{\partial u_\alpha^h}{\partial x_\beta}\overset{dr}{\rightharpoonup}g_\alpha^\beta\in L^2\left(\omega\times(-1,1)\right),\ \alpha,\beta=1,2. \tag{4.21}$$

Let $\varphi \in C_c^\infty(\omega \times (-1,1))$. Then, using (4.21),

$$\begin{aligned}
\lim_{h\to\infty} \frac{1}{\varepsilon_h} \int_{\Omega_h} \frac{\partial u_\alpha^h}{\partial x_\beta} \varphi(x', x_3/\varepsilon_h)\, dx &= \int_{-1}^{1}\int_\omega g_\alpha^\beta \varphi(x', z)\, dx' dz \\
&= -\lim_{h\to\infty} \frac{1}{\varepsilon_h} \int_{\Omega_h} u_\alpha^h \frac{\partial \varphi}{\partial x_\beta}(x', x_3/\varepsilon_h)\, dx \\
&= -\int_{-1}^{1}\int_\omega U_\alpha \frac{\partial \varphi}{\partial x_\beta}(x', z)\, dx' dz \\
&= \int_{-1}^{1}\int_\omega \frac{\partial U_\alpha}{\partial x_\beta} \varphi(x', z)\, dx' dz,
\end{aligned} \tag{4.22}$$

which implies that $g_\alpha^\beta = \dfrac{\partial U_\alpha}{\partial x_\beta}$. Moreover, using (4.20), we have

$$\begin{aligned}
&\int_{-1}^{1}\int_\omega \left(\frac{\partial U_\alpha}{\partial x_\beta}(x', z)\right)^2 dx' dz \\
&= \left(\int_\omega \left(\frac{\partial^2 u_3}{\partial x_\beta \partial x_\alpha}(x')\right)^2 dx'\right) \int_{-1}^{1} z^2 dz \\
&+ 2\int_\omega \left(\frac{\partial u_\alpha}{\partial x_\beta}(x')\right)^2 dx' + 2\left(\int_\omega \left(u_\alpha \frac{\partial^2 u_3}{\partial x_\beta \partial x_\alpha}\right)(x')\, dx'\right) \int_{-1}^{1} z dz \\
&= \frac{2}{3}\int_\omega \left(\frac{\partial^2 u_3}{\partial x_\beta \partial x_\alpha}(x')\right)^2 dx' + 2\int_\omega \left(\frac{\partial u_\alpha}{\partial x_\beta}(x')\right)^2 dx',
\end{aligned}$$

from which we deduce that $\dfrac{\partial u_\alpha}{\partial x_\beta} \in L^2(\omega)$ and $\dfrac{\partial^2 u_3}{\partial x_\beta \partial x_\alpha} \in L^2(\omega)$. Taking $\varphi \in C^\infty(\overline{\omega})$, we deduce from the above computations that

$$\begin{aligned}
\lim_{h\to\infty} \frac{1}{\varepsilon_h} \int_{\Omega_h} \frac{\partial u_\alpha^h}{\partial x_\beta} \varphi dx &= 2\int_{-1}^{1}\int_\omega \frac{\partial U_\alpha}{\partial x_\beta} \varphi dx' dz \\
&= -2\int_{-1}^{1}\int_\omega U_\alpha \frac{\partial \varphi}{\partial x_\beta} dx' dz \\
&\quad + 2\int_{-1}^{1}\int_{\partial\omega} U_\alpha \nu_\beta \varphi ds dz,
\end{aligned} \tag{4.23}$$

where ν is the outward unit normal to $\partial\omega$. Moreover, as $u_\alpha^h = 0$ on Γ_h; $\alpha = 1, 2$,

$$\begin{aligned}
\lim_{h\to\infty} \frac{1}{\varepsilon_h} \int_{\Omega_h} \frac{\partial u_\alpha^h}{\partial x_\beta} \varphi dx &= \lim_{h\to\infty} -\frac{1}{\varepsilon_h} \int_{\Omega_h} u_\alpha^h \frac{\partial \varphi}{\partial x_\beta} dx \\
&= -2\int_{-1}^{1}\int_\omega U_\alpha \frac{\partial \varphi}{\partial x_\beta} dx' dz.
\end{aligned} \tag{4.24}$$

Combining (4.23) and (4.24), we conclude that $\int_{\partial\omega} U_\alpha \nu_\beta \varphi ds = 0$; hence, $U_\alpha = 0$ on $\partial\omega \times (-1, 1)$ and $u_\alpha = 0$ on $\partial\omega$; $\alpha = 1, 2$. Taking into account (4.21), it follows that $(u_1, u_2) \in H_0^1(\omega, \mathbb{R}^2)$. Similarly, as $u_3^h = 0$ on Γ_h, we deduce, according to Lemma 4, that, for $\varphi \in C^\infty(\overline{\omega})$ and $\alpha = 1, 2$,

$$\begin{aligned}
\lim_{h\to\infty} \frac{1}{\varepsilon_h} \int_{\Omega_h} \frac{\partial\left(\varepsilon_h u_3^h\right)}{\partial x_\alpha} \varphi dx &= 2\int_\omega \frac{\partial u_3}{\partial x_\alpha} \varphi dx' \\
&= \lim_{h\to\infty} -\frac{1}{\varepsilon_h} \int_{\Omega_h} u_3^h \frac{\partial \varphi}{\partial x_\alpha} dx \\
&= -2\int_\omega u_3 \frac{\partial \varphi}{\partial x_\alpha} dx'.
\end{aligned} \tag{4.25}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

This implies that $\int_{\partial\omega} u_3 \nu_\alpha \varphi ds = 0$; hence, $u_3 = 0$ on $\partial\omega$. On the other hand, using (4.20), we deduce that $\dfrac{\partial u_3}{\partial x_\alpha} = 0$ on $\partial\omega$; $\alpha = 1, 2$; thus, $u_3 \in H_0^2(\omega)$.

Let $\mathcal{M}\left(\mathbb{R}^3\right)$ be the space of Radon measures on $\mathbb{R}^3$. We have the following result:

Lemma 9 *Let $v_h \in L^2(\Omega) \cap L^2(T_h)$, such that*

$$\sup_h \frac{1}{2r_h} \int_{-r_h}^{r_h} \int_\Sigma v_h^2 dm_h dx_3 < +\infty,$$

where m_h is the measure defined in (3.11). Then, there exists a subsequence of $(v_h)_h$, still denoted $(v_h)_h$, such that

$$v_h \frac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v \mathbf{1}_\Sigma(s) \frac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(\Sigma)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),$$

with $v(s, 0) \in L^2_{\mathcal{H}^d}(\Sigma)$.

Proof. According to Lemma 1, the sequence $(m_h)_h$ weakly converges in $C(\Sigma)^*$ to the measure $m = \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)}$. One can then easily check that, for every $\varphi \in C_0\left(\mathbb{R}^3\right)$,

$$\lim_{h\to\infty} \int_{\mathbb{R}^3} \varphi(x) \frac{\mathbf{1}_{T_h}(x)}{2r_h} dm_h dx_3 = \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma \varphi(s, 0)\, d\mathcal{H}^d(s). \tag{4.26}$$

Since $\sup_h \dfrac{1}{2r_h} \int_{-r_h}^{r_h} \int_\Sigma |v_h|^2 dm_h dx_3 < +\infty$, the sequence $\left(v_h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3\right)_h$ is uniformly bounded in variation, hence $*$-weakly relatively compact. Possibly passing to a subsequence, we can suppose that the sequence $\left(v_h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3\right)_h$ converges to some χ. Let $\varphi \in C_0\left(\mathbb{R}^3\right)$, we have, using Fenchel's inequality

$$\begin{aligned}
&\liminf_{h\to\infty} \frac{1}{2r_h} \int_{-r_h}^{r_h} \int_\Sigma |v_h|^2 dm_h dx_3 \\
&\geq \liminf_{h\to\infty} \left(\int_{\mathbb{R}^3} v_h \varphi \frac{\mathbf{1}_{T_h}(x)}{2r_h} dm_h dx_3 - \frac{1}{2} \int_{\mathbb{R}^3} \varphi^2 \frac{\mathbf{1}_{T_h}(x)}{2r_h} dm_h dx_3 \right) \\
&\geq \langle \chi, \varphi \rangle - \frac{1}{2\mathcal{H}^d(\Sigma)} \int_\Sigma \varphi^2(s, 0)\, d\mathcal{H}^d(s).
\end{aligned}$$

As the left hand side of this inequality is bounded, we deduce that

$$\sup \left\{ \langle \chi, \varphi \rangle ; \varphi \in C_0\left(\mathbb{R}^3\right), \frac{1}{\mathcal{H}^d(\Sigma)} \int_\Sigma \varphi^2(s, 0)\, d\mathcal{H}^d(s) \leq 1 \right\} < +\infty,$$

from which we deduce, according to Riesz' representation Theorem, that there exists v, such that $v \in L^2_{\mathcal{H}^d}(\Sigma)$ and $\chi = v \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(\Sigma)}$.

Proposition 10 *Let $\left(u^h\right)_h$; $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$, be a sequence such that*

$$\sup_h \int_{T_h} \sigma_{ij}^{*h}\left(u^h\right) e_{ij}\left(u^h\right) ds dx_3 < +\infty.$$

Then, under the assumption (2.20), there exists a subsequence, still denoted $\left(u^h\right)_h$, *such that*

1. $u_\alpha^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v_\alpha \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)}$, *with* $v_\alpha(s) \in L^2_{\mathcal{H}^d}(\Sigma)$; $\alpha = 1, 2$,
2. $\varepsilon_h u_3^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} 0.$

Proof. According to Lemma $5_{2,3}$, we have, up to some subsequence,

$$u_\alpha^h \frac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v_\alpha \mathbf{1}_\Sigma(s) \frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)}; \alpha = 1, 2,$$
$$\varepsilon_h u_3^h \frac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} 0,$$

with $v_\alpha \in L^2_{\mathcal{H}^d}(\Sigma)$; $\alpha = 1, 2$.

5 The main result

In this section, we state the main result of this work. According to Propositions 8 and 10, we introduce the following topology τ:

Definition 11 *We say that a sequence* $\left(u^h\right)_h$; $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$, τ*-converges to* (u, v); $v = (v_1, v_2)$, *if u is independent of* $z \in (-1, 1)$, $u_\alpha(x') = \int_{-1}^{1} U_\alpha(x', z)\, dz$; $\alpha = 1, 2$, *and*

$$\begin{cases} u_\alpha^h \overset{dr}{\rightharpoonup} U_\alpha \quad L^2\left(\omega\times(-1,1), \mathbb{R}^3\right); \alpha = 1, 2, \\ \varepsilon_h u_3^h \overset{dr}{\rightharpoonup} u_3 \quad L^2\left(\omega\times(-1,1), \mathbb{R}^3\right), \\ u_\alpha^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v_\alpha \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right); \alpha = 1, 2, \\ \varepsilon_h u_3^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} 0 \text{ in } \mathcal{M}\left(\mathbb{R}^3\right). \end{cases}$$

We state our main result of the Γ-convergence in the topology τ of the sequence of functionals F_h to the functional F_∞ defined in (1.1) as follows:

Theorem 12 *If* $\gamma \in (0, +\infty)$ *then, under the assumptions (2.20) and (2.25),*

1. *(*$\lim - \sup$ *inequality) For every* $(u, v) \in H\left(\omega, \mathbb{R}^3\right)$, *there exists a sequence* $\left(u^h\right)_h$; $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$, *such that* $\left(u^h\right)_h$ τ*-converges to* (u, v) *and*
$$\limsup_{h\to\infty} F_h\left(u^h\right) \le F_\infty(u, v).$$
2. *(*$\liminf$ *inequality) For every* $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$ *such that* $\left(u^h\right)_h$ τ*-converges to* (u, v), *we have* $(u, v) \in H\left(\omega, \mathbb{R}^3\right)$ *and*
$$\liminf_{h\to\infty} F_h\left(u^h\right) \ge F_\infty(u, v).$$

Let us write the associated homogenised problem obtained at the limit as $h \longrightarrow \infty$.

Corollary 13 *Problem (2.28) admits a unique solution* u^h *which, under the hypotheses of Theorem 12,* τ*-converges to* $(u, v) \in H\left(\omega, \mathbb{R}^3\right)$ *such that*

$$\lim_{h\to\infty} F_h\left(u^h\right) = F_\infty(u, v)$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

and (u, v) *is the solution of the problem*

$$\begin{cases} -\eta_{\alpha\beta,\beta}(\overline{u}) &= \widetilde{f}_\alpha;\ \alpha = 1, 2, & \textit{in } \omega, \\ \dfrac{\partial^2 \varpi_{\alpha\beta}(u_3)}{\partial x_\alpha \partial x_\beta} &= \widetilde{f}_3 & \textit{in } \omega, \\ -\mu^* \Delta_{\alpha,\Sigma}(v) &= \mu\gamma A_{\alpha\alpha}(s)(u_\alpha - v_\alpha);\ \alpha = 1, 2, & \textit{in } \Sigma, \\ \left[\eta_{\alpha\beta}(\overline{u})\nu_\beta\right]_\Sigma &= \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} A_{\alpha\alpha}(s)(u_\alpha - v_\alpha)\mathcal{H}^d & \textit{on } \Sigma, \\ \left[\varpi_{\alpha\beta}(u_3)\nu_\beta\right]_\Sigma &= 0 & \textit{on } \Sigma, \\ \left[\dfrac{\partial \varpi_{\alpha\beta}(u_3)}{\partial x_\alpha}\nu_\beta\right]_\Sigma &= \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(1+\kappa)(\ln 2)^2} u_3 & \textit{on } \Sigma, \\ u &= 0 & \textit{on } \partial\omega, \\ v &= 0 & \textit{on } \Sigma \cap \partial\omega, \end{cases}$$

where ν *is the unit normal on* Σ *and* $\widetilde{f}_i(x_1, x_2) = \int_{-1}^{1} f_i(x_1, x_2, x_3)\, dx_3$*;* $i = 1, 2, 3$.

Proof. One can easily check that problem (2.28) has a unique solution $u^h \in H_0^1(\Omega, \mathbb{R}^3) \cap H^1(T_h, \mathbb{R}^3)$. Now, observing that

$$F_h(u^h) - 2\int_{\Omega_h} f.u^h dx \leq F_h(0) = 0,$$

we deduce, using the fact that $\lim\limits_{h\to\infty} c_h = +\infty$, and the inequalities (4.1) and (4.2) of the proof of Lemma 4, that

$$\begin{aligned} &\sum_{\alpha,\beta=1,2} \frac{1}{\varepsilon_h} \int_{\Omega_h} (u_\alpha^h)^2 dx + \frac{1}{\varepsilon_h} \int_{\Omega_h} (\varepsilon_h u_3^h)^2 dx \\ &\leq F_h(u_h) \leq 2\int_{\Omega_h} f.u_h dx \\ &\leq 2\left(\frac{1}{\varepsilon_h}\int_{\Omega_h} |f|^2 dx\right)^{1/2} \left(\frac{1}{\varepsilon_h}\int_{\Omega_h}^2 \left|\varepsilon_h u^h\right|^2 dx\right)^{1/2} \\ &\leq C\left(\frac{1}{\varepsilon_h}\int_{\Omega_h}^2 \left|\varepsilon_h u^h\right|^2 dx\right)^{1/2}, \end{aligned}$$

from which we deduce, in particular, that $\sup_h F_h(u^h) < +\infty$. Then, in view of Propositions 8, 10, and Theorem 12, we deduce, according to [11, Theorem 7.8]), that the sequence $(u^h)_h$ τ-converges to the solution $(u, v) \in H(\omega, \mathbb{R}^3)$ of the problem

$$\min_{(\xi,\zeta)\in H(\omega,\mathbb{R}^3)} \left\{ \begin{array}{l} \int_\omega \eta_{\alpha\beta}(\xi)\, e_{\alpha\beta}(\xi)\, dx' + \mu^* \int_\Sigma d\mathcal{L}_\Sigma(\zeta) \\ + \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A_{\alpha\alpha}(s)(\xi_\alpha - \zeta_\alpha)^2 d\mathcal{H}^d(s) \\ + \dfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A_{33}(s)\, \xi_3^2 d\mathcal{H}^d(s) \\ + \int_\omega \varpi_{\alpha\beta}(\xi_3) \dfrac{\partial^2 \xi_3}{\partial x_\alpha \partial x_\beta} dx' \\ -2\int_\omega \widetilde{f}_i \xi_i dx', \end{array} \right\} \tag{5.1}$$

and

$$\lim_{h\to\infty} F_h(u^h) = F_\infty(u, v).$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

The trace of an element of $H^1(\omega, \mathbb{R}^2)$ on $\omega \cap \Sigma$ exists for $\mathcal{H}^d$-almost-every $x \in \omega \cap \Sigma$ and belongs to the Besov space $B^2_{d/2}(\Sigma, \mathbb{R}^2)$ defined by

$$B^2_{d/2}(\Sigma, \mathbb{R}^2) = \left\{ \begin{array}{l} v : \Sigma \longrightarrow \mathbb{R}^2;\ \int_\Sigma |v(x)|^2 \, d\mathcal{H}^d(x) \\ + \int_\Sigma \int_{\substack{\Sigma \\ |x-y|<1}} \dfrac{|v(x) - v(y)|^2}{|x-y|^{2d}} d\mathcal{H}^d(x)\, d\mathcal{H}^d(y) < +\infty \end{array} \right\}, \tag{5.2}$$

see [24, Theorem 6]. More details on Besov spaces $B^{p,q}_\alpha(K)$, $\alpha > 0$, $1 \le p, q \le \infty$, defined for a large class of closed subsets K of $\mathbb{R}^N$ including fractal subsets, can be found in [22, Chapters 5 and 6]. In our case $K = \Sigma$, $\alpha = d/2$, and $p = q = 2$. The trace Theorem [24, Theorem 6] can be applied to a more geometrically complex domain which, supplied with a positive Borel measure, is a d-set preserving Markov's inequality [24, pp. 193–195]. Typical examples of d-sets are self-similar fractals (see for instance [24, pp. 194]). According to [22, Theorem 3, p. 39], if $K \subset R^N$ is a d-set with $d > N - 1$, then K preserves Markov's inequality. In particular, the Sierpinski carpet Σ is a d-set preserving Markov's inequality where d is the fractal dimension of Σ given in (1.4). Then, using Lemma 3, we obtain from (5.1) that $v \in \mathcal{D}_{\Delta_\Sigma}$ and for every $(\xi, \zeta) \in H(\omega, \mathbb{R}^3)$,

$$\begin{aligned} & \int_\omega \left(-\eta_{\alpha\beta,\beta}(\overline{u}) - \widetilde{f}_\alpha\right) \xi_\alpha dx' - \frac{\mu^*}{\mathcal{H}^d(\Sigma)} \int_\Sigma \left(\Delta_{\alpha,\Sigma} v\right) \zeta_\alpha d\mathcal{H}^d \\ & + \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)} \sum_{\alpha=1,2} \int_\Sigma A_{\alpha\alpha}(s)(u_\alpha - v_\alpha)(\xi_\alpha - \zeta_\alpha)\, d\mathcal{H}^d(s) \\ & + \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)} \int_\Sigma A_{33}(s)\, u_3 \xi_3 d\mathcal{H}^d(s) \\ & + \int_\omega \left(\frac{\partial^2 \varpi_{\alpha\beta}(u_3)}{\partial x_\alpha \partial x_\beta} - \widetilde{f}_3 \right) \xi_3 dx' \\ & + \left\langle \left[\varpi_{\alpha\beta}(u_3)\, \nu_\beta\right]_\Sigma, \frac{\partial \xi_3}{\partial x_\alpha} \right\rangle_{B^2_{-d/2}(\Sigma,\mathbb{R}^3), B^2_{d/2}(\Sigma,\mathbb{R}^3)} \\ & - \left\langle \left[\frac{\partial \varpi_{\alpha\beta}(u_3)}{\partial x_\alpha} \nu_\beta \right]_\Sigma, \xi_3 \right\rangle_{B^2_{-d/2}(\Sigma,\mathbb{R}^3), B^2_{d/2}(\Sigma,\mathbb{R}^3)} \\ & + \left\langle \left[\eta_{\alpha\beta}(\overline{u})\, \nu_\beta\right]_\Sigma, \xi_\alpha \right\rangle_{B^2_{-d/2}(\Sigma,\mathbb{R}^3), B^2_{d/2}(\Sigma,\mathbb{R}^3)} = 0, \end{aligned} \tag{5.3}$$

where $B^2_{-d/2}(\Sigma, \mathbb{R}^3)$ is the dual space of $B^2_{d/2}(\Sigma, \mathbb{R}^3)$ (see [23, p. 291]).

6 Proof of the main result

This section is devoted to the proof of the main results. We first study a local problem which is related to boundary layers due to the local interactions between the constituent materials. The solution of this local problem is crucial in constructing appropriate test functions in order to pass to the limit in the original problem.

6.1 Local problems

We consider here some local problems associated with boundary layers in the vicinity of the ribbons. We denote w^m; $m = 1, 2$, the solution of the following boundary value problem:

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma(w^m)(y) = 0 & \forall y \in \mathbb{R}^{2+} \\ w^m(y_1, 0) = (\delta_{1m}, \delta_{2m}) & \forall y_1 \in]-1, 1[\,, \\ \sigma_{i2}(w^m)(y) = 0 & \forall y \in (\mathbb{R} \setminus]-1, 1[) \times \{0\}\,, \\ w^m_m(y) = -\dfrac{\ln|y|}{\ln 2} & \text{when } |y| \to \infty,\ y_2 > 0, \\ \left|w^m_p\right|(y) \le C & \text{when } \begin{cases} p = 2 \text{ if } m = 1, \\ p = 1 \text{ if } m = 2, \end{cases} \end{array} \right. \tag{6.1}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

where $\sigma_{ij}(w^m) = \lambda e_{kk}(w^m)\,\delta_{ij} + 2\mu e_{ij}(w^m)$; $i, j = 1, 2$ and

$$\mathbb{R}^{2+} = \left\{y = (y_1, y_2) \in \mathbb{R}^2;\, y_2 > 0\right\}.$$

The displacement w^m; $m = 1, 2$, which belongs to the space $H^1_{loc}\left(\mathbb{R}^{2+}, \mathbb{R}^2\right)$, is given (see for instance [12, 14, 29]) by

$$\begin{cases} w_1^1(y) = -\dfrac{1+\kappa}{4\pi\mu} \int_{-1}^{1} \theta(t) \ln\left(\sqrt{(y_1-t)^2 + y_2^2}\right) dt \\ \qquad + \dfrac{1}{4\pi\mu} \int_{-1}^{1} \theta(t) \dfrac{2y_2^2}{(y_1-t)^2 + y_2^2} dt, \\ w_2^1(y) = -\dfrac{(1-\kappa)}{4\pi\mu} \int_{-1}^{1} \theta(t) \arctan\left(\dfrac{y_2}{y_1 - t}\right) dt \\ \qquad + \dfrac{1}{4\pi\mu} \int_{-1}^{1} \theta(t) \dfrac{2y_2(y_1-t)}{(y_1-t)^2 + y_2^2} dt \end{cases} \tag{6.2}$$

and

$$\begin{cases} w_1^2(y) = \dfrac{(1-\kappa)}{4\pi\mu} \int_{-1}^{1} \theta(t) \arctan\left(\dfrac{y_2}{y_1 - t}\right) dt \\ \qquad + \dfrac{1}{4\pi\mu} \int_{-1}^{1} \theta(t) \dfrac{2y_2(y_1-t)}{(y_1-t)^2 + y_2^2} dt, \\ w_2^2(y) = -\dfrac{(1+\kappa)}{4\pi\mu} \int_{-1}^{1} \theta(t) \ln\left(\sqrt{(y_1-t)^2 + y_2^2}\right) dt \\ \qquad - \dfrac{1}{4\pi\mu} \int_{-1}^{1} \theta(t) \dfrac{2y_2^2}{(y_1-t)^2 + y_2^2}, \end{cases} \tag{6.3}$$

where

$$\theta(t) = \begin{cases} \dfrac{4\mu}{(1+\kappa)\ln 2} \dfrac{1}{\sqrt{1-t^2}} & \text{if } t \in \left]-1, 1\right[, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

One can check that $w^m(y)$; $m = 1, 2$, is also the solution of problem (6.1) posed in the half-plane $\mathbb{R}^{2-}$:

$$\mathbb{R}^{2-} = \left\{y = (y_1, y_2) \in \mathbb{R}^2;\, y_2 < 0\right\}.$$

We introduce the following scalar problem:

$$\begin{cases} \Delta w(y) = 0 & \forall y \in \mathbb{R}^{2+}, \\ w(y_1, 0) = 1 & \forall y_1 \in \left]-1, 1\right[, \\ \dfrac{\partial w}{\partial y_2}(y_1, 0) = 0 & \forall y_1 \in \mathbb{R} \setminus \left]-1, 1\right[, \\ w(y) = -\dfrac{\ln|y|}{\ln 2} \text{ as } |y| \to \infty,\, y_2 > 0. \end{cases} \tag{6.5}$$

The solution of (6.5) is given by

$$w(y) = -\frac{1}{\pi \ln 2} \int_{-1}^{1} \frac{\ln\left(\sqrt{(y_1-t)^2 + y_2^2}\right)}{\sqrt{1-t^2}} dt. \tag{6.6}$$

Observe that $w(y)$ is also the solution of problem (6.5) posed in the half-plane $\mathbb{R}^{2-}$. We now state the following preliminary result in this subsection:

Proposition 14 *([12, Proposition 7]). One has*

1. $\lim_{R\to+\infty} \dfrac{1}{\ln R} \int_{B(0,R)\cap\mathbb{R}^{2\pm}} \sigma_{ij}(w^m)\, e_{ij}\left(w^l\right) dy = \delta_{ml} \dfrac{2\mu\pi}{(1+\kappa)(\ln 2)^2}$,

2. $\lim_{R\to+\infty}\frac{1}{\ln R}\int_{B(0,R)\cap\mathbb{R}^{2\pm}}|\nabla w|^2\,dy=\frac{\pi}{(\ln 2)^2}$, *where* $D(0,R)$ *is a disk of radius* R *centred at the origin.*

We define the rotation matrix $\mathcal{R}\left(x_h^k\right)$; $x_h^k=\left(x_{1h}^k,x_{2h}^k\right)$ being the centre of S_h^k; $k\in I_h$, by

$$\mathcal{R}\left(x_h^k\right)=\begin{cases} Id_{\mathbb{R}^3} & \text{if } n^k=\pm e_2,\\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} & \text{if } n^k=\pm e_1,\end{cases} \tag{6.7}$$

where $Id_{\mathbb{R}^3}$ is the 3×3 identity matrix and n^k is the unit normal to the line segment S_h^k, in the plane xOy. Let φ_h^k; $k\in I_h$, be the truncation function defined on $\mathbb{R}^2$ by

$$\varphi_h^k(x)=\begin{cases}\dfrac{4\left(3^{-2h}-4R_{k,h}^2(x)\right)}{3^{-2h+1}} & \text{if } 3^{-h}/4\le R_h^k(x)\le 3^{-h}/2,\\ 1 & \text{if } R_h^k(x)\le 3^{-h}/4,\\ 0 & \text{if } R_h^k(x)\ge 3^{-h}/2,\end{cases} \tag{6.8}$$

where $R_h^k(x)=\sqrt{\left(\left(x-x_h^k\right).n^k\right)^2+x_3^2}$. We define, for $k\in I_h$,

$$D_h^k(s_h)=\left\{\left(\left(x-x_h^k\right).n^k,x_3\right)\in\mathbb{R}^2;\ R_h^k(x)<3^{-h}/2,\ \forall x\in\mathbb{R}^3\right\} \tag{6.9}$$

and the cylinder

$$Z_h^k=\mathcal{R}\left(x_h^k\right)S_h^k\times D_h^k(s_h)\,;\ k\in I_h. \tag{6.10}$$

We then set

$$Z_h=\bigcup_{k\in I_h}Z_h^k. \tag{6.11}$$

Let $k\in I_h$. We set $\Phi_h^k(x)=\varphi_h^k(x)\,\mathcal{R}\left(x_h^k\right)$ and define the function $w_h^{mk}(x)$; $m=1,2,3$, by

$$w_h^{1k}(x)=\Phi_h^k(x)\left(e_1-\frac{1}{\ln r_h}\begin{pmatrix}1-w\left(\dfrac{x_3}{r_h},\dfrac{\left(x-x_h^k\right).n^k}{r_h}\right)\\0\\0\end{pmatrix}\right), \tag{6.12}$$

$$w_h^{2k}(x)=\Phi_h^k(x)\left(e_2-\frac{1}{\ln r_h}\begin{pmatrix}0\\1-w_1^1\left(\dfrac{x_3}{r_h},\dfrac{\left(x-x_h^k\right).n^k}{r_h}\right)\\w_2^1\left(\dfrac{x_3}{r_h},\dfrac{\left(x-x_h^k\right).n^k}{r_h}\right)\end{pmatrix}\right) \tag{6.13}$$

and

$$w_h^{3k}(x)=\Phi_h^k(x)\left(e_3-\frac{\varepsilon_h}{\ln r_h}\begin{pmatrix}0\\w_1^2\left(\dfrac{x_3}{r_h},\dfrac{\left(x-x_h^k\right).\nu^k}{r_h}\right)\\w_2^2\left(\dfrac{x_3}{r_h},\dfrac{\left(x-x_h^k\right).\nu^k}{r_h}\right)\end{pmatrix}\right). \tag{6.14}$$

where $e_i=(\delta_{1i},\delta_{2i},\delta_{3i})$. We define now the local perturbations w_h^m; $m=1,2,3$, through

$$w_h^m(x)=w_h^{mk}(x)\,,\ \forall k\in I_h,\ \forall x\in\omega\times(-1,1)\,. \tag{6.15}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

Lemma 15 *If $\gamma \in (0,+\infty)$ then, under the assumption (2.25), for every $\Psi \in C^1\left(\overline{\omega} \times [-1,1], \mathbb{R}^3\right)$ and $\Psi^h = (\Psi_1, \Psi_2, \Psi_3/\varepsilon_h)$, we have*

$$\lim_{h\to\infty}\left(\int_{Z_h} \sigma^h_{ij}\left(w^l_h \Psi^h_m\right) e_{ij}\left(w^{l'}_h \Psi^h_l\right) dx\right) = \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma A(s)\,\Psi(s).\Psi(s)\, d\mathcal{H}^d(s),$$

where $A(s)$ is the matrix defined in (1.6).

Proof. Let us introduce the local variables $y = (y_1, y_2)$ with

$$\begin{cases} y_1 = x_3/r_h, \\ y_2 = \left(x - x^k_h\right).n^k/r_h. \end{cases} \tag{6.16}$$

Then, using the smoothness of Ψ, the assumption (2.25), and Proposition 14, we obtain that

$$\begin{aligned}
&\lim_{h\to\infty}\int_{Z_h}\sigma^h_{ij}\left(w^l_h\Psi^h_m\right)e_{ij}\left(w^{l'}_h\Psi^h_l\right)dx\\
&=\lim_{h\to\infty}\sum_{k\in I_h}\int_{Z^k_h}\sigma^h_{ij}\left(w^{lk}_h\right)e_{ij}\left(w^{l'k}_h\right)\Psi^h_m\Psi^h_l dx\\
&=\lim_{h\to\infty}\frac{b}{\frac{3^h}{8^h}\varepsilon_h\ln^2 r_h}\int_{D\left(0,\frac{3^{-h}}{2r_h}\right)\setminus D(0,1)}\sigma_{ij}\left(w^l\right)e_{ij}\left(w^{l'}\right)dy\\
&\times\left(\sum_{k=1}^{N^v_h}\frac{a}{bN^v_h}\left(\mathcal{R}\left(x^k_h\right)\Psi\right)_m\left(\mathcal{R}\left(x^k_h\right)\Psi\right)_l\left(x^k_{1h},x^k_{2h}\right)\right)\\
&=\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma\left(D_{iag}\mathcal{R}(s)\Psi(s)\right)_m\left(\mathcal{R}(s)\Psi(s)\right)_l d\mathcal{H}^d(s)\\
&=\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma\mathcal{R}^t(s)D_{iag}\mathcal{R}(s)\Psi(s).\Psi(s)\,d\mathcal{H}^d(s),
\end{aligned}\tag{6.17}$$

where $D\left(0,\frac{3^{-h}}{2r_h}\right)$ is the disk of radius $\frac{3^{-h}}{2r_h}$ centred at the origin, $D(0,1)$ is the disk of radius 1 centred at the origin, $D_{iag} = \mathrm{Diag}\left(1, \frac{2}{(1+\kappa)}, \frac{2}{(1+\kappa)}\right)$, and $\mathcal{R}(s)$ is the rotation matrix defined by $\mathcal{R}(s) = Id_{\mathbb{R}^3}$ on the face of Σ which is perpendicular to the vector e_2 and by $\mathcal{R}(s) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the face of Σ which is perpendicular to the vector e_1. Observing that in (6.17)

$$\mathcal{R}^t(s)\,D_{iag}\mathcal{R}(s) = \mathcal{R}(s)\,D_{iag}\mathcal{R}(s) = A(s),$$

we obtain the desired result.

6.2 Proof of Theorem 12

The proof of Theorem 12 is given in two steps.

6.2.1 Step 1: Lim-sup inequality

Here we prove the lim-sup property of the Γ-convergence stated in Theorem 12. We first construct a test function on each line segment S^k_h; $k \in I_h$, with extremities $p^k_h = \left(p^k_{h1}, p^k_{h2}\right)$, $q^k_h = \left(q^k_{h1}, q^k_{h2}\right)$, and centre $x^k_h = \left(x^k_{h1}, x^k_{h2}\right)$. Let $(v_1, v_2, v_3) \in C^2_c\left(\omega, \mathbb{R}^3\right)$. We consider the sequence $\left(v^{h,k}\right)_h$ of test functions defined,

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

for every $x = (x_1, x_2) \in S_h^k$, by

$$\begin{aligned} v_1^{h,k}(x') &= v_1\left(x_{1h}^k, x_{2h}^k\right) + 3^h \vartheta_h^{1,k}(x')\left|v_1\left(p_h^k\right) - v_1\left(q_h^k\right)\right|, \\ v_2^{h,k}(x') &= v_2\left(x_{1h}^k, x_{2h}^k\right) + 3^h \vartheta_h^{2,k}(x')\left|v_2\left(p_h^k\right) - v_2\left(q_h^k\right)\right|, \\ v_3^{h,k}(x') &= v_3\left(x_{1h}^k, x_{2h}^k\right), \end{aligned} \tag{6.18}$$

where $\vartheta_h^{i,k}(x')$; $i = 1, 2, 3$, is defined by

$$\begin{cases} \vartheta_h^{1,k}(x') = \sqrt{\dfrac{\mu_h^*}{2}} \dfrac{\left(x_1 - p_{h,1}^k\right) + \left(x_1 - q_{h,1}^k\right)}{\sqrt{\lambda_h^* + 2\mu_h^*}} \\ \qquad - \dfrac{\left(x_2 - p_{h,2}^k\right) + \left(x_2 - q_{h,2}^k\right)}{\sqrt{2}}, \\ \vartheta_h^{2,k}(x') = \sqrt{\dfrac{\mu_h^*}{2}} \dfrac{\left(x_2 - p_{h,2}^k\right) + \left(x_2 - q_{h,2}^k\right)}{2\sqrt{\lambda_h^* + 2\mu_h^*}} \\ \qquad - \dfrac{\left(x_1 - p_{h,1}^k\right) + \left(x_1 - q_{h,1}^k\right)}{\sqrt{2}}. \end{cases} \tag{6.19}$$

Let us now introduce the intervals $J_h^{p_h^k}$ and $J_h^{q_h^k}$ which are centred at the points p_h^k and $q_{h,}^k$ respectively, such that

$$S_h^k \cap J_h^{p_h^k} = \left[p_h^k, p_h^k + \mathbf{l}_h\right), S_h^k \cap J_h^{q_h^k} = \left(q_h^k - \mathbf{l}_h, q_h^k\right], \tag{6.20}$$

where $\mathbf{l}_h = \begin{pmatrix} l_h \\ l_h \end{pmatrix}$ so that $\lim\limits_{h\to\infty} 3^h l_h = 0$. Let ψ_h^k be a $C_c^\infty\left(S_h^k \cup J_h^{p_h^k} \cup J_h^{q_h^k}\right)$-mollifier such that

$$\psi_h^k = \begin{cases} 1 \text{ on } S_h^k \backslash J_h^{p_h^k} \cup J_h^{q_h^k}, \\ 0 \text{ on } J_h^{p_h^k} \cup J_h^{q_h^k} \backslash \left(\left(p_h^k, p_h^k + \mathbf{l}_h\right) \cup \left(q_h^k - \mathbf{l}_h, q_h^k\right)\right). \end{cases} \tag{6.21}$$

We define the test function v_0^h on T_h by

$$v^h = \psi_h^k v^{h,k} \text{ on } T_h^k, \forall k \in I_h. \tag{6.22}$$

We have the following result:

Lemma 16 *Under the assumption (2.20) we have*

1. $v_\alpha^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v_\alpha \mathbf{1}_\Sigma(s) \dfrac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(\Sigma)}$ *in* $\mathcal{M}\left(\mathbb{R}^3\right)$*;* $\alpha = 1, 2,$
2. $\varepsilon_h v_3^h \dfrac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} 0$ *in* $\mathcal{M}\left(\mathbb{R}^3\right)$*,*
3. $\lim\limits_{h\to\infty} \int_{T_h} \sigma_{ij}^{*h}\left(v^h\right) e_{ij}\left(v^h\right) dx = \mu^* \lim\limits_{h\to\infty} \rho^h \sum\limits_{\substack{\alpha=1,2 \\ p,q \in \mathcal{V}_h, \\ [p,q] \in S_h^\alpha}} \left(v_\alpha(p) - v_\alpha(q)\right)^2.$

Proof. 1. Let $\varphi \in C_0\left(\mathbb{R}^3\right)$. Then,

$$\begin{aligned} \lim_{h\to\infty} \int_{\mathbb{R}^3} \varphi v_\alpha^h \frac{\mathbf{1}_{T_h}(x)}{2r_h} m_h dx_3 &= \lim_{h\to\infty} \sum_{k\in I_h} \frac{1}{N_h^e} v\left(x_{h1}^k, x_{h2}^k\right) \varphi\left(x_{h1}^k, x_{h2}^k, 0\right) \\ &\quad + \lim_{h\to\infty} \sum_{\substack{\alpha=1,2 \\ k\in I_h}} \frac{\vartheta_h^{\alpha,k}\left(x_{h1}^k, x_{h2}^k\right)}{N_h^e} \varphi\left(x_{h1}^k, x_{h2}^k, 0\right). \end{aligned}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

According to [17, Theorem 6.1], we have

$$\lim_{h\to\infty}\sum_{k\in I_h}\frac{1}{N_h^e}v\left(x_{h1}^k,x_{h2}^k\right)\varphi\left(x_{h1}^k,x_{h2}^k,0\right)=\lim_{h\to\infty}\sum_{p\in\mathcal{V}_h}\frac{v\left(p\right)\varphi\left(p,0\right)}{N_h^v}$$
$$=\frac{1}{\mathcal{H}^d\left(\Sigma\right)}\int_\Sigma v\left(s\right)\varphi\left(s,0\right)d\mathcal{H}^d\left(s\right).$$

Observing that, for every $h\in\mathbb{N}^*$ and every $k\in I_h$,

$$\left|v_\alpha\left(p_h^k\right)-v_\alpha\left(q_h^k\right)\right|\leq C\left|p_h^k-q_h^k\right|,$$

and $\left|p_h^k-q_h^k\right|=3^{-h}/2$, we deduce that $\left|\vartheta_h^{\alpha,k}\left(p_h^k\right)\right|\leq 3^{-h}$ and

$$\lim_{h\to\infty}\sum_{\substack{\alpha=1,2\\k\in I_h}}\frac{\vartheta_h^{\alpha,k}\left(x_{h1}^k,x_{h2}^k\right)}{N_h^e}\varphi\left(x_{h1}^k,x_{h2}^k,0\right)=0.$$

2. We immediately obtain that

$$\varepsilon_h v_3^h\frac{\mathbf{1}_{T_h}\left(x\right)}{\left|T_h\right|}dx\underset{h\to\infty}{\overset{*}{\rightharpoonup}}0\ \text{ in }\mathcal{M}\left(\mathbb{R}^3\right).$$

3. We have, after straightforward computations, that

$$\sigma_{ij}^{*h}\left(v^{h,k}\right)e_{ij}\left(v^{h,k}\right)=\left(\lambda_h^*+2\mu_h^*\right)\left(\frac{\partial v_1^{h,k}\left(x'\right)}{\partial x_1}\right)^2+\mu_h^*\left(\frac{\partial v_2^{h,k}\left(x'\right)}{\partial x_1}\right)^2,$$

for $S_h^k\in S_h^1$, and

$$\sigma_{ij}^{*h}\left(v^{h,k}\right)e_{ij}\left(v^{h,k}\right)=\left(\lambda_h^*+2\mu_h^*\right)\left(\frac{\partial v_2^{h,k}\left(x'\right)}{\partial x_2}\right)^2+\mu_h^*\left(\frac{\partial v_1^{h,k}\left(x'\right)}{\partial x_2}\right)^2,$$

for $S_h^k\in S_h^2$. Then, according to (6.18) and (6.19), we have that

$$\sigma_{ij}^{*h}\left(v^{h,k}\right)e_{ij}\left(v^{h,k}\right)=\mu_h^*3^{2h}\left(\left|v_1\left(p_h^k\right)-v_1\left(q_h^k\right)\right|^2+\left|v_2\left(p_h^k\right)-v_2\left(q_h^k\right)\right|^2\right),$$

on each S_h^k. We deduce from this, using the hypothesis (2.20), that

$$\begin{aligned}&\lim_{h\to\infty}\int_{T_h}\sigma_{ij}^{*h}\left(v^h\right)e_{ij}\left(v^h\right)dsdx_3\\&=\mu^*\lim_{h\to\infty}c_h\sum_{k\in I_h,\alpha=1,2}r_h3^h\left|v_\alpha\left(p_h^k\right)-v_\alpha\left(q_h^k\right)\right|^2\\&=\mu^*\lim_{h\to\infty}\rho^h\sum_{k\in I_h,\alpha=1,2}\left|v_\alpha\left(p_h^k\right)-v_\alpha\left(q_h^k\right)\right|^2\\&=\mu^*\lim_{h\to\infty}\rho^h\sum_{\substack{\alpha=1,2\\p,q\in\mathcal{V}_h\\|p-q|=3^{-h}/2}}\left|v_\alpha\left(p\right)-v_\alpha\left(q\right)\right|^2.\end{aligned}$$

Let $u\in C_c^4\left(\omega,\mathbb{R}^3\right)$ and $\left(v_1,v_2,v_3\right)\in C_c^3\left(\omega,\mathbb{R}^3\right)$. We define the sequence $\left(u_{00}^h\right)_h$ of scaled Kirchhoff-Love displacements by

$$\begin{cases}\left(u_{00}^h\right)_\alpha\left(x\right)=u_\alpha\left(x_1,x_2\right)-\dfrac{x_3}{\varepsilon_h}\dfrac{\partial u_3}{\partial x_\alpha};\ \alpha=1,2,\\\left(u_{00}^h\right)_3\left(x\right)=u_3\left(x_1,x_2\right)/\varepsilon_h\\\qquad\qquad-x_3\dfrac{\lambda_h}{2\mu_h+\lambda_h}\left(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\right)\\\qquad\qquad+\dfrac{x_3^2}{2\varepsilon_h}\dfrac{\lambda_h}{2\mu_h+\lambda_h}\Delta_{x'}u_3.\end{cases}\tag{6.23}$$

We then compute

$$
\begin{cases}
e_{\alpha\alpha}\left(u_{00}^{h}\right)=\dfrac{\partial u_{\alpha}}{\partial x_{\alpha}}-\dfrac{x_3}{\varepsilon_h}\dfrac{\partial^2 u_3}{\partial x_{\alpha}^2};\alpha=1,2,\\
e_{12}\left(u_{00}^{h}\right)=e_{12}(u)-\dfrac{x_3}{2\varepsilon_h}\left(\dfrac{\partial^2 u_3}{\partial x_1\partial x_3}+\dfrac{\partial^2 u_3}{\partial x_2\partial x_3}\right),\\
e_{\alpha 3}\left(u_{00}^{h}\right)=-\dfrac{\lambda_h}{2\mu_h+\lambda_h}x_3\left(\dfrac{\partial^2 u_1}{\partial x_1\partial x_{\alpha}}+\dfrac{\partial^2 u_2}{\partial x_2\partial x_{\alpha}}\right)\\
\qquad +\dfrac{x_3^2}{2\varepsilon_h}\dfrac{\lambda_h}{2\mu_h+\lambda_h}\dfrac{\partial\left(\Delta_{x'}u_3\right)}{\partial x_{\alpha}};\alpha=1,2,\\
e_{33}\left(u_{00}^{h}\right)=-\dfrac{\lambda_h}{2\mu_h+\lambda_h}\left(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\right)\\
\qquad +\dfrac{x_3}{\varepsilon_h}\dfrac{\lambda_h}{2\mu_h+\lambda_h}\Delta_{x'}u_3,
\end{cases}
\tag{6.24}
$$

from which we deduce, using the expression (2.18) of the stress tensor in $\Omega_h\backslash T_h$, that

$$
\begin{cases}
\sigma_{\alpha\alpha}^{h}\left(u_{00}^{h}\right)e_{\alpha\alpha}\left(u_{00}^{h}\right)=\dfrac{2\mu_h\lambda_h}{2\mu_h+\lambda_h}\left(e_{11}(\overline{u})+e_{22}(\overline{u})\right)^2\\
\qquad +2\mu_h\left(e_{11}(\overline{u})\right)^2+2\mu_h\left(e_{22}(\overline{u})\right)^2\\
\qquad +\left(\dfrac{x_3}{\varepsilon_h}\right)^2\dfrac{2\mu_h\lambda_h}{2\mu_h+\lambda_h}\left(\Delta_{x'}u_3\right)^2\\
\qquad +O(1)\dfrac{x_3}{\varepsilon_h},\\
\sigma_{12}^{h}\left(u_{00}^{h}\right)e_{12}\left(u_{00}^{h}\right)=4\mu_h\left(e_{12}(\overline{u})\right)^2\\
\qquad +4\left(\dfrac{x_3}{\varepsilon_h}\right)^2\mu_h\left(\dfrac{\partial^2 u_3}{\partial x_1\partial x_2}\right)^2+O(1)\dfrac{x_3}{\varepsilon_h},\\
\sigma_{3\alpha}^{h}\left(u_{00}^{h}\right)e_{3\alpha}\left(u_{00}^{h}\right)=O(\varepsilon_h);\alpha=1,2,\\
\sigma_{33}^{h}\left(u_{00}^{h}\right)e_{33}\left(u_{00}^{h}\right)=0,
\end{cases}
\tag{6.25}
$$

where $O(1)$ is a function of u and its derivatives up to order 3. We now define the sequence of test functions $\left(u_0^h\right)_h$ in Ω_h by

$$
u_0^h=u_{00}^h-w_h^l\left(\left(u_{00}^h\right)_l-\left(v^h\right)_l\right).
\tag{6.26}
$$

We are now in a position to prove the first assertion of Theorem 12.

Proposition 17 *If $\gamma\in(0,+\infty)$ then, under the assumptions (2.20) and (2.25), for every $(u,v)\in H\left(\omega,\mathbb{R}^3\right)$, there exists a sequence $\left(u^h\right)_h$; $u^h\in H^1\left(\Omega_h,\mathbb{R}^3\right)$, such that $\left(u^h\right)_h$ τ-converges to (u,v) and*

$$
\limsup_{h\to\infty}F_h\left(u^h\right)\leq F_\infty(u,v).
$$

Proof. Let $(u,v)\in H\left(\omega,\mathbb{R}^3\right)$. Let us consider the sequence $(u^n,v^n)_n$, such that $u^n\in C_c^4\left(\omega,\mathbb{R}^3\right)$, $v^n\in C_c^2\left(\omega,\mathbb{R}^3\right)$, $\overline{u}^n\underset{n\to\infty}{\longrightarrow}\overline{u}$ $H^1\left(\omega,\mathbb{R}^3\right)$-strong, $u_3^n\underset{n\to\infty}{\longrightarrow}u_3$ $H^2(\omega)$-strong, and $\left(v_1^n,v_2^n\right)\underset{n\to\infty}{\longrightarrow}v$ strongly with respect to the norm (3.5). Let us consider the sequence $\left(u_0^{h,n}\right)_{h,n}$ constructed in (6.26) for u^n and v^n through

$$
u_0^{h,n}=u_{00}^{h,n}-w_h^l\left(\left(u_{00}^{h,n}\right)_l-\left(v^{h,n}\right)_l\right).
\tag{6.27}
$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

Then, $u_0^{h,n} \in H^1_{\Gamma_h}\left(\Omega_h, \mathbb{R}^3\right)$, and, according to Lemmas 15, 16, observing that the measure $|Z_h|$ of the set Z_h tends to zero as h tends to ∞, the sequence $\left(u_0^{h,n}\right)_h$ τ-converges to $\left(u^n, v_1^n, v_2^n\right)$ as h tends to ∞. Let us write $F_h\left(u_0^{h,n}\right)$ as

$$\begin{aligned} F_h\left(u_0^{h,n}\right) &= \int_{\Omega_h\setminus Z_h} \sigma^h_{ij}\left(u_0^{h,n}\right) e_{ij}\left(u_0^{h,n}\right) dx \\ &\quad + \int_{Z_h} \sigma^h_{ij}\left(u_0^{h,n}\right) e_{ij}\left(u_0^{h,n}\right) dx + \int_{T_h} \sigma^{*h}_{ij}\left(v^{h,n}\right) e_{ij}\left(v^{h,n}\right) dx. \end{aligned} \tag{6.28}$$

Then, observing that

$$\int_{-\varepsilon_h}^{\varepsilon_h} \frac{x_3}{\varepsilon_h} dx_3 = 0 \;\text{ and }\; \int_{-\varepsilon_h}^{\varepsilon_h} \left(\frac{x_3}{\varepsilon_h}\right)^2 dx_3 = \frac{2}{3}\varepsilon_h,$$

we have, using (6.25), that

$$\begin{aligned} \lim_{h\to\infty} \int_{\Omega_h\setminus Z_h} \sigma^h_{ij}\left(u_0^{h,n}\right) e_{ij}\left(u_0^{h,n}\right) dx &= \int_\omega \eta_{\alpha\beta}\left(\overline{u}^n\right) e_{\alpha\beta}\left(\overline{u}^n\right) dx' \\ &\quad + \int_\omega \varpi_{\alpha\beta}\left(u_3\right) \frac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} dx'. \end{aligned} \tag{6.29}$$

It follows from Lemma 15 that

$$\begin{aligned} &\lim_{h\to\infty} \int_{Z_h} \sigma^h_{ij}\left(u_0^{h,n}\right) e_{ij}\left(u_0^{h,n}\right) dx \\ &= \lim_{h\to\infty} \int_{Z_h} \left(\sigma^h_{ij}\left(w_h^l F_l^{h,n}\right) e_{ij}\left(w_h^l F_l^{h,n}\right)\right) dx \\ &= \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \sum_{\alpha=1,2} \int_\Sigma A_{\alpha\alpha}(s)\left(u_\alpha^n - v_\alpha^n\right)^2 d\mathcal{H}^d(s) \\ &+ \frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A_{33}(s)\left(u_3^n\right)^2 d\mathcal{H}^d(s), \end{aligned} \tag{6.30}$$

where $F_l^{h,n} = \left(u_0^{h,n}\right)_l - \left(v^{h,n}\right)_l$. On the other hand, using Lemma 16 and Proposition 2, we have

$$\begin{aligned} \lim_{h\to\infty} \int_{T_h} \sigma^{*h}_{ij}\left(v^{h,n}\right) e_{ij}\left(v^{h,n}\right) dx &= \pi\mu^* \lim_{h\to\infty} \rho^h \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h,\\ [p,q]\in S_h^\alpha}} \left(v_\alpha^{h,n}(p) - v_\alpha^{h,n}(q)\right)^2 \\ &= \mu^* \mathcal{E}_\Sigma\left(v_1^n, v_2^n\right) \\ &= \mu^* \int_\Sigma d\mathcal{L}_\Sigma\left(v_1^n, v_2^n\right). \end{aligned} \tag{6.31}$$

Therefore, combining (6.28)–(6.31), we obtain that

$$\begin{aligned} \lim_{h\to\infty} F_h\left(u_0^{h,n}\right) &= \int_\omega \eta_{\alpha\beta}\left(\overline{u}^n\right) e_{\alpha\beta}\left(\overline{u}^n\right) dx' + \mu\mu^* \int_\Sigma d\mathcal{L}_\Sigma\left(v_1^n, v_2^n\right) \\ &\quad + \int_\omega \varpi_{\alpha\beta}\left(u_3\right) \frac{\partial^2 u_3}{\partial x_\alpha \partial x_\beta} dx' \\ &\quad + \tfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \sum_{\alpha=1,2} \int_\Sigma A_{\alpha\alpha}(s)\left(u_\alpha^n - v_\alpha^n\right)^2 d\mathcal{H}^d(s) \\ &\quad + \tfrac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2} \int_\Sigma A_{33}(s)\left(u_3^n\right)^2 d\mathcal{H}^d(s) \\ &= F_\infty\left(u^n, v^n\right). \end{aligned} \tag{6.32}$$

The continuity of F_∞ implies that $\lim_{n\to\infty}\lim_{h\to\infty} F_h\left(u_0^{h,n}\right) = F_\infty(u, v)$. The topology τ being metrisable, we deduce, according to the diagonalisation of [2, Corollary 1.18], that the sequence $\left(u^h\right)_h = \left(u_0^{h,n(h)}\right)_h$; $\lim_{h\to\infty} n(h) = +\infty$, τ-converges to (u, v) and

$$\limsup_{h\to\infty} F_h\left(u^h\right) \le F_\infty(u, v).$$

6.2.2 Step 2: Lim-inf inequality

In this part, we prove the second assertion of Theorem 12. Let us define the functional G_h on $L^2\left(S_h, \mathbb{R}^2\right)$ through

$$G_h(\psi) = \begin{cases} \mathcal{E}^h_\Sigma(\psi, \psi) & \text{if } z \in H^1\left(S_h, \mathbb{R}^2\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{6.33}$$

We consider the topology τ_g defined in the following:

Definition 18 *A sequence $\left(\psi^h\right)_h$; $\psi^h \in H^1\left(S_h, \mathbb{R}^2\right)$, τ_g-converges to ψ if*

$$\psi^h \mathbf{1}_{S_h}\left(x'\right) m_h \underset{h\to\infty}{\overset{*}{\rightharpoonup}} \psi \mathbf{1}_\Sigma(s) \frac{d\mathcal{H}^d(s)}{\mathcal{H}^d(\Sigma)} \text{ in } \mathcal{M}\left(\mathbb{R}^2\right) \text{ and } \psi \in L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right),$$

where m_h is the measure defined in (3.11).

We have the following convergence:

Proposition 19 *The sequence $(G_h)_h$ Γ-converges in the topology τ_g to the functional G_∞ defined by*

$$G_\infty(\psi) = \begin{cases} \mathcal{E}_\Sigma(\psi, \psi) & \text{if } \psi \in \mathcal{D}_{\Sigma,\mathcal{E}}, \\ +\infty & \text{otherwise,} \end{cases}$$

Proof. According to [11, Theorems 8.5, 11.10], there exist a subsequence $\left(G_{h_k}\right)_k$ of the sequence $(G_h)_h$ and a non-negative quadratic form $\mathcal{E}^*_\Sigma$ such that $\left(G_{h_k}\right)_k$ Γ-convergences in the topology τ_g to the functional $G^*_\infty(\psi)$ defined by

$$G^*_\infty(\psi) = \begin{cases} \mathcal{E}^*_\Sigma(\psi, \psi) & \text{if } \psi \in \mathcal{D}_{\Sigma,\mathcal{E}^*}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{D}_{\Sigma,\mathcal{E}^*}$ is the domain of $\mathcal{E}^*_\Sigma$. Using [11, Proposition 6.8 and Proposition 12.16], we deduce that $\mathcal{E}^*_\Sigma$ is a closed form on $L^2_{\mathcal{H}^d}\left(\Sigma, \mathbb{R}^2\right)$ and $\mathcal{D}^*_\Sigma$ is a Hilbert space with the scalar product associated to the norm

$$\|z\|_{\mathcal{D}^*_\Sigma} = \left\{\mathcal{E}^*_\Sigma(z) + \|z\|^2_{L^2_{\mathcal{H}^d}(\Sigma,\mathbb{R}^2)}\right\}^{1/2}.$$

Using [20, Proposition 10.2 -Theorem 10.4], we can obtain the characterisation of $\left(\mathcal{E}^*_\Sigma, \mathcal{D}_{\Sigma,\mathcal{E}^*}\right)$ as $\mathcal{E}^*_\Sigma = \mathcal{E}_\Sigma$ and $\mathcal{D}_{\Sigma,\mathcal{E}^*} = \mathcal{D}_{\Sigma,\mathcal{E}}$; thus $G^*_\infty = G_\infty$. On the other hand, using the test function (6.22), the fact that the topology τ_g is metrisable, and a diagonalisation argument, we can prove that

$$\Gamma - \limsup_{h\to\infty} G_h = G_\infty,$$

in the topology τ_g. Therefore, the whole sequence $(G_h)_h$ Γ-converges in the topology τ_g to the functional G_∞.

We now prove the second assertion of Theorem 12.

Proposition 20 *If $\gamma \in (0, +\infty)$ then, under the assumptions (2.20) and (2.25), for every sequence $\left(u^h\right)_h$; $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$, such that $\left(u^h\right)_h$ τ-converges to (u, v), we have $(u, v) \in H\left(\omega, \mathbb{R}^3\right)$ and*

$$\liminf_{h\to\infty} F_h\left(u^h\right) \geq F_\infty(u, v).$$

Proof. Let $\left(u^h\right)_h$; $u^h \in H\left(\Omega_h, \mathbb{R}^3\right)$, such that $\left(u^h\right)_h$ τ-converges to (u, v). We suppose that $\sup_h F_h\left(u^h\right) < +\infty$; otherwise, there is nothing to prove. Then, according to Proposition 8, $\overline{u} = (u_1, u_2) \in H^1_0\left(\omega, \mathbb{R}^2\right)$ and $u_3 \in H^2_0(\omega)$. Let us define $\overline{u}^h = \left(u^h_1, u^h_2\right)$ by

$$\tilde{\overline{u}}^h = \frac{1}{2r_h} \int_{-r_h}^{r_h} \left(\overline{u}^h(., x_3)\right) dx_3. \tag{6.34}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

Then, according to Lemma 5_1, we have

$$\begin{aligned}&\mu^* \sup_h \mathcal{E}^h_\Sigma\left(\widetilde{\overline{u}}^h, \widetilde{\overline{u}}^h\right)\\&= \mu^* \sup_h \sum_{\substack{\alpha=1,2\\ p,q\in\mathcal{V}_h\\ |p-q|=3^{-h}/2}} \rho^h \left(\frac{1}{2r_h}\int_{-r_h}^{r_h}\left(u^h_\alpha(p,x_3)-u^h_\alpha(q,x_3)\right)dx_3\right)^2\\&\leq \sup_h \int_{T_h}\sigma^h_{ij}\left(u^h\right)e_{ij}\left(u^h\right)ds<+\infty.\end{aligned} \tag{6.35}$$

On the other hand, since

$$\overline{u}^h\frac{\mathbf{1}_{T_h}(x)}{2r_h}m_h dx_3 \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v\mathbf{1}_\Sigma(s)\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(\Sigma)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),$$

the sequence $\left(\widetilde{\overline{u}}^h\right)_h$ τ_g-converges to v and, according to (6.35) and to Proposition 19,

$$G_\infty(v)\leq \liminf_{h\to\infty} G_h\left(\widetilde{\overline{u}}^h\right)<+\infty. \tag{6.36}$$

Thus, $v\in\mathcal{D}_{\Sigma,\mathcal{E}}$ and

$$\liminf_{h\to\infty}\int_{T_h}\sigma^h_{ij}\left(u^h\right)e_{ij}\left(u^h\right)ds\geq\mu^*\mathcal{E}_\Sigma(v,v). \tag{6.37}$$

Let us consider the sequence $(u^n, v^n)_n$, such that $u^n\in C^4_c\left(\omega,\mathbb{R}^3\right)$, $v^n\in C^2_c\left(\omega,\mathbb{R}^3\right)$, $\overline{u}^n\underset{n\to\infty}{\longrightarrow}\overline{u}$ $H^1\left(\omega,\mathbb{R}^2\right)$-strong, $u^n_3\underset{n\to\infty}{\longrightarrow}u_3$ $H^2(\omega)$-strong, and $\left(v^n_1,v^n_2\right)\underset{n\to\infty}{\longrightarrow}v$ strongly with respect to the norm (3.5). Let $\left(u^{h,n}_0\right)_{h,n}$ be the sequence constructed in (6.27). We have from the definition of the subdifferentiability of convex functionals

$$\begin{aligned}\int_{\Omega_h\setminus T_h}\sigma^h_{ij}\left(u^h\right)e_{ij}\left(u^h\right)dx&\geq\int_{\Omega_h\setminus T_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^{h,n}_0\right)dx\\&\quad+2\int_{\Omega_h\setminus T_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx.\end{aligned} \tag{6.38}$$

We have for the second integral in the right-hand side of the inequality (6.38)

$$\begin{aligned}&\int_{\Omega_h\setminus T_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx\\&=\int_{\Omega_h\setminus Z_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx\\&\quad+\int_{Z_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx.\end{aligned} \tag{6.39}$$

Then, due to the structure of the sequence $\left(u^{h,n}_0\right)_h$, we have

$$\begin{aligned}\int_{Z_h}\sigma^h_{ij}\left(u^{h,n}_0\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx&=\int_{Z_h}\sigma^h_{ij}\left(u^n\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx\\&\quad-\int_{Z_h}\sigma^h_{ij,j}\left(w^l_h\left(u^n-v^{h,n}\right)_l\right)\left(u^h-u^{h,n}_0\right)_i dx.\end{aligned} \tag{6.40}$$

Since $|Z_h|$ tends to zero as h tends to ∞, we have that

$$\lim_{h\to\infty}\int_{Z_h}\sigma^h_{ij}\left(u^n\right)e_{ij}\left(u^h-u^{h,n}_0\right)dx=0. \tag{6.41}$$

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

Using the definition (6.15) of the local perturbation w_h^l; $l=1,2,3$, and the expressions (6.2), (6.3), and (6.6), we obtain the following estimate:

$$\left|\int_{Z_h}\sigma_{ij,j}^h\left(w_h^l\left(u^n-v^{h,n}\right)_l\right)\left(u^h-u_0^{h,n}\right)_i dx\right|$$
$$\leq C_n\sum_\alpha\left\{\begin{array}{c}\left(\int_{Z_h}\left|\left(u_\alpha^h-\left(u_0^{h,n}\right)_\alpha\right)\right|^2dx\right)^{1/2}\\ \times\left(1+\left(\int_{Z_h}\left|\nabla w_h^l(x)\right|^2dx\right)^{1/2}\right)\end{array}\right\}$$
$$+C_n\left\{\begin{array}{c}\left(\int_{Z_h}\left|\left(u_\alpha^h-\left(\varepsilon_h u_0^{h,n}\right)_3\right)\right|^2dx\right)^{1/2}\\ \times\left(1+\left(\int_{Z_h}\left|\nabla w_h^l(x)\right|^2dx\right)^{1/2}\right)\end{array}\right\}, \tag{6.42}$$

where C_n is a positive constant which may depend of n, which implies, using the fact that $\int_{Z_h}\left|\nabla w_h^l(x)\right|^2dx$ is bounded, that

$$\lim_{h\to\infty}\int_{Z_h}\sigma_{ij,j}^h\left(w_h^l\left(u^n-v^{h,n}\right)_l\right)\left(u^h-u_0^{h,n}\right)_i dx=0. \tag{6.43}$$

According to (6.30), we have

$$\lim_{h\to\infty}\int_{Z_h}\sigma_{ij}^h\left(u_0^{h,n}\right)e_{ij}\left(u_0^{h,n}\right)dx$$
$$=\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\sum_{\alpha=1,2}\int_\Sigma A_{\alpha\alpha}(s)\left(u_\alpha^n-v_\alpha^n\right)^2d\mathcal{H}^d(s)$$
$$+\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma A_{33}(s)\left(u_3^n\right)^2d\mathcal{H}^d(s). \tag{6.44}$$

Using the construction of u_0^{nh}, we deduce that

$$\lim_{h\to\infty}2\int_{\Omega_h\setminus Z_h}\sigma_{ij}^h\left(u_0^{n,h}\right)e_{ij}\left(u^h-u_0^{n,h}\right)dx$$
$$=2\int_\omega\eta_{\alpha\beta}(\overline{u}^n)e_{\alpha\beta}(\overline{u}-\overline{u}^n)dx'+2\int_\omega\varpi_{\alpha\beta}(u_3)\frac{\partial^2\left(\overline{u}_3-\overline{u}_3^n\right)}{\partial x_\alpha\partial x_\beta}dx'. \tag{6.45}$$

Combining (6.37)–(6.45), we deduce that

$$\liminf_{h\to\infty}F_h\left(u^h\right)\geq\int_\omega\eta_{\alpha\beta}(\overline{u}^n)e_{\alpha\beta}(\overline{u}^n)dx'+\int_\omega\varpi_{\alpha\beta}\left(u_3^n\right)\frac{\partial^2u_3^n}{\partial x_\alpha\partial x_\beta}dx'$$
$$+2\int_\omega\varpi_{\alpha\beta}(u_3)\frac{\partial^2\left(\overline{u}_3-\overline{u}_3^n\right)}{\partial x_\alpha\partial x_\beta}dx'+2\int_\omega\eta_{\alpha\beta}(\overline{u}^n)e_{\alpha\beta}(\overline{u}-\overline{u}^n)dx'$$
$$+\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\sum_{\alpha=1,2}\int_\Sigma A_{\alpha\alpha}(s)\left(u_\alpha^n-v_\alpha^n\right)^2d\mathcal{H}^d(s)$$
$$+\frac{\pi\mu\gamma}{\mathcal{H}^d(\Sigma)(\ln 2)^2}\int_\Sigma A_{33}(s)\left(u_3^n\right)^2d\mathcal{H}^d(s)+\mathcal{E}_\Sigma(v). \tag{6.46}$$

Letting n tend to $+\infty$ in the right-hand side of (6.46), we conclude that

$$\liminf_{h\to\infty}F_h\left(u^h\right)\geq F_\infty(u,v).$$

Conflicts of interest. There is no conflicts of interest.

References

[1] Adkins, J. E. & Rivlin, R. S. (1955) Large elastic deformations of isotropic materials X. Reinforcement by inextensible cords. *Philos. Trans. R. Soc. London A* **248**, 201–223.

[2] Attouch, H. (1984) *Variational Convergence for Functions and Operators, Applied Mathematical Sciences*, London, Pitman.

https://doi.org/10.1017/S0956792523000025 Published online by Cambridge University Press

[3] Barlow, M. T. & Bass, R. F. (1990) On the resistance of the Sierpinski carpet. *Proc. Math. Phys. Sci.* **431**(1882), 345–360.
[4] Barlow, M. T., Bass, R. F. & Sherwood, J. D. (1990) Resistance and spectral dimension of the Sierpinski carpets. *J. Phys. A* **23**, L253–L258.
[5] Bellieud, M. & Bouchitté, G. (1998) Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effects. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **26**(3), 407–436.
[6] Capitanelli, R., Lancia, M. R. & Vivaldi, M. A. (2013) Insulating layers of fractal type. *Differ. Integr. Equations* **26**(9/10), 1055–1076.
[7] Capitanelli, R. & Vivaldi, M. A. (2011) Insulating layers and Robin problems on Koch mixtures. *J. Differ. Equations* **251**(4–5), 1332–1353.
[8] Capitanelli, R. & Vivaldi, M. A. (2015) Reinforcement problems for variational inequalities on fractal sets. *Calc. Var.* **54**, 2751–2783.
[9] Capitanelli, R. & Vivaldi, M. A. (2016) Dynamical quasi-filling fractal layers. *SIAM J. Math. Anal.* **48**(6), 3931–3961.
[10] Creo, S. (2021) Singular p-homogenization for highly conductive fractal layers. *Z. Anal. Anwend.* **40**(4), 401–424.
[11] Dal Maso, G. (1993) *An Introduction to* Γ*-Convergence*, PNLDEA, Vol. **8**, Birkhäuser, Basel.
[12] El Jarroudi, M. (2019) Homogenization of an elastic material reinforced with thin rigid von Kármán ribbons. *Math. Mech. Solids* **24**(7), 1965–1991.
[13] El Jarroudi, M. (2022) Homogenization of a quasilinear elliptic problem in a fractal-reinforced structure. *SeMA* **79**, 571–592. https://doi.org/10.1007/s40324-021-00250-5.
[14] El Jarroudi, M. & Er-Riani, M. (2019) Homogenization of elastic materials containing self-similar microcracks. *Q. Jl Mech. Appl. Math.* **72**(2), 131–155.
[15] El Jarroudi, M., Er-Riani, M., Lahrouz, A. & Settati, A. (2018) Homogenization of elastic materials reinforced by rigid notched fibres. *Appl. Anal.* **97**(5), 705–738.
[16] El Jarroudi, M., Filali, Y., Lahrouz, A., Er-Riani, M. & Settati, A. (2022) Asymptotic analysis of an elastic material reinforced with thin fractal strips. *Netw. Heterog. Media* **17**(1), 47–72.
[17] Falconer, K. (1997) *Techniques in Fractal Geometry*, J. Wiley and Sons, Chichester.
[18] Freiberg, U. R. & Lancia, M. R. (2004) Energy form on a closed fractal curve. *Z. Anal. Anwendungen* **23**(1), 115–137.
[19] Fukushima, M., Oshima, Y. & Takeda, M. (1994) *Dirichlet Forms and Symmetric Markov Processes*, *de Gruyter Studies in Mathematics*, Vol. **19**, Eds. Bauer Kazdan, Zehnder, Berlin.
[20] Grigor'yan, A. & Yang, M. (2019) Local and non-local Dirichlet forms on the Sierpinski carpet. *Trans. AMS* **372**(6), 3985–4030.
[21] Izotova, O. V., Nazarov, S. A. & Sweers, G. H. (2008) Asymptotic sharp weight Korn's inequality for thin-walled elastic structures. *J. Math. Sci. (N.Y)* **150**(1), 1807–1855.
[22] Jonsson, A. & Wallin, H. (1984) *Function Spaces on Subsets of* $\mathbb{R}^n$, Mathematical *Reports*, Vol. **2** (Part 1), Harwood Academic Publisher, London.
[23] Jonsson, A. & Wallin, H. (1995) The dual of Besov spaces on fractals. *Stud. Math.* **112**(3), 285–300.
[24] Jonsson, A. & Wallin, H. (1997) Boundary value problems and Brownian motion on fractals. *Chaos Soli. Frac.* **8**(2), 191–205.
[25] Kato, T. (1966) *Perturbation Theory for Linear Operators*, Springer, Berlin.
[26] Lancia, M. R., Mosco, U. & Vivaldi, M. A. (2008) Homogenization for conductive thin layers of pre-fractal type. *J. Math. Anal. Appl.* **347**(1), 354–369.
[27] Le Dret, H. (1991) *Problèmes Variaonnels dans les Multi-Domaines : Modélisation des Jonctions et Applications, RMA*, Vol. **19**, Masson, Paris.
[28] Le Jean, Y. (1978) Measures associées á une forme de Dirichlet. *Applications. Bull. Soc. Math. France* **106**, 61–112.
[29] Lobo, M. & Perez, E. (1992) Boundary homogenization of certain elliptic problems for cylindrical bodies. *Bull. Sci. Math. Sér 2* **116**, 399–426.
[30] Malo, R. J. (2015) *Discrete Extremal Lengths of Graph Approximations of Sierpinski Carpets*. PhD thesis, Montana State University.
[31] Mosco, U. (2000) Energy functionals on certain fractal structures. *J. Conv. Anal.* **9**, 581–600.
[32] Mosco, U. & Vivaldi, M. A. (2007) An example of fractal singular homogenization. *Georgian Math. J.* **14**(1), 169–194.
[33] Mosco, U. & Vivaldi, M. A. (2009) Fractal reinforcement of elastic membranes. *Arch. Rat. Mech. Anal.* **194**, 49–74.
[34] Mosco, U. & Vivaldi, M. A. (2013) Thin fractal fibers. *Math. Meth. Appl. Sci.* **36**, 2048–2068.
[35] Mosco, U. & Vivaldi, M. A. (2015) Layered fractal fibers and potentials. *J. Math. Pures Appl.* **103**, 1198–1227.

Cite this article: El Jarroudi M., El Merzguioui M., Er-Riani M., Lahrouz A. and El Amrani J. (2023). Dimension reduction analysis of a three-dimensional thin elastic plate reinforced with fractal ribbons. *European Journal of Applied Mathematics*, **34**, 838–869. https://doi.org/10.1017/S0956792523000025