SOME PRINCIPLES UNDERLYING THE CONSTRUCTION OF MEASURES

CHARLES A. HAYES, JR.

1. Introduction. Measures may be obtained from suitable non-negative valued functions in a number of ways. It is the purpose of this paper to present an abstract formulation of certain principles which may be used to construct measures, and to show that the various methods most frequently encountered in the literature are in fact all special applications of these principles.

The basic requirements A1–A4 are set forth in § 2, and it is there shown that a measure can be defined when these are fulfilled.* These requirements are satisfied in every case known to the writer. Usually, however, conditions stronger than A1–A4 hold, and it is these extra restrictions which yield information on the class of measurable sets and other matters. In §§ 3, 4, and 5 certain abstracts form of such restrictions are considered, and results are derived thereforom. The paper concludes with an analysis of how a number of measures occur as special cases of the theory.

We begin by stating a number of definitions and conventions which will be used throughout the paper.

If $\mathfrak{F}$ is any family of subsets of a set S, we agree to let $\bigcup \mathfrak{F}$ denote the *union* of $\mathfrak{F}$; that is, the set of all points which belong to at least one member of $\mathfrak{F}$. For a sequence of sets A we sometimes use the notation $\bigcup_{n=1}^{\infty} A_n$ to denote the union. If A and B are sets, then by $A - B$ we shall mean the set of those points which are in A but not in B. We agree to denote by $U(x)$ the set whose sole member is x. We further agree to let 0 denote the null set as well as zero.

When discussing metric spaces, the terms sphere, closed, open, Borel, etc., will be understood to have their customary meanings with reference to the metric of the space. If A and B are subsets of a metric space, the diameter of A will be denoted by diam A, with the convention that diam $0 = 0$, and the distance between A and B will be denoted by dist (A, B). We keep in mind that $\inf 0 = \infty$ and empty sums are zero.

We shall say that the *function* ϕ *measures* S if, and only if,

$$0 \leqslant \phi(A) \leqslant \sum_{\beta \in \mathfrak{F}} \phi(\beta)$$

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*These principles were formulated as an outgrowth of an observation by A. P. Morse, who noticed that a certain measure occurring in (2), discussed in 6.3 of the present paper, satisfies conditions which are special cases of the above.

whenever $\mathfrak{F}$ is a finite or countably infinite family and $A \subset \cup \mathfrak{F} \subset S$. This definition is equivalent to the usual properties, namely $\phi(0) = 0$; $\phi(A) \leqslant \phi(B)$ whenever $A \subset B \subset S$; and

$$\phi(\cup\mathfrak{F}) \leqslant \sum_{\beta \epsilon \mathfrak{F}} \phi(\beta)$$

whenever $\mathfrak{F}$ is a finite or countably infinite family and $\cup \mathfrak{F} \subset S$.

The set $A \subset S$ is said to be *ϕ-measurable* if, and only if, ϕ measures S and

$$\phi(E) = \phi(A \cap E) + \phi((S - A) \cap E)$$

whenever $E \subset S$.

A family of sets $\mathfrak{F}$ is said to be a *ring* if, and only if, $(E \cup F) \in \mathfrak{F}$ and $(E - F) \in \mathfrak{F}$ whenever $E \in \mathfrak{F}$ and $F \in \mathfrak{F}$.

A family of sets $\mathfrak{F}$ is said to *cover* a set A if, and only if, $A \subset \cup \mathfrak{F}$.

We say that F is a *blanket* if, and only if, F is such a function that for x in its domain:

(i) S is a metric space;

(ii) $x \in S$ and $F(x)$ is a non-vacuous family of subsets of S;

(iii) diam $\beta < \infty$ whenever $\beta \in F(x)$;

(iv) $\inf_{\beta \epsilon F(x)} \operatorname{diam} (\beta \cup U(x)) = 0$.

If F is a blanket with domain A, then $\cup_{x \epsilon A} F(x)$ is called the *spread* of F.

2. The general theory. At this point we introduce certain basic sets and functions which we shall understand to be fixed throughout the paper, along with the properties A1–A4 given below. In § 6, we will give certain examples which may serve to furnish motivation for the work to follow.

We let S and $\mathfrak{F}_0$ denote, respectively, a non-empty point set and a non-empty class of subsets* of S. It will turn out that S is our measure space. We let T denote another non-empty set which may or may not bear any relation to S.

We further fix a function $\mathbf{M}$ with the following properties:

A1. For each set $A \subset S$ and each point $t \in T$, $\mathbf{M}(A, t)$ is a collection (possibly empty) of finite or countably infinite subfamilies of $\mathfrak{F}_0$.

A2. $0 \in \mathbf{M}(0, t)$ for each $t \in T$.

A3. $\mathbf{M}(B, t) \subset \mathbf{M}(A, t)$ whenever $A \subset B \subset S$ and $t \in T$.

A4. If $t \in T$ and $\mathfrak{H}$ is any finite or countably infinite collection of subsets of S, then there exists a function H on $\mathfrak{H}$ to T such that if Q is any function on $\mathfrak{H}$ with $Q(\gamma) \in \mathbf{M}(\gamma, H(\gamma))$ whenever $\gamma \in \mathfrak{H}$, then

$$\bigcup_{\gamma \epsilon \mathfrak{H}} Q(\gamma) \in \mathbf{M}(\cup\mathfrak{H}, t).$$

Finally, we fix a non-negative finite-valued function f whose domain is

*One could take $\mathfrak{F}_0$ as a class of subsets of a space other than S without affecting the validity of any results of § 2.

$\mathfrak{F}_0$. With f we associate the function $\bar{f}$ whose domain is the class of all subsets of S, so defined that whenever $A \subset S$,

$$\bar{f}(A) = \sup_{t \in T} \left\{ \inf_{\mathfrak{G} \in \mathbf{M}(A,\, t)} \sum_{\gamma \in \mathfrak{G}} f(\gamma) \right\}.$$

We propose to investigate some of the properties of $\bar{f}$.

2.1. LEMMA. *If $A \subset S$, $\epsilon > 0$, and $t \in T$, then there exists a family $\mathfrak{K} \in \mathbf{M}(A, t)$ for which*

$$\sum_{\alpha \in \mathfrak{K}} f(\alpha) \leqslant \bar{f}(A) + \epsilon$$

Proof. Evidently

$$\inf_{\mathfrak{G} \in \mathbf{M}(A,\, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant \bar{f}(A);$$

hence there exists a family $\mathfrak{K} \in \mathbf{M}(A, t)$ for which, as required,

$$\sum_{\alpha \in \mathfrak{K}} f(\alpha) \leqslant \left(\inf_{\mathfrak{G} \in \mathbf{M}(A,\, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \right) + \epsilon \leqslant \bar{f}(A) + \epsilon.$$

2.2. THEOREM. *$\bar{f}$ measures S.*

Proof. From A2 above we clearly have

$$0 \leqslant \inf_{\mathfrak{G} \in \mathbf{M}(0,\, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant \sum_{\alpha \in 0} f(\alpha) = 0$$

for each $t \in T$; consequently $\bar{f}(0) = 0$.

If $A \subset B \subset S$, then from A3, we see that for each $t \in T$,

$$0 \leqslant \inf_{\mathfrak{G} \in \mathbf{M}(A,\, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant \inf_{\mathfrak{G} \in \mathbf{M}(B,\, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant \bar{f}(B),$$

whence it follows that $0 \leqslant \bar{f}(A) \leqslant \bar{f}(B)$.

Finally, we consider a finite or countably infinite family $\mathfrak{H}$ of subsets of S. Given $\epsilon > 0$, we select any convenient positive-valued function η with domain $\mathfrak{H}$ for which

$$\sum_{\gamma \in \mathfrak{H}} \eta(\gamma) < \epsilon. \tag{1}$$

If $t \in T$, we may use A4 to select a function H on $\mathfrak{H}$ to T, enjoying the properties therein specified. Further, we may use Lemma 2.1 to find such a function Q that for each $\gamma \in \mathfrak{H}$,

$$Q(\gamma) \in \mathbf{M}(\gamma, H(\gamma)), \sum_{\alpha \in Q(\gamma)} f(\alpha) \leqslant \bar{f}(\gamma) + \eta(\gamma). \tag{2}$$

We let $\mathfrak{K} = \bigcup_{\gamma \in \mathfrak{H}} Q(\gamma)$. From (2) and the choice of H under A4 we infer that

$$\mathfrak{K} \in \mathbf{M}(\bigcup \mathfrak{H}, t). \tag{3}$$

Thus, using (1), (2), and (3), we see that

$$\inf_{\mathfrak{G} \in \mathbf{M}(\cup \mathfrak{H}, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant \sum_{\alpha \in \mathfrak{K}} f(\alpha) \leqslant \sum_{\gamma \in \mathfrak{H}} \left(\sum_{\alpha \in Q(\gamma)} f(\alpha) \right)$$
$$\leqslant \sum_{\gamma \in \mathfrak{H}} (\bar{f}(\gamma) + \eta(\gamma)) < \sum_{\gamma \in \mathfrak{H}} \bar{f}(\gamma) + \epsilon.$$

From the arbitrary nature of both t and ϵ in this last relation, we conclude that

$$\bar{f}(\cup \mathfrak{H}) \leqslant \sum_{\gamma \in \mathfrak{H}} \bar{f}(\gamma),$$

thus completing the proof of the theorem.

3. A specialization of the general theory. In order to obtain further specific information concerning $\bar{f}$ we shall observe the effect on $\bar{f}$ when tee functions $\mathbf{M}$ and f are subjected to certain restrictions in addition to those already imposed.

For the remainder of this section we shall assume that T is directed by $\prec$ and that the following holds:

$A3'$. $\mathbf{M}(B, t_1) \subset \mathbf{M}(A, t_2)$ whenever $A \subset B \subset S$, $t_1 \in T$, $t_2 \in T$ and $t_2 \prec t_1$.

Clearly, A3′ is a stronger restriction than $A3$. In this case it is easy to see that for any set $A \subset S$, $\bar{f}(A)$ may be expressed as a Moore-Smith limit. For any $t \in T$, we let

$$\bar{f}_t(A) = \inf_{\mathfrak{G} \in \mathbf{M}(A, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha);$$

and then it follows that

$$\bar{f}(A) = \sup_{t \in T} \bar{f}_t(A) = \lim_{t \prec} \bar{f}_t(A).$$

3.1. Definition. *If $\mathfrak{G}$ is any family of sets and β is any set, then by $\mathfrak{G} \odot \beta$ we shall mean that family of sets $\mathfrak{G}'$ for which $\gamma \in \mathfrak{G}'$ if, and only if, $\gamma = \alpha \cap \beta$, where α is some member of $\mathfrak{G}$.*

3.2. Definition. *If $A \subset S$, then we shall write $\tilde{A} = S - A$.*

We are now ready to formulate a condition on $\mathbf{M}$ which bears on the nature of the class of $\bar{f}$-measurable sets.

A5. If $\beta \in \mathfrak{F}_0$, $A \subset S$, and $t \in T$, then $\mathfrak{G} \odot \beta \in \mathbf{M}(A \cap \beta, t)$ and $\mathfrak{G} \odot \tilde{\beta} \in \mathbf{M}(A \cap \tilde{\beta}, t)$ whenever $\mathfrak{G} \in \mathbf{M}(A, t)$.

This involves possible restrictions on the class $\mathfrak{F}_0$.

3.3. Definition. W*e shall say that f is weakly quasi-additive on its domain $\mathfrak{F}_0$ if, and only if,*

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(i) $\alpha \cap \beta \in \mathfrak{F}_0$ *and* $\alpha \cap \tilde{\beta} \in \mathfrak{F}_0$ *whenever* $\alpha \in \mathfrak{F}_0$ *and* $\beta \in \mathfrak{F}_0$;
(ii) $f(\alpha) \geqslant f(\alpha \cap \beta) + f(\alpha \cap \tilde{\beta})$ *whenever* $\alpha \in \mathfrak{F}_0$ *and* $\beta \in \mathfrak{F}_0$.

3.4. THEOREM. *If* $\mathbf{M}$ *satisfies A5 and f is weakly quasi-additive on* $\mathfrak{F}_0$, *then each member of* $\mathfrak{F}_0$ *is* $\bar{f}$*-measurable.*

Proof. We consider any set $A \subset S$, arbitrary members β and t of $\mathfrak{F}_0$ and T respectively, and an arbitrary member $\mathfrak{H}$ of $\mathbf{M}(A, t)$. Then, using A5 and Definition 2.3 we clearly have

$$
\begin{aligned}
(1) \qquad \sum_{\alpha \in \mathfrak{H}} f(\alpha) &\geqslant \sum_{\alpha \in \mathfrak{H}} f(\alpha \cap \beta) + \sum_{\alpha \in \mathfrak{H}} f(\alpha \cap \tilde{\beta}) \\
&\geqslant \sum_{\gamma \in \mathfrak{H} \odot \beta} f(\gamma) + \sum_{\gamma' \in \mathfrak{H} \odot \tilde{\beta}} f(\gamma') \\
&\geqslant \inf_{\mathfrak{G} \in \mathbf{M}(A \cap \beta,\, t)} \sum_{\gamma \in \mathfrak{G}} f(\gamma) + \inf_{\mathfrak{G}' \in \mathbf{M}(A \cap \tilde{\beta},\, t)} \sum_{\gamma' \in \mathfrak{G}'} f(\gamma') \\
&= \bar{f}_t(A \cap \beta) + \bar{f}_t(A \cap \tilde{\beta}).
\end{aligned}
$$

From the arbitrary nature of $\mathfrak{H}$ in (1) we obtain at once

$$
(2) \qquad \bar{f}_t(A) \geqslant \bar{f}_t\,(A \cap \beta) + \bar{f}_t\,(A \cap \tilde{\beta}).
$$

Since t is arbitrary in (2), we obtain at once from the theory of Moore-Smith limits that

$$
\begin{aligned}
\bar{f}(A) = \lim_{t, \succ} \bar{f}_t(A) &\geqslant \lim_{t, \succ} \bar{f}_t(A \cap \beta) + \lim_{t, \succ} \bar{f}_t(A \cap \beta) \\
&= f(\bar{A} \cap \beta) + \bar{f}(A \cap \tilde{\beta}),
\end{aligned}
$$

so that β is $\bar{f}$-measurable and the proof is complete.

4. Another form of specialization.

We consider now another specialization of $\mathbf{M}$ and $\mathfrak{F}_0$ which is encountered in some applications. In such cases, f is not subjected to any requirements over and above those of § 2, except as the specialization just mentioned affects its domain $\mathfrak{F}_0$.

4.1. DEFINITION. The element $t \in T$ is said to be an ϵ*-approximator* for $\bar{f}(A)$ if, and only if, $A \subset S$ and

$$
\bar{f}(A) \leqslant \sum_{\alpha \in \mathfrak{G}} f(\alpha) + \epsilon
$$

whenever $\mathfrak{G} \in \mathbf{M}(A, t)$.

4.2. LEMMA. *If* $\epsilon > 0$, $A \subset S$, and $\bar{f}(A) < \infty$ *then there exists some* $t \in T$ *which is an* ϵ*-approximator for* $\bar{f}(A)$.

Proof. It follows from the definition of $\bar{f}$ in § 2 that there exists some $t \in T$ for which

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$$\bar{f}(A) \leqslant \inf_{\mathfrak{G} \in \mathbf{M}(A,t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) + \epsilon,$$

from which it is clear that t is the desired element.

A6. There exists a relation R whose members are certain ordered pairs of subsets of S, with the following properties:

(i) whenever $A' \subset A \subset S$, $B' \subset B \subset S$, and $(A, B) \in R$, then $(A', B') \in R$;

(ii) if $A \subset S$, $B \subset S$, $C \subset S$, $(A, C) \in R$, and $(B, C) \in R$, then $(A \cup B, C) \in R$;

(iii) whenever $\bar{f}(A) < \infty$, $\bar{f}(B) < \infty$, $(A, B) \in R$, and $\epsilon > 0$, then there exist elements t, t', t'' of T such that t' and t'' are, respectively, ϵ-approximators for $\bar{f}(A)$ and $\bar{f}(B)$; in addition, whenever $\mathfrak{G} \in \mathbf{M}(A \cup B, t)$ then there exist two disjointed families $\mathfrak{G}'$ and $\mathfrak{G}''$ with $\mathfrak{G}' \in \mathbf{M}(A, t')$, $\mathfrak{G}'' \in \mathbf{M}(B, t'')$, such that $\mathfrak{G}' \cup \mathfrak{G}'' \subset \mathfrak{G}$.

4.3. THEOREM. *If* $\mathbf{M}$ *and* $\mathfrak{F}_0$ *satisfy* A6 *and* $(A, B) \in R$ *then* $\bar{f}(A \cup B) = \bar{f}(A) + \bar{f}(B)$.

Proof. Since $\bar{f}$ measures S by Theorem 2.2, we need prove only that $\bar{f}(A \cup B) \geqslant \bar{f}(A) + \bar{f}(B)$. Since this inequality clearly holds if either $\bar{f}(A)$ or $\bar{f}(B)$ is infinite, we may restrict ourselves to the case where $\bar{f}(A)$ and $\bar{f}(B)$ are both finite. We take $\epsilon > 0$, determine elements t, t', t'' of T in accordance with A6 (iii), and note that

$$\inf_{\mathfrak{H} \in \mathbf{M}(A \cup B, t)} \sum_{\alpha \in \mathfrak{H}} f(\alpha) \leqslant \bar{f}(A \cup B). \tag{1}$$

We select $\mathfrak{G} \in \mathbf{M}(A \cup B, t)$ so that

$$\inf_{\mathfrak{H} \in \mathbf{M}(A \cup B, t)} \sum_{\alpha \in \mathfrak{H}} f(\alpha) \geqslant \sum_{\alpha \in \mathfrak{G}} f(\alpha) - \epsilon. \tag{2}$$

Applying A6 (iii) we determine two disjointed families $\mathfrak{G}'$ and $\mathfrak{G}''$ with $\mathfrak{G}' \in \mathbf{M}(A, t')$, $\mathfrak{G}'' \in \mathbf{M}(B, t'')$, $\mathfrak{G}' \cup \mathfrak{G}'' \subset \mathfrak{G}$, put (1) and (2) together and use the nature of the choices of t' and t'' to obtain

$$\begin{aligned} \bar{f}(A \cup B) &\geqslant \sum_{\alpha \in \mathfrak{G}} f(\alpha) - \epsilon \\ &\geqslant \sum_{\alpha \in \mathfrak{G}'} f(\alpha) + \sum_{\alpha \in \mathfrak{G}'} f(\alpha) - \epsilon \\ &\geqslant (\bar{f}(A) - \epsilon) + (\bar{f}(B) - \epsilon) - \epsilon = \bar{f}(A) + \bar{f}(B) - 3\epsilon. \end{aligned} \tag{3}$$

The arbitrary nature of ϵ permits the desired conclusion.

A useful result which may be proved by simple induction based on A6 (ii) is stated in the following lemma.

4.4. LEMMA. *If* $B \subset S$, $A_i \subset S$, *and* $(A_i, B) \in R$ *for* $i = 1, 2, \ldots, n$, *then*

$$\left(\bigcup_{i=1}^{n} A_i, B\right) \in R.$$

We now state another useful result which follows by simple induction on Theorem 4.3 and Lemma 4.4.

4.5. THEOREM. *If* **M** *and* $\mathfrak{F}_0$ *satisfy* $A6$, $A_i \subset S$ *for* $i = 1, 2, \ldots, n$, *and* $(A_i, A_j) \in R$ *for each pair of positive integers* i *and* j *such that* $1 \leqslant i < j \leqslant n$, *then*

$$\bar{f}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \bar{f}(A_i).$$

4.6. LEMMA. *If* **M** *and* $\mathfrak{F}_0$ *satisfy* $A6$, $E \subset S$, $\beta \subset S$, *and there exists a positive integer* p *and a sequence of sets* Q *such that*

(a) $\beta = \bigcup_{i=1}^{\infty} Q_i$;

(b) $(Q_i, \tilde{\beta}) \in R$ *for each positive integer* i;

(c) $(Q_i, Q_{i+pj}) \in R$ *for each positive integer* i *and* j,

then

$$\bar{f}(\beta \cap E) = \lim_n \bar{f}\left(\bigcup_{i=1}^{n} (Q_i \cap E)\right).$$

Proof. For each positive integer n, we let

$$A_n = \bigcup_{i=1}^{n} (Q_i \cap E) \subset \beta \cap E,$$

from which it follows that

$$\lim_n \bar{f}(A_n) \leqslant \bar{f}(\beta \cap E). \tag{1}$$

Now for each positive integer i, $0 \leqslant i \leqslant p - 1$, we consider the sets of the form Q_{i+pj}, $j = 1, 2, \ldots$. From (c) of the hypotheses, we see that $(Q_{i+pj}, Q_{i+pj'}) \in R$ whenever $1 \leqslant j < j'$, so that Theorem 4.5 yields

$$\sum_{j=1}^{n} \bar{f}(Q_{i+pj} \cap E) = \bar{f}\left(\bigcup_{j=1}^{n} (Q_{i+pj} \cap E)\right) \leqslant \bar{f}(A_{i+pn}) \leqslant \bar{f}(\beta \cap E). \tag{2}$$

In case

$$\sum_{j=1}^{\infty} \bar{f}(Q_{i+pj} \cap E) = \infty$$

for some i, $0 \leqslant i \subset p - 1$, then we see from (2) that $\lim_n \bar{f}(A_{i+pn}) = \infty$, hence $\lim_n \bar{f}(A_n) = \bar{f}(\beta \cap E) = \infty$, whence the desired result holds. If, however,

$$\sum_{j=1}^{\infty} \bar{f}(Q_{i+qj} \cap E) < \infty$$

for each i, $0 \leqslant i \leqslant p - 1$, then for anygiven $\epsilon > 0$ there exists such a positive integer N that for each i, $0 \leqslant i \leqslant p - 1$,

$$\sum_{j=N}^{\infty} \bar{f}(Q_{i+pj} \cap E) < \epsilon/p,$$

whence it follows that

$$\sum_{n=Np}^{\infty} \bar{f}(Q_n \cap E) < \epsilon. \tag{3}$$

Since $\bar{f}$ measures S and

$$\beta \cap E = A_{Np-1} \cup \bigcup_{i=Np}^{\infty} (Q_i \cap E),$$

we have, using (1) and (3),

$$\begin{aligned}\bar{f}(\beta \cap E) &\leqslant \bar{f}(A_{Np-1}) + \sum_{i=Np}^{\infty} \bar{f}(Q_i \cap E)\\ &\leqslant \bar{f}(A_{Np-1}) + \epsilon \leqslant \lim_n \bar{f}(A_n) + \epsilon \leqslant \bar{f}(\beta \cap E) + \epsilon.\end{aligned}$$

From this last relation and the arbitrary nature of ϵ, the desired result follows.

4.7. THEOREM. *Under the hypotheses of Lemma* 4.6, β *is* $\bar{f}$*-measurable.*

Proof. Let E denote an arbitrary subset of S. Using again the notation of Lemma 4.6, we showed there that

$$\lim_n \bar{f}(A_n) = \bar{f}(\beta \cap E), A_n \subset \beta \cap E \text{ for } n = 1, 2, \ldots . \tag{1}$$

From (b) of the hypotheses of Lemma 4.6, A6 (i), the definition of A_n, and Lemma 4.4 we have

$$(A_n, \tilde{\beta} \cap E) \in R, \qquad n = 1, 2, \ldots . \tag{2}$$

Using (1), (2), and Theorem 4.3, we see that

$$\bar{f}(E) = \bar{f}(\beta \cap E) \cup (\tilde{\beta} \cap E)) \geqslant \bar{f}(A_n \cup (\tilde{\beta} \cap E)) = \bar{f}(A_n) + \bar{f}(\tilde{\beta} \cap E)$$

for each positive integer n; applying (2) we have finally

$$\bar{f}(E) \geqslant \bar{f}(\beta \cap E) + \bar{f}(\tilde{\beta} \cap E),$$

which establishes the $\bar{f}$-measurability of β.

5. Further considerations. Unrelated to questions of measurability but of interest in itself is the relationship between $\bar{f}(\beta)$ and $f(\beta)$ for $\beta \in \mathfrak{F}_0$. A relatively simple restriction on $\mathbf{M}$, valid in many applications and yielding information on this point is now given.

A7. $U(\beta) \in \mathbf{M}(\beta, t)$ whenever $\beta \in \mathfrak{F}_0$ and $t \in T$.

5.1. THEOREM. *If* $\mathbf{M}$ *satisfies* A7 *then* $\bar{f}(\beta) \leqslant f(\beta)$ *whenever* $\beta \in \mathfrak{F}_0$.

Proof. For each $t \in T$ we have

$$\inf_{\mathfrak{G} \in \mathbf{M}(\beta, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \leqslant f(\beta);$$

thus $\bar{f}(\beta) \leqslant f(\beta)$.

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Although $\mathbf{M}$ satisfies A1–A4 as well as A7 in the theorem just proved, only A1 and A7 and the definition of $\bar{f}$ enter the proof.

To reverse the inequality just proved requires relatively specific properties of both $\mathbf{M}$ and f, and there appears to be no clear-cut restriction valid in most applications which yields the desired result. We do obtain the following result, however.

5.2. THEOREM. If $\mathbf{M}$ *satisfies* A7 *then a necessary and sufficient condition that* $\bar{f}(\beta) = f(\beta)$ *for each* $\beta \in \mathfrak{F}_0$ *is that for each* $\epsilon > 0$, *there exists some* $t_0 \in T$ *for which*

$$f(\beta) \leqslant \sum_{\alpha \epsilon \mathfrak{G}} f(\alpha) + \epsilon$$

whenever $\mathfrak{G} \in \mathbf{M}(\beta, t_0)$.

Proof. If the stated condition holds, we clearly have

$$f(\beta) \leqslant \inf_{\mathfrak{G} \epsilon \mathbf{M}(\beta,\, t_0)} \sum_{\alpha \epsilon \mathfrak{G}} f(\alpha) + \epsilon \leqslant \bar{f}(\beta) + \epsilon; \tag{1}$$

since ϵ is arbitrary then $f(\beta) \leqslant \bar{f}(\beta)$. The reverse inequality follows from Theorem 5.1.

If $f(\beta) = \bar{f}(\beta)$ and $\epsilon > 0$, then by the definition of $\bar{f}(\beta)$, there exists $t_0 \in T$ for which, since f is finite-valued,

$$f(\beta) = \bar{f}(\beta) \leqslant \inf_{\mathfrak{G} \epsilon \mathbf{M}(\beta,\, t_0)} \sum_{\alpha \epsilon \mathfrak{G}} f(\alpha) + \epsilon,$$

from which follows at once the stated condition.

6. Applications of the theory.

6.1. The most common application of the theory just set forth consists of the case where S, $\mathfrak{F}_0$, and f satisfy the conditions of § 2, and for any set $A \subset S$, $\bar{f}(A)$ is the infimum of all numbers of the form $\Sigma_{\alpha \epsilon \mathfrak{G}} f(\alpha)$, where $\mathfrak{G}$ denotes a finite or countably infinite subfamily of $\mathfrak{F}_0$ which covers A. Thus defined, $\bar{f}$ may not immediately seem to be in conformity with the definition in § 2. However, we may take for T any set consisting of a single element, for example, $T = U(0)$, and so define $\mathbf{M}$ that for each set $A \subset S$ and each $t \in T$, $\mathbf{M}(A, t) = \mathbf{M}(A, 0)$ consists of all finite or countably infinite subfamilies of $\mathfrak{F}_0$ which cover A. It is easily verified that A1–A4 are satisfied; in A4 one takes $H(\gamma) = 0 \in T$ for each $\gamma \in \mathfrak{H}$. With this interpretation it is also easily seen that $\bar{f}$ as defined in § 2 agrees with the definition given more briefly above. It is clear that $U(\beta) \in \mathbf{M}(\beta, t)$ for each $t \in T$ and each $\beta \in \mathfrak{F}_0$, hence by Theorem 5.1, $\bar{f}(\beta) \leqslant f(\beta)$ whenever $\beta \in \mathfrak{F}_0$.

In case f is a measure* and $\mathfrak{F}_0$ is a ring of sets, then A3′ and A5 hold and

*Here we use the term *measure* as defined in (**3**, p. 30).

Theorem 3.4 assures us that $\mathfrak{F}_0$ is contained in the class of $\bar{f}$-measurable sets. Furthermore, the criterion formulated in Theorem 5.2 is fulfilled, and consequently $\bar{f}$ agrees with f on $\mathfrak{F}_0$.

6.2. We now consider a second well-known method of defining a measure, where S is a metric space, $\mathfrak{F}_0$ is an arbitrary non-empty family of subsets of S, and f satisfies the requirements of § 2. Here we take T to be the set of all positive real numbers. For each set $A \subset S$ and each $t \in T$ we define $\mathbf{M}(A, t)$ to be the collection (possibly empty) of all finite or countably infinite subfamilies of $\mathfrak{F}_0$ which cover A, each of whose members is of diameter less than t. It is easily checked that A1–A4 are satisfied; in A4 one may take $H(\gamma) = t$ for each $\gamma \in \mathfrak{H}$. One may so define $\bar{f}_t$ for a given $t \in T$ that for each $A \subset S$

$$\bar{f}_t(A) = \inf_{\mathfrak{G} \in \mathbf{M}(A, t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha),$$

whence in accordance with the definition in § 2,

$$\bar{f}(A) = \sup_{t \in T} \bar{f}_t(A).$$

It is easily checked that for any given $t \in T$, $\bar{f}_t$ measures S; in fact, with t fixed, $\bar{f}_t$ is itself merely an example of the kind discussed in 6.1, with the added restriction on the covering families that their members be of diameter less than t. Also, in this case $\mathbf{M}$ satisfies condition A3′ of § 3 with $t_2 \prec t_1$ if, and only if, $t_2 \geqslant t_1$, and accordingly for any set $A \subset S$ we may write

$$\bar{f}(A) = \sup_{t \in T} \bar{f}_t(A) = \lim_{t \to 0+} \bar{f}_t(A),$$

which is the usual form of representation for $\bar{f}$.

If we introduce the relation

$$R = \{(A, B) : A \subset S, B \subset S, \text{ and } \operatorname{dist}(A, B) > 0\},$$

then A6 is valid for $\mathbf{M}$ and $\mathfrak{F}_0$. For A6 (i) and (ii) clearly hold. If $\bar{f}(A) < \infty$, $\bar{f}(B) < \infty$, $\epsilon > 0$, and $(A, B) \in R$, then one can evidently find $t \in T$ such that

$$0 < t < \operatorname{dist}(A, B),\ \bar{f}_t(A) > \bar{f}(A) - \epsilon,\ \bar{f}_t(B) > \bar{f}(B) - \epsilon,$$

from which t is seen to be an ϵ-approximator for both $\bar{f}(A)$ and $\bar{f}(B)$. If $\mathfrak{G} \in \mathbf{M}(A \cup B, t)$, we may define $\mathfrak{G}'$ as the subfamily of $\mathfrak{G}$ whose members intersect A, $\mathfrak{G}''$ as the remainder of $\mathfrak{G}$, and then $\mathfrak{G}' \in \mathbf{M}(A, t)$, $\mathfrak{G}'' \in \mathbf{M}(B, t)$, so that A6 (iii) holds.

If β is any open set and i is any positive integer, we may define $Q_i = \{x : 1/(1 + i) \leqslant \operatorname{dist}(x, \tilde{\beta}) < 1/i\}$, from which it follows easily that the hypotheses (a), (b), and (c) of Lemma 4.6 hold, with $p = 2$. Thus Theorem 4.7 applies and all open sets are $\bar{f}$-measurable, thus so are all Borel sets as well.

As a special example of the above, we mention the case where $\mathfrak{F}_0$ consists of all open sets in S, q is a fixed positive number, and

$$f(\beta) = (\operatorname{diam} \beta)^q$$

whenever $\beta \in \mathfrak{F}_0$. The resulting $\bar{f}$ in this case is Hausdorff q-dimensional measure; all Borel subsets of S are $\bar{f}$-measurable. Since A7 holds only for the null set and sets consisting of a single point, one cannot expect to make any general statement concerning the relative magnitudes of $f(\beta)$ and $\bar{f}(\beta)$ when $\beta \in \mathfrak{F}_0$.

Lebesgue measure in n-dimensional Euclidean space may be regarded as a further example of the above. One may take $\mathfrak{F}_0$ as the class of all open intervals in S, and f as that function on $\mathfrak{F}_0$ whose values are the hypervolumes of members of $\mathfrak{F}_0$. As above, one may first define $\bar{f}_t$ for each $t > 0$ and then take

$$\bar{f} = \lim_{t \to 0+} \bar{f}_t.$$

Due to the nature of Euclidean space, $\bar{f}_t$ does not depend upon the value of t; thus $\bar{f} = \bar{f}_\infty$. However $\bar{f}_\infty$ is clearly a measure-function of the type discussed in 6.1, with $\bar{f}_\infty(\beta) \leqslant f(\beta)$ for each $\beta \in \mathfrak{F}_0$. Furthermore $\bar{f}_\infty$ is in conformity with the usual definition of Lebesgue outer measure. Thus $\bar{f}$ is ordinary Lebesgue outer measure and $\bar{f}(\beta) \leqslant f(\beta)$ for each $\beta \in \mathfrak{F}_0$. Since f satisfies the criterion of Theorem 5.2, we have in fact $\bar{f}(\beta) = f(\beta)$ for each $\beta \in \mathfrak{F}_0$. From above we see that all members of $\mathfrak{F}_0$ are $\bar{f}$-measurable, and consequently so are all Borel subsets of S.

6.3. A somewhat more complicated situation occurs in the generation of Borel measures. Here* we have a locally compact Hausdorff space S, the class $\mathfrak{C}$ of all compact subsets of S, and the class $\mathfrak{U}$ of all open sets belonging to $\mathfrak{C}$. The function λ is a *content* defined on $\mathfrak{C}$, and from λ is derived the function λ_* on $\mathfrak{U}$, so defined that for $G \in \mathfrak{U}$,

$$\lambda_*(G) = \sup_{G \supset C \in \mathfrak{C}} \lambda(C).$$

It turns out that λ^* is subadditive on $\mathfrak{U}$! From λ^* is derived the outer measure μ^*, where for $A \subset S$,

$$\mu^*(A) = \inf_{A \subset G \in \mathfrak{U}} \lambda_*(G).$$

To show that μ^* may be interpreted as an application of our general theory, we take $\mathfrak{F}_0 = \mathfrak{U}$, $T = U(0)$; for each set $A \subset S$ and $t = 0 \in T$ we take $\mathbf{M}\ (A, t)$ as the class of all finite or countably infinite subfamilies $\mathfrak{G}$ of $\mathfrak{U}$ which cover A. As in 6.1 it follows that A1–A4 are satisfied. From the subadditivity of λ_* in $\mathfrak{U}$ and the fact that $U(G) \in \mathbf{M}\ (A, 0)$ for each $G \in \mathfrak{U}$ such that $A \subset G$, it follows readily that

*The notation and terminology of (3, pp. 231–6) are used in this section.

$$\mu^*(A) = \inf_{A \subset G \in \mathfrak{U}} \lambda_*(G) = \inf_{\mathfrak{G} \in \mathbf{M}(A,0)} \sum_{\alpha \in \mathfrak{G}} \lambda_*(\alpha)$$
$$= \sup_{t \in T} \inf_{\mathfrak{G} \in \mathbf{M}(A,t)} \sum_{\alpha \in \mathfrak{G}} \lambda_*(\alpha),$$

so that μ^* is an outer measure by virtue of Theorem 2.2. Since A5 and the hypotheses of Theorem 5.2 are valid we have $\mu^*(G) = \lambda_*(G)$ whenever $G \in \mathfrak{U}$. In this example, the measurability of the Borel sets and the fact that μ^* induces a regular measure μ follow from the particular properties of λ and are not taken up here.

6.4. Other applications arise in the theory of differentiation of set functions. In these situations, $\mathfrak{F}_0$ is usually the spread of a blanket F, and f satisfies conditions considerably stronger than those of § 2. We let ϕ denote a function which measures S. For each $A \subset S$ one may define $\bar{f}(A)$ as the infimum of numbers of the form $\sum_{\alpha \in \mathfrak{G}} f(\alpha)$, where $\mathfrak{G}$ denotes a finite or countably infinite subset of $\mathfrak{F}_0$ for which $\phi(A - \cup \mathfrak{G}) = 0$ (that is, $\mathfrak{G}$ is a 0-covering of A). By making a rather obvious modification in the definition of M as given in 6.1, it is easy to check that the conditions A1–A4 are satisfied and $\bar{f}$ measures S. Another function $\bar{f}$ arises in some situations. Here for each fixed $t > 0$ and each set $A \subset S$ we define $\bar{f}_t(A)$ as the infimum of numbers of the form $\sum_{\alpha \in \mathfrak{G}} f(\alpha)$, where $\mathfrak{G}$ is a finite or countably infinite subset of $\mathfrak{F}_0$ for which $\phi(A - \cup \mathfrak{G}) < t$ (that is, $\mathfrak{G}$ is a t-covering of A). Then one defines

$$\bar{f}(A) = \lim_{t \to 0+} \bar{f}_t(A).$$

If one takes for T the set of all positive real numbers, and so defines $\mathbf{M}$ that for each set $A \subset S$ and each $t \in T$, $\mathbf{M}(A, t)$ is the collection of all t-coverings of A, then it is not difficult to show that $\mathbf{M}$ satisfies A1–A4. In verifying A4, when $t \in T$ and $\mathfrak{H}$ are given, one chooses a positive-valued function H such that $\sum_{\gamma \in \mathfrak{H}} H(\gamma) < t$. If $\gamma \in \mathfrak{H}$ we choose any family $Q(\gamma) \in \mathbf{M}(\gamma, H(\gamma))$ and let $\mathfrak{Q} = \cup_{\gamma \in \mathfrak{H}} Q(\gamma)$, then $\phi(\gamma - \cup Q(\gamma)) < H(\gamma)$ whenever $\gamma \in \mathfrak{H}$ and

$$\phi(\cup \mathfrak{H} - \cup \mathfrak{Q}) \leqslant \phi\Big(\bigcup_{\gamma \in \mathfrak{H}} (\gamma - \cup Q(\gamma))\Big) \leqslant \sum_{\gamma \in \mathfrak{H}} \phi(\gamma - Q(\gamma)) < \sum_{\gamma \in \mathfrak{H}} H(\gamma) < t;$$

thus $\mathfrak{Q} \in \mathbf{M}(\cup \mathfrak{H}, t)$ and A4 holds.

In this example, condition A3′ of § 3 is again valid, which makes possible the equation

$$\bar{f}(A) = \lim_{t \to 0+} f_t(A) = \sup_{t \in T} \Big\{ \inf_{\mathfrak{G} \in \mathbf{M}(A,t)} \sum_{\alpha \in \mathfrak{G}} f(\alpha) \Big\}.$$

In the above, applications to differentiation theory A7 holds, so that by Theorem 5.1, $\bar{f}(\beta) \leqslant f(\beta)$ whenever $\beta \in \mathfrak{F}_0$.

As was mentioned earlier, f and F are usually subject to various technical requirements which make it possible to use the more specialized parts of the above theory to obtain information concerning the class of $\bar{f}$-measurable sets and other matters. These are too lengthy for consideration here.

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University of California,
Davis, California

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