ON A PAPER OF MAURICE SION

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1. Let M_0 be the set of measures μ on the real line such that open sets are μ^*-measurable. While attempting to find out whether a set μ^*-measurable for all μ in M_0 is mapped into a similar set by a continuous function of bounded variation, Maurice Sion develops a theory for what he calls variational measure **(4)**. As an application of the theory, he gets conditions on a function f and a set of measures M in order that f map a set, which is μ^*-measurable for all $\mu \in M$, into a set of the same kind. In particular he proves for his class M_2 (def. 2.5), the following theorem **(4**, § 8.11**)**.

THEOREM. *If A is measurable for all measures in M_2 and if f is continuous from the irrationals to* [0, 1], *then $f(A)$ is measurable for all measures in M_2.*

Since all projective sets are continuous images of the irrationals **(2**, p. 39**)** and since the existence of a non-measurable projective set is consistent with the axioms of set theory if they are consistent, **(1)**, Sion concludes that Lebesgue measure is not in M_2.

We prove Sion's result in another way and more importantly, we characterize M_2 completely with respect to open regular measures. As an application, we prove, without the continuum hypothesis, the existence of a function discontinuous on every set of positive outer measure (Lebesgue).

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2. Notation and definitions.

2.1. A *partition*, $P(S)$, of a set S is a collection of sets, $E \subset S$, finite in number, pairwise disjoint and whose union is S.

2.2. A *refinement of a partition*, P_1, is a second partition, P_2, such that each set in P_2 is a subset of some set in P_1.

2.3. An *open regular measure* is a measure such that each μ^*-measurable set has a measurable cover which is a G_δ set.

2.4. $M_0 = |\mu: \mu$ is a measure on [0, 1] and open sets are μ^*-measurable$|$.

2.5. A sequence with property A is a sequence of partitions $P_n(S)$ such that

(a) $S \subset [0, 1]$ and $0 < \mu^*(S) < \infty$;
(b) P_{n+1} is a refinement of P_n;
(c) if $B \subset S$ and $\mu^*(B) > 0$, then

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$$\lim_{n\to\infty} \sum_{E \in P_n} \mu^*(B \cap E) = \infty.$$

2.6. $M_2 = |\mu : \mu \in M_0$ and there does not exist a set S with a partition sequence having property $A|$.

2.7. $M_3 = |\mu : \mu \in M_0$, μ is open regular, there exists a partition sequence $P_n([0, 1])$ with property $A|$.

2.8. A measure will be called *non atomic* if no single point has positive outer measure.

3. Conditions implying a measure is not in M_2 or is in M_3.

3.1. THEOREM. *If there exists a set $S \subset [0, 1]$ of positive outer measure and a bounded function f defined on S discontinuous on every set $E \subset S$ for which $\mu^*(E) > 0$ and if μ is open regular, then $\mu \notin M_2$.*

The proof is long and will be given in § 5.

3.2. COROLLARY. *If $S = [0, 1]$ in the theorem, then $\mu \in M_3$.*

This corollary is an immediate consequence of the proof of the theorem (see 5).

3.3. The following lemma is obtained by a minor modification of the proof of the similar theorem (without the word "bounded") due to Sierpinski and Zygmund **(3)**.

LEMMA. *There exists a bounded function from the reals to the reals which is discontinuous on every set having the power of the continuum.*

3.4. Then we can prove this

THEOREM. *If μ is such that every set of positive outer measure has the power of the continuum and μ is open regular, then μ is in M_3.*

Proof. The theorem follows immediately from 3.2 and 3.3.

3.5. COROLLARY. *If there exists one set of positive outer measure such that all subsets of positive outer measure have the power of the continuum and if μ is open regular for all subsets of this set, then $\mu \notin M_2$.*

3.6. COROLLARY. *Under the continuum hypothesis: If μ is non-atomic and open regular, then μ is in M_3. If there exists a subset of positive outer measure such that every single point subset has measure zero, then $\mu \notin M_2$.*

Proof. If a measure is non-atomic then every countable set has measure zero. The continuum hypothesis then implies every set having positive outer measure has the power of the continuum and 3.4 and 3.5 prove the theorem.

3.7. Every measure on the subsets of the unit interval is either in M_3 or it is not. The definition of M_3, which enables one to decide whether or not a measure is in M_3, does not depend on the continuum hypothesis, that is, the

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definition makes sense if the hypothesis is true or false. Now, if there exists a non-atomic, open regular measure not in M_3, then this can be shown by a set theoretic argument. Such an argument with corollary 3.6 would be a proof from set theory of the proposition: *the continuum hypothesis is false.* Gödel **(1)** has shown that this cannot be proven with such an argument. Therefore, all open regular, non-atomic measures are in M_3, that is, we can improve 3.6 to the following

THEOREM. *If μ is open regular and non-atomic, then $\mu \in M_3$.*

3.8. We can restate this by this

THEOREM. *There are no σ-finite open regular measures in M_2. Lebesgue measure is not in M_2 but it is in M_3.*

4. A converse to Theorem 3.1.

4.1. THEOREM. *If μ is an open regular measure not in M_2 and S is a set with a partition sequence having property A, then there is a function defined on S which is discontinuous on every subset E of S such that $\mu^*(E) > 0$.*

Proof. For μ, there exists a sequence $P_m(S)$ of partitions with property A. Let $F_{11}, \ldots, F_{k1}$ be a numbering of the sets of P_1. Let n_1 be the smallest integer larger than $\log_2 k$. Define

$$f_1(x) = (i-1)/2^{n_1}$$

for $x \in F_{i1}$, for $i = 1, \ldots, k$.

Suppose for $m - 1$ we have defined n_{m-1}, a numbering, $F_{i,m-1}$, for the partition P_{m-1}, and f_{m-1}. The induction step will be defined as follows:
Let $p_m = \max_q j_q$, where j_q is the number of sets in P_m which are subsets of $F_{q,m-1}$. Let h_m be the smallest integer greater than $\log_2(p_m + 2)$. Let $n_m = h_m + n_{m-1}$. Let F_{im}, for

$$i = (q-1)2^{hm} + 1, \ldots, (q-1)2^{hm} + j_q$$

and

$$q = 1, \ldots, 2^{n_{m-1}},$$

be a numbering of the sets of P_m which are subsets of $F_{q,m-1}$. If F_{im} does not appear in this numbering, then $F_{im} = \phi$. Then define

$$f_m(x) = (i-1)/2^{n_m} \qquad \text{for} \qquad x \in F_{im}.$$

The sequence f_m is monotonically non-decreasing and is uniformly bounded by one. Therefore there exists a limit function f_0.

For m fixed, our choice of h_m assures us that

$$|x: f_m(x) = (q2^{hm} - 1)/2^{n_m}| = \phi$$

since it would equal

$$F_{q2^{hm},m}$$

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which is empty. As a consequence, we have: if

$$f_0(x) < (q2^{hm})/2^{nm}$$

for any m and

$$q = 1, \ldots, 2^{nm-1},$$

then

$$f_0(x) \leqslant (q2^{hm} - 1)/2^{nm}.$$

Therefore

$$F_{im} = |x: [i - 1 - (1/2^{nm+1})]/2^{nm} < f_0(x) < i/2^{nm}|.$$

Now suppose there exists a set $B \subset S$ such that $\mu^*(B) > 0$ and such that f_0 is continuous on B. Since F_{im} is the inverse image of an open set, $F_{im} \cap B$ is open in B, that is, there exists an open set U_{im} such that $F_{im} \cap B = U_{im} \cap B$. Let $U_{ijm} = U_{im} \cap U_{jm}$ for $i \neq j$ and $U_{jjm} = \phi$ and let $V_{im} = U_{im} - \bigcup_j U_{ijm}$. Clearly V is pairwise disjoint for m fixed. Also, since F is pairwise disjoint for m fixed, no point of $F_{jm} \cap B$ can be in U_{im} for $i \neq j$. Therefore, we have $V_{im} \cap B = F_{im} \cap B$. Hence we can choose a measurable cover of $F_{im} \cap B$, C_{im}, which is a subset of V_{im}. Therefore, C is pairwise disjoint and

$$\sum_i \mu^*(F_{im} \cap B) = \sum_i \mu(C_{im}) = \mu(U_i C_{im}) = \mu^*(B).$$

Since m is arbitrary, we have

$$\lim_{m\to\infty} \sum_i \mu^*(F_{im} \cap B) = \mu^*(B)$$

but this is just

$$\lim_{m\to\infty} \sum_{E \epsilon P_m} \mu^*(B \cap E) = \mu^*(B) \leqslant \mu^*(S) < \infty.$$

This contradiction proves the theorem:

4.2. Corollary. *For every $\mu \in M_3$, there exists a function discontinuous on every set having positive outer measure. In particular there exists such a function for Lebesgue measure.*

5. Proof of Theorem 3.1. Let f be the function described in the theorem. Since f is bounded we can suppose that $0 \leqslant f(x) \leqslant 1$. Let $E_{ni} = |x: i/2^n \leqslant f(x) < (i+1)/2^n|$, $i = 0, \ldots, 2^n$. Set $P_n(S) = \{E_{ni}\}$; we shall prove that this sequence has the property A. The facts that P is a partition and that P_{n+1} is a refinement of P_n are clear. We need only show that, for any $B \subset S$ for which $\mu^*(B) > 0$, we have

$$\lim_{n\to\infty} \sum_{i=0}^{2^n} \mu^*(E_{ni} \cap E) = \infty.$$

Assume that there exists a set $E \subset S$ such that

(1) $$0 < \lim_{n\to\infty} \sum_i \mu^*(E_{ni} \cap E) = a < \infty.$$

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We then shall prove that there exists a subset of E having positive outer measure and on which f is continuous and this contradiction will prove the theorem.

Subadditivity of μ^* implies that the limit in (1) approaches a from below. Therefore, there exists an N such that $n \geqslant N$ implies, for $a/10 > \epsilon > 0$,

$$a - \epsilon \leqslant \sum_i \mu^*(E_{ni} \cap E) \leqslant a$$

and in particular

$$a - \epsilon \leqslant \sum_j \mu^*(E_{Nj} \cap E) \leqslant a.$$

Let B_{Nj} be a measurable cover of $E \cap E_{Nj}$. If E_{ni} is a subset of E_{Nj}, we shall write E_{nij} and we shall designate a measurable cover of $E \cap E_{nij}$ by B_{nij}. It is easily shown that the sets B_{nij}, $n \geqslant N$, can be so determined that if I is a set of integers such that $\cup_{i\epsilon I} E_{mi} = E_{nk}$, then $\cup_{i\epsilon I} B_{mij} = B_{nkj}$.

We next derive measurable sets H_{nij} contained in B_{nij}, disjoint for each fixed pair n, j and with

$$a - 2\epsilon \leqslant \sum_{j,i} H_{nij} \leqslant a.$$

Let

(2) $$B_{nikj} = B_{nij} \cap B_{nkj} \qquad i \neq k$$
$$= \phi \qquad i = k$$

and let $H_{nij} = B_{nij} - \cup_k B_{nikj}$. Since, for $n > N$,

$$a - \epsilon \leqslant \sum_j \mu(B_{Nj}) = \sum_j \mu(\cup_i B_{nij}) \leqslant a$$

and

$$a - \epsilon \leqslant \sum_{j,i} \mu(B_{nij}) \leqslant a,$$

we have

$$\sum_j \left[\sum_i \mu(B_{nij}) - \mu(\cup_i B_{nij}) \right] = \sum_{j,i} \mu(\cup_k B_{nikj}) \leqslant \epsilon.$$

From this and the definition of H, we have

(3) $$a - 2\epsilon \leqslant \sum_{j,i} \mu(H_{nij}) \leqslant a$$

for all $n > N$. By the choice of B, $\cup_i H_{nij}$ is monotonically decreasing as a function of n for each j. Letting $H_j = \cap_n \cup_i H_{nij}$, we have from (3)

(4) $$\sum_j \mu(H_j) \geqslant a - 2\epsilon.$$

We next obtain formulas analagous to (3) and (4) with the sets H_{nij} replaced by open sets. Let V_{nij} be an open cover of H_{nij} such that

$$\mu(V_{nij}) \subset \mu(H_{nij}) + \epsilon/2^i(2^N + 1)$$

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and if I is a set of integers such that $\bigcup_{i\epsilon I} E_{mi} = E_{nk}$, then $\bigcup_{i\epsilon I} V_{mij} \subset V_{nkj}$. Such a cover exists because of the open regularity of μ. Then for every $n > N$ we have

$$a - 2\epsilon \leqslant \sum_{i,j} \mu(H_{nij}) = \sum_{j} \mu(\bigcup_i H_{nij}) \leqslant \sum_{j} \mu(\bigcup_i V_{nij})$$
$$\leqslant \sum_{i,j} \mu(V_{nij}) \leqslant \sum_{i,j} \mu(H_{nij}) + \epsilon \leqslant a + \epsilon.$$

Therefore

$$\sum_{i,j} \mu(V_{nij}) - \sum_{j} \mu(\bigcup_i V_{nij}) \leqslant 3\epsilon.$$

Using notation analogous to that of (2), letting $U_{nij} = V_{nij} - \bigcup_k V_{nikj}$, and using the same argument which leads to (3) and (4), we have

(5) $$\sum_{j,i} \mu(\bigcup_k V_{nikj}) \leqslant 3\epsilon \quad \text{and} \quad a - 5\epsilon \leqslant \sum_{j,i} \mu(U_{nij}) \leqslant a + \epsilon$$

for all $n \geqslant N$.

By the choice of V, $\bigcup_i U_{nij}$ is monotonically decreasing as a function of n for each j. Letting $U_j = \bigcap_n \bigcup_i U_{nij}$, we have from (5)

(6) $$\lim_{n\to\infty} \sum_{j,i} \mu(U_{nij}) = \sum_{j} \lim_{n\to\infty} \mu(\bigcup_i U_{nij}) = \sum_{j} \mu(U_j) \geqslant a - 5\epsilon.$$

We shall now show

(7) $$\sum_{j} \mu(U_j \cap H_j) \geqslant a - 8\epsilon.$$

Since

$$\mu(V_{Nj}) = \mu(V_{Nj} - (U_j \cup H_j)) + \mu(\mathrm{H}_j \cup U_j)$$

and

$$\mu(H_j \cup U_j) = \mu(\mathrm{H}_j) + \mu(U_j) - \mu(H_j \cap U_j),$$

we have

$$\sum_{j} [\mu(H_j) + \mu(U_j) - \mu(H_j \cap U_j)] \leqslant \sum \mu(V_{Nj}) \leqslant a + \epsilon$$

or

$$a - 2\epsilon + a - 5\epsilon - \sum_{j} \mu(H_j \cap U_j) \leqslant a + \epsilon.$$

This yields (7).

Pick a j such that $\mu(U_j \cap H_j) > 0$. Then

$$\mu^*(\mathrm{E} \cap E_{Nj} \cap U_j) \geqslant \mu(U_j \cap H_j) > 0.$$

Let $C = E \cap E_{Nj} \cap U_j$. Then for arbitrary but fixed $n > N$ we have

$$V_{nij} \cap V_{nkj} \cap C = \phi \textit{ for } i = k.$$

We shall show that f is continuous on C. Let $\delta > 0$ be given. Then there exists an n such that $2^{-n} < \delta$. Let x_0 be in C. Then

$$|f(x_0) - f(x)| < 2^{-n} < \delta$$

for all x in $V_{nij} \cap C$ where i is such that x_0 is in E_{ni}. Therefore f is continuous on C contrary to hypothesis on f. This contradiction proves the theorem.

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