Demythologising Mathematics: Gödel, Escher, Bach Revisited

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Why should a journal like *New Blackfriars* be looking retrospectively at books about mathematics? A short answer is that mathematics, or at least the myth of mathematics, affects the way we think. I shall be looking at two books which demythologise mathematics: *The Mathematical Experience*[1] and *Mathematics: The Loss of Certainty*[2] by Morris Kline. These help examine the myths of proof—the myths about how mathematicians think. Part of the demythologisation process is to look at alternatives to the myths; in particular I shall look at the role of intuition, and will illustrate this with reference to *Catastrophe Theory*[3] and *Gödel, Escher, Bach*[4].

The prompting for this investigation is the emergence of a new genre of books about mathematics following in the wake of *Gödel, Escher, Bach,* of which *The Mathematical Experience* is an example. It would be a fascinating aside to trace the rise of this genre from *Fourteen Funny Bits of Mathematics in Avuncular Style* as found on the shelves of most public libraries, alongside popularisations of pedestrian seriousness. No doubt it would mention the science-as-a-human-activity school as exemplified by Koestler's *The Sleepwalkers*, or Watson's *The Double Helix*, a vein continued by *Catastrophe Theory*. It would take a venture into catastrophe theory itself to explain the emergence of *Gödel, Escher, Bach*—a rogue book which won the Pulitzer Prize in 1980 for general non-fiction, a picaresque which takes in the art of fugue, Zen koans, and the Jabberwocky and leads to questions at the heart of some of the problems in the foundations of mathematics and of artificial intelligence, while remaining accessible to a child of twelve (albeit the one I know is quite bright). The audience the genre aims at seems to be not so much the interested layman, but more the new technical elite whose computer skills and mathematical awareness can be used to outflank the experience of their seniors. In my years in radar design it was interesting to watch the gradual take-over of certain departments by mathematicians in spite of the active opposition of the electronics-trained managers. In my last department over half the people there had copies of *Gödel, Escher, Bach*.

The starting point for this investigation is to look at the myth of mathematics—a myth that portrays mathematics as a discipline

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which, starting from self-evident premises, proceeds with inexorable logic to uncover certainties about the world—if mathematics proves it, then it must be true. We start absorbing this myth at primary school—or now even earlier—when we are trained that 1 + 1 = 2. Later we absorb our paradigm of proof, Euclid's Geometry—all of which seems perfectly natural. And, as ever, it is the apparently natural which most strongly straight-jackets the way we look at the world, if only because we do not think things could be otherwise than how we perceive them.

To follow through the history of mathematics is to see both the rise of the myth and the developments that led to its falsification. Both *The Mathematical Experience* and *Mathematics: The Loss of Certainty* follow this approach. The former is a parson's egg of a book, published I would guess to cash in on the excitement generated by *Gödel, Escher, Bach*, in which the authors seem to have written about what they know without any particular readership in mind, with the result that the style varies from over-simple to sections in which a degree in mathematics is essential. A symptom of its inadequacies is the disproportionate size of the book in relation to its contents—it uses an exceptionally large typeface and very wide margins which occasionally contain minute pictures of people referred to—a book to be mined rather than read. In tracing the history, Kline is easier to follow, giving a solid account (in both the good and bad sense), albeit in an overliterary style. His particular virtue is the abundance of quotations from mathematicians and philosophers throughout the history of mathematics, although this is offset by their being inadequately cited, and so references below will be to Kline, rather than to the original authors.

The story starts—almost inevitably—with the Greeks, and with geometry, when in about 600 B.C. Thales proved that a diameter of a circle divides it into two equal parts—perhaps the most significant innovation in Western thought. The Greek's developments in geometry are epitomised by Euclid's geometry, which for two millennia provided the paradigm of proof, and indeed still today creates many people's expectation of what counts as proof, and what its significance is.It is worth looking in some detail at the geometries' starting point, since it contains both the seed of the myth of mathematics, and the flaw which led to the destruction of the myth.

Euclid's system starts from five axioms, which are held to be self-evident. The first four may be simply stated:

(i) A straight line may be drawn between any two points.
(ii) Any straight line may be extended indefinitely.
(iii) A circle may be drawn with any point as its centre, and with any given radius.
(iv) All right angles are equal.

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The fifth axiom requires a more complex formulation, but is equivalent to our expectation that 'Parallel lines never meet', Euclid avoiding the problem caused by infinity by saying something like 'non-parallel lines meet somewhere'. From these simple beginnings simple results are obtained, and then complex theorems are built up, so that eventually even such complex theorems as Pythagoras's may be obtained.

The role of the diagram or figure in the proof is significant. Although it is never formally part of the proof, it is an essential aid to visualisation. Its thick lines and large dots represent only very roughly the 'ideal figure', and it is this view that gives rise to the 'Platonic' view of mathematics—i.e. that mathematics gives a privileged view of the true nature of the world, and that the objects of mathematics have a real existence. "The knowledge which geometry aims at is knowledge of the eternal and not of aught perishing or transient" (Plato *The Republic* [Kline p. 16]).

Let us skip directly to the seventeenth century, where we find Descartes, a mathematician and philosopher, saying in his 'Meditations' (1641) "I count as most certain the truths which I conceived clearly as regards figures, numbers, ... and in general abstract mathematics. ...Only mathematicians contrived to reach certainty and evidence, since they started from what is easiest and simplest" (Kline p. 42). Descartes (1596—1650) might be taken as a spokesman for his age, which led on to that of Pascal (1623—1662) and Leibniz (1646—1716), who both combined the roles of philosopher and mathematician.

Eighteenth-century science confirmed the view that mathematics gave a privileged insight into the nature of the universe through the very success of mathematics in framing the laws of nature, especially through Newton's work on gravitation and planetary motion. In accepting this premise, the thrust of natural science turned away from finding the causes of phenomena to concentrate more on describing them in a precise mathematical way. This is still largely the case today, and the distrust of the social sciences is perhaps due to their failure to produce neat mathematical laws—even if this is a false expectation.

The eighteenth century was the high point of the myth of mathematics—where mathematics was used to deduce truths about the world. The simplicity of the starting points of mathematics and the certainty of its argument vouched for the certainty of scientific knowledge. A myth so long and so deeply held could hardly be expected to disappear at once after it had been discredited. Unfortunately the process by which it was discredited requires much greater mathematical sophistication than the process by which the myth was inculcated, and the myth still lies at the heart of science and mathematics teaching today, being overlayed rather than displaced by modern views.

The investigations which led to the discrediting of the myth had begun before the eighteenth century, and although work was done on the problem throughout the century, it was not until the nineteenth century that the significance of the work was realised. The itch that mathematicians felt the need to scratch was Euclid's fifth axiom; in particular, its complexity when compared to the other four. Attempts to improve formulation, or to deduce the fifth axiom from the other four, were unsuccessful, and so a favourite tool was brought to bear—the *reductio ad absurdum*. If an axiom which contradicted the fifth axiom was used to construct geometry, then it was thought that a contradiction would be found, and so the necessity of the fifth axiom would be proved. Several mathematicians thought that they had found the contradiction, but later mathematicians realised they had not, and were in fact constructing a new geometry. The acceptance of this new non-Euclidean geometry began around 1830, and to Gauss (1777—1855) is attributed the insight that the new geometry could be physically significant, and that deciding which geometry applied to the world was a matter of experiment. One of the implications of the new geometry was that the angles of a triangle would add up to less than 180°, and there is a story that Gauss sent people up three mountains to do the experiment. Like most good stories, it is true but for the facts, for which see Kline p. 85. The curvature of space, which demonstrates that the geometry of space is non-Euclidean, was not shown until 1919, in some observations of the solar eclipse concerned with testing the theory of relativity.

The discovery of non-Euclidean geometry, coupled with the construction of quarterions by Hamilton (1805—1865) in 1843 led mathematicians to re-examine the foundations of their subject. In geometry not only was there no reason for preferring one or other version of the fifth axiom, but it was found that the logical methods, even when used, were rather too informal, and that a number of axioms had slipped in unnoticed. Arithmetic was in a far worse state and had virtually no foundations at all. The collapse of the myth disturbed few outside the world of mathematics, and even mathematicians thought that with a proper reconstruction job they could leave most of mathematics intact. The rebuilding work went ahead vigorously and apparently rigorously, so that by 1900 Poincaré (1854—1912) could boast to the Second International Conference "One may say today that absolute rigor has been attained". Kline follows this quotation with another from Voltaire's *Candide*, in which the philosopher Dr Pangloss, even when he is about to be hanged, says "This is the best of all possible worlds".

Kline identifies four separate schools which arose at the beginning of the twentieth century: the formalists, led by Hilbert,

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claimed that mathematics is the manipulation of meaningless or uninterpreted symbols according to the rules of logic; the intuitionism of Brouwer required that mathematics be intuitively sound, rejected the axiom of the excluded middle and hence the method of *reductio ad absurdum*, and required that a claim of the existence of something required its construction; the logicist school, whose ideas are found in Russell and Whitehead's *Principia Mathematica*, sought to construct mathematics from logic, although some of the axioms required could not be justified as pertaining strictly to logic; the set-theoretic school, which is often lumped together with the logicist school, sought to make a similar construction starting from set theory. These schools disputed not only the nature of mathematics, but also what constituted valid proof. Then after 1930 things got worse. In 1931 Kurt Gödel (1906—1978) showed "the consistency of any mathematical system that is extensive enough to embrace even the arithmetic of the whole numbers cannot be established by the logical principles adopted by the several foundation schools, the logicists, the formalists and the set-theorists". (Kline p. 261).

After this point in the story Kline ceases to narrate history and becomes part of the debate, wishing to move mathematics away from the problems of its foundations and back to solving scientific problems. He wants mathematical theories to be treated in the same way as scientific theories—that is, to be treated not as proven eternal facts but as contingent, to be accepted only until they are falsified. Even logical principles are to be taken as induction from experience and possibly falsifiable. He is not alone in this view, and the development of this view is discussed further in *The Mathematical Experience*.

It is worth asking at this point what in fact do mathematicians think they are doing. *The Mathematical Experience* takes the line that although for the purposes of philosophical discussion most mathematicians claim to be formalists, in practice they act as if they were Platonists, that is, they think they are discovering rather than inventing new theories, which implies that they believe in a mathematical reality external to themselves. To get to the heart of what they are doing we have to investigate 'mathematical intuition'.

In films the cipher for an intellectual is the chess-player—the man who thinks out each step carefully and logically—an image that might also be appropriate for the mathematician. However, players' performance in ten-seconds-per-move games is closely correlated to their performance in more leisurely games, and moves are made on the basis of experience or aesthetics, with the help of a few precepts such as 'control the centre', or 'maintain the pace'. Similarly, mathematical intuition is the driving force in new mathematics, and suggests approaches to problems. It is built around mathematical ideas and allows these ideas to be manipulated. Admittedly it is a strange sort of intuition, which is developed by learning to believe at least three impossible things before breakfast everyday (to quote one mathematician), but without it mathematics is virtually impossible. The problem with intuition is that it is likely to be wrong, and logic and rigorous argument are its policemen, just as experimental results police scientific theories.

Only with some grasp of what is meant by intuition is it possible to understand how mathematicians think, and although *The Mathematical Experience* discusses intuition, it fails to give a clear idea of what it means, and at this point the reader may find it helpful to turn to *Catastrophe Theory*. Catastrophe theory was developed in the 1960's primarily by Thom of the French Institute for Advanced Scientific Studies and Zeeman, then at Cambridge, both of whom are described as sharing a belief "in the importance of spatial intuition" (p. 29). The theory is intended to provide a mathematical description of discontinuities in the behaviour of phenomena physical, biological, psychological etc. For example, if a load is slowly increased on a steel beam, at first the beam bends in proportion to the load, but at some point it buckles and behaves quite differently, and the objective of the catastrophe theory is to describe the transition between the two ways of behaving. The main mathematical significance of the theory is concerned with the topological proof of the uniqueness of the curves used to describe the catastrophe, a subject rather too complex to deal with in an introductory book. For our present purposes, the book illustrates two things: firstly, it shows how a new mathematical theory is produced, although the account is little more than a sketch; secondly, it shows the use of visualisation, a frequent element of mathematical intuition. Half the book is given over to examples of applications of the theory, showing how the various curves the theory produces may be used to describe a wide range of phenomena.

Thinking visually is not the only technique used by mathematicians, and where *Catastrophe Theory* gives a mild taste of the possibilities of mathematical intuition, *Gödel, Escher, Bach* is Vindaloo. One of the main themes of *Gödel, Escher, Bach* is the construction and interpretation of formal systems—part of the demonstration of Gödel's theorem for the general reader. A large part of the book is concerned with constructing the intuition need to understand Gödel's proof. Mathematicians value proofs such as this for the ideas they contain, and are little interested in the logic of the proof in itself—if it were the other way round it would be as if students of literature read books to enjoy the grammatical correctness of sentences.

Each chapter of the book is prefaced by a dialogue, usually a comic exchange between Achilles and the tortoise, the dialogue often being structured in imitation of one of J.S. Bach's canons or fugues. The dialogue will illustrate the theme of the chapter, and other aids to understanding include numerous drawings—often taken from the graphic work of the Dutch artist Escher, who is known for his illustrations of spatial puzzles. One is introduced to the structure and operation of DNA, which, though interesting in itself, is there primarily to illustrate certain points of the proof and some of the philosophical problems of artificial intelligence. One of the central points of the book is the separation between a formal system and its interpretation, which brings us back to the formalist view of mathematics.

One can illustrate this by looking at numbers, or rather the various systems of numbers commonly in use. Our perception of numbers dates back to our earliest schooling, when we were trained to count. Looking back, counting seems very natural—in fact, the numbers 1, 2, 3, ... are usually called the natural numbers—and naturalness is a warning to look for deeply embedded conventions. For from counting we learn arithmetic as if it were a speeded-up version of counting, pick up the concept of zero, and fill in the gaps with fractions. After that we learn how to extend the system to get the integers (...,-3, -2, -1, 0, 1, 2, 3, ...) or rather we learn that the natural numbers are just part of a bigger system. In fact mathematicians treat these various stages as different systems with different properties, and even have a special notation for each system, there being only a family resemblance between the various systems covered by the term 'number'. These systems are used in different contexts—we count apples, say, using the natural numbers, not the integers, for we cannot count negative number of apples. What about the situation when I am owed apples—I can have negative apples then? But here the situation has changed from counting apples to counting apples *which I own*—if I let this change go unremarked I may find myself having to account for irrational, transcendental and even complex apples, just to account for some of the simpler number systems.

We grasp abstract formal systems through their exemplifications—we have an intuitive understanding of what they are. What we often fail to notice when we think of numbers in the abstract is that when we learned how to use them, we also learned when to use them. Children's jokes may give a clue here—"One drop of water plus one drop of water makes one" or "Man plus woman equals three"—the joke lies in the obvious misapplication of the system.

One might wish to say that formal systems provide models of the world. In scientific theories the model is not intended as a precise reproduction of the world, but is chosen for ease of manipulation and as an aid to understanding—a model of the Gulf Stream may be

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perfectly adequate even if it assumes the Atlantic Ocean is square. The use of mathematical models is favoured by science because they provide systems that are comparatively easy to manipulate. The most influential description of scientific activity is of a three-step process: (i) examine the phenemenon and induce a hypothesis to describe it (this often involves producing a mathematical model); (ii) predict some of the consequences of the hypothesis, and do experiments to test for them; (iii) modify the hypothesis in the light of the results, then go back to step (ii).

The move suggested by Kline and others is to treat mathematics as a science open to empirical verification. I find this rather odd, since the phenomenon to be hypothesised about is mathematical intuition, and the empirical testing the proofs of mathematics. The proponents of this theory would perhaps prefer to say that the hypotheses refer to the 'Platonic world' of mathematics, but this world is perceived within intuition, rather than through it, the latter way of speaking being a *metaphor* of sight. For me, mathematics abstract is the manipulation of uninterpreted symbols by agreed rules and mathematics concrete is the intuition, though neither are separable except in language. Axioms represent bifurcation points, where mathematical systems diverge; however, since mathematics is built up with a particular interpretation in mind (just as counting apples is an interpretation of the natural numbers) it is unnecessary to know beforehand what axioms are used, and one may view the discovery of the axioms as an opportunity to create new systems.

Several things emerge from this investigation of the world of *Gödel, Escher, Bach*. Firstly, the role of proof in mathematics is not to show how things are, but to show the consistency of the system. In a more general theatre of argument it has long been accepted and often forgotten that the correctness of an argument does not imply the truth of its conclusions. One could go further and agree with Wittgenstein "For we can avoid ineptness or emptiness in our assertions only by presenting the model as what it is, as an object of comparison—as, so to speak, a measuring-rod; not as a preconceived idea to which reality *must* correspond. (The dogmatism into which we fall so easily in doing philosophy)"[5]. Secondly, the idea of formal systems, of languages detached from direct reference to the world, is becoming part of the assumptions of a significant sector of our society. To this group a 'proof of the existence of God' is an absurd notion. Thirdly, there is an audience for quite deep speculative books, "if only academics didn't write such dull books" (i.e. books aimed at other academics) or assumed everyone needed everything predigested to a pap.

The myth of mathematics was the myth of certainty through argument. The death of the myth is at last becoming public knowledge, and must lead to a questioning of old certainties based on old arguments. For a theologian this will mean having to find new answers to old questions, or rather he will need new ways of answering—logic will not be enough, a more substantial vision will need to be visible in his work.

References

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Reviews

A LITTLE WAY TO GOD, by Gaston Roberge SJ. *Gujarat Sahitya Prakash,* **Anand, India, 1984. Pp xxii + 151. $6.**

This book offers a sort of introduction to the "little way" of St Thérèse of Lisieux, focused particularly on the idea of the heart. Roberge suggests that Thérèse effectively represents a spirituality of the devotion to the Sacred Heart, in spite of her evident aversion to contemporary forms of the devotion. He also examines her idea of being in the Heart of the Church, her devotion to the Heart of Mary, and her "discovery" of the Heart of the Neighbour. There is some fairly meticulous discussion of precise Teresian texts, which gives the book a genuine solidity, in spite of the fact that the author has not made use of the critical editions which have so enlarged and facilitated our access to the saint herself and her milieu (one consequence of which is that he is unaware of the fact that it seems to have been Pauline, not Thérèse herself, who initiated the image of being Jesus' *jouet*). But the lack of historical perspective (such as we find in the fascinating books by Jean-Francois Six), and the failure to discuss adequately the real difficulties and the sometimes serious opposition which faced Thérèse make this a disappointingly bland book and leave the saint curiously disembodied. Although her increasing descent into inner darkness is alluded to, with her consequent complete identification of herself with "my brothers the sinners", the over-all impression given by this book is of a fairly commonplace and competent nun, and this belies the real significance of Thérèse.

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