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Duality for generalized problems in complex programming

D.G. Mahajan and M.N. Vartak

Weak duality and direct duality theorems are proved, under appropriate assumptions, for the following pair of programming problems in complex space:

$$\begin{aligned}&\text{minimize} \quad F(z, \bar{z}) = \operatorname{Re} f(z, \bar{z}) + \max\{\operatorname{Re} k^H z \mid k \in K\}\\ &\text{subject to} \quad Az - b + m \in S \quad \text{for some} \quad m \in M\ , \quad z \in T\ ;\end{aligned}$$

$$\text{maximize} \quad g(u, \bar{u}, v) = \operatorname{Re}\left[f(u, \bar{u}) - u^t\nabla_1 f(u, \bar{u}) - u^H\nabla_2 f(u, \bar{u}) + b^H v\right] - \max\{\operatorname{Re} m^H v \mid m \in M\}$$

$$\text{subject to} \quad -A^H v + \overline{\nabla_1 f(u, \bar{u})} + \nabla_2 f(u, \bar{u}) + k \in T^* \quad \text{for some} \quad k \in K\ , \quad v \in S^*\ .$$

The objective function may be nondifferentiable and the constraints are of a more general nature than those considered earlier by various authors. Several well-known results are shown to be special cases of the results proved here.

Introduction

Duality relations for various classes of complex programming problems have appeared in literature [1-15]. Here we establish weak duality and direct duality theorems for a pair of programming problems in complex space whose objective function and constraints are of a more general nature than those considered recently by Mond [11]. The primal, dual

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problems and the weak and direct duality theorems established in [2-15] turn out to be special cases of the results proved in this paper.

Notations and terminology

For a complex function $f(w^1, w^2)$ analytic in the $2n$ variables (w^1, w^2) at the point $(z^0, \overline{z^0}) \in C^n \times C^n$, we define

$$\nabla_1 f(z^0, \overline{z^0}) \equiv \nabla_z f(z^0, \overline{z^0}) \equiv \left(\frac{\partial f}{\partial w_i^1}(w^1, w^2)\right)_{w^1=z^0, w^2=\overline{z^0}} \quad \text{for } i = 1, \ldots, n ,$$

and

$$\nabla_2 f(z^0, \overline{z^0}) \equiv \nabla_{\overline{z}} f(z^0, \overline{z^0}) \equiv \left(\frac{\partial f}{\partial w_i^2}(w^1, w^2)\right)_{w^1=z^0, w^2=\overline{z^0}} \quad \text{for } i = 1, \ldots, n .$$

The superscripts H and t will denote complex conjugate transpose and transpose respectively, when applied to vectors or matrices. The superscript $*$ will be used to denote polar of a polyhedral cone. For $x, y \in C^n$ let (x, y) denote their inner product; that is, $(x, y) = x^H y$. A nonempty set $S \subset C^n$ is called a polyhedral cone if, for some positive integer k and $A \in C^{n\times k}$,

$$S = AR_+^k = \{Ax \mid x \in R_+^k\} ;$$

that is, S is generated by finitely many vectors (the columns of A). The polar of a polyhedral cone $S \subset C^n$ is denoted by S^* and is defined as

$$S^* = \{z \in C^n \mid w \in S \Rightarrow \text{Re } z^H w \geq 0\} .$$

A polyhedral cone in C^n is a closed convex cone.

Abrams [1] has defined convexity of a complex valued function as follows.

DEFINITION. Let $f : C^n \times C^n \to C$ and let $S \subset C$ be a closed convex cone. Then f is convex with respect to S on the manifold $W = \{(w^1, w^2) \in C^{2n} \mid w^2 = \overline{w^1}\}$ if

$$(1)\quad \lambda f\left(z^1, \overline{z^1}\right) + (1-\lambda)f\left(z^2, \overline{z^2}\right) - f\left(\lambda z^1+(1-\lambda)z^2, \lambda\overline{z^1}+(1-\lambda)\overline{z^2}\right) \in S$$
$$\text{for all } 0 \le \lambda \le 1 , \quad z^1, z^2 \in C^n .$$

When $f\left(w^1, w^2\right)$ is analytic, a condition equivalent to (1) is

$$f\left(z^1, \overline{z^1}\right) - f\left(z^2, \overline{z^2}\right) - \left(z^1-z^2\right)^t\nabla_1 f\left(z^2, \overline{z^2}\right) - \left(z^1-z^2\right)^H\nabla_2 f\left(z^2, \overline{z^2}\right) \in S .$$

If f is real and $S = R_+$ then (1) and (2) reduce to the classical definition of convexity. When referring to the objective function of a programming problem, convexity of the real part will be of interest. Thus, if $S \subset R$, the real part of an analytic function $f\left(w^1, w^2\right)$ is convex with respect to S on the manifold $W = \left\{\left(w^1, w^2\right) \in C^{2n} \mid w^2 = \overline{w^1}\right\}$ if, for any z^1, z^2,

$$(2)\quad \operatorname{Re}\left[f\left(z^1, \overline{z^1}\right)-f\left(z^2, \overline{z^2}\right)-\left(z^1-z^2\right)^t\nabla_1 f\left(z^2, \overline{z^2}\right)-\left(z^1-z^2\right)^H\nabla_2 f\left(z^2, \overline{z^2}\right)\right] \in S .$$

With $S = R_+$, (2) is the definition of convexity of a complex valued function given by Hanson and Mond [6], and Mond [11].

The complex programs considered in this paper are the following.

PROBLEM P (Primal):

$$\text{minimize}\quad F(z, \overline{z}) = \operatorname{Re} f(z, \overline{z}) + \max\left\{\operatorname{Re} k^H z \mid k \in K\right\}$$
$$(3)\quad \text{subject to}\quad Az - b + m \in S \quad \text{for some } m \in M ,$$
$$(4)\quad z \in T ;$$

PROBLEM D (Dual):

$$\text{maximize}\quad g(u, \overline{u}, v) = \operatorname{Re}\left[f(u, \overline{u})-u^t\nabla_1 f(u, \overline{u})-u^H\nabla_2 f(u, \overline{u})+b^H v\right]$$
$$- \max\left\{\operatorname{Re} m^H v \mid m \in M\right\}$$
$$(5)\quad \text{subject to}\quad -A^H v + \overline{\nabla_1 f(u, \overline{u})} + \nabla_2 f(u, \overline{u}) + k \in T^* \quad \text{for some } k \in K ,$$
$$(6)\quad v \in S^* ;$$

where $A \in C^{m\times n}$, $b \in C^m$, z and $u \in C^n$, $v \in C^m$; $K \subset C^n$, $M \subset C^m$ are bounded closed convex sets; $S \subset C^m$, $T \subset C^n$ are polyhedral cones; $f : C^{2n} \to C$ is analytic and has convex real part with respect to R_+ on the manifold

$$W = \{(w^1, w^2) \in C^{2n} \mid w^2 = \overline{w^1}\} .$$

Preliminary results

Mahajan and Vartak [8] studied the following pair of symmetric problems:

PRIMAL PROBLEM I:

$$\begin{aligned} &\text{maximize} \quad \Phi(z) = \text{Re}(c, z) + \min\{\text{Re}(z, k) \mid k \in K\} \\ &\text{subject to} \quad -Az + b - m \in S \quad \text{for some} \quad m \in M , \\ &\qquad\qquad\qquad z \in T ; \end{aligned}$$

DUAL PROBLEM II:

$$\begin{aligned} &\text{minimize} \quad \psi(y) = \text{Re}(y, b) - \min\{\text{Re}(y, m) \mid m \in M\} \\ &\text{subject to} \quad A^H y - c - k \in T^* \quad \text{for some} \quad k \in K , \\ &\qquad\qquad\qquad y \in S^* . \end{aligned}$$

They have also established, among other results, the following.

RESULT 1. The supremum of $\Phi(x)$ over the constraint set of Primal Problem I is less than, or equal to, the infimum of $\psi(y)$ over the constraint set of Dual Problem II.

RESULT 2. If Primal Problem I has an optimal solution, then Dual Problem II also has an optimal solution, and the two extrema are equal, if the following hypothesis is satisfied.

HYPOTHESIS H1. For all $y \in D_y$, $\min\{\text{Re}(y, m) \mid m \in M\}$ is attained at a point $m_0 \in P_M$, where

$$D_y = \{y \mid y \text{ satisfies the dual constraints for some } k \in K\} ,$$

$$P_M = \{m \in M \mid m \text{ satisfies the primal constraints for some } z \in T\} .$$

RESULT 3. If Dual Problem II has an optimal solution, then Primal Problem I also has an optimal solution, and the two extrema are equal, if a hypothesis dual to H1 is satisfied.

In what follows, we shall need Result 2 in a slightly different form, which is, therefore, stated below for easy reference and use.

THEOREM 1. *Let* z^0 *be an optimal solution of the problem*

$$\begin{array}{ll} \text{minimize} & \Phi(z) = \operatorname{Re}(c, z) + \max\{\operatorname{Re}(z, k) \mid k \in K\} \\ \text{subject to} & Az - b + m \in S \text{ for some } m \in M , \\ & z \in T . \end{array}$$

Then the problem

$$\begin{array}{ll} \text{maximize} & \psi(y) = \operatorname{Re}(y, b) - \max\{\operatorname{Re}(y, m) \mid m \in M\} \\ \text{subject to} & -A^H y + c + k \in T^* \text{ for some } k \in K , \\ & y \in S^* , \end{array}$$

has an optimal solution y^0 , *and* $\Phi(z^0) = \psi(y^0)$, *if the following hypothesis is satisfied:*

> *for all* $y \in D_y$, $\max\{\operatorname{Re}(y, m) \mid m \in M\}$ *is attained at a point* $m_0 \in P_M$.

Theorem 1 is easily deducible from Result 2 by converting the minimum problem into a maximum problem.

Duality

THEOREM 2. *The infimum of Problem P is greater than, or equal to, the supremum of Problem D.*

Proof. Let $(z^0, \overline{z^0}, m^0)$ be a feasible solution for Problem P and $(u^0, \overline{u^0}, v^0, k^0)$ be a feasible solution for Problem D. Then

$$F(z^0, \overline{z^0}) - g(u^0, \overline{u^0}, v^0)$$

$$= \operatorname{Re}\left[f(z^0, \overline{z^0}) - f(u^0, \overline{u^0}) + u^{0^t}\nabla_1 f(u^0, \overline{u^0}) + u^{0^H}\nabla_2 f(u^0, \overline{u^0})\right] - \operatorname{Re} b^H v^0$$

$$+ \max\{\operatorname{Re} k^H z^0 \mid k \in K\} + \max\{\operatorname{Re} m^H v^0 \mid m \in M\}$$

$$\geq \operatorname{Re}\left[(z^0-u^0)^t\nabla_1 f(u^0, \overline{u^0}) + (z^0-u^0)^H\nabla_2 f(u^0, \overline{u^0}) + u^{0^t}\nabla_1 f(u^0, \overline{u^0}) + u^{0^H}\nabla_2 f(u^0, \overline{u^0})\right]$$

$$- \operatorname{Re} b^H v^0 + \operatorname{Re} k^{0^H} z^0 + \operatorname{Re} m^{0^H} v^0 \quad \text{(by (2))}$$

$$= \operatorname{Re}\left[z^{0^t}\nabla_1 f(u^0, \overline{u^0}) + z^{0^H}\nabla_2 f(u^0, \overline{u^0}) + k^{0^H} z^0\right] - \operatorname{Re} b^H v^0 + \operatorname{Re} m^{0^H} v^0$$

$$\geq \operatorname{Re}\left[z^{0^H} A^H v^0\right] - \operatorname{Re} b^H v^0 + \operatorname{Re} m^{0^H} v^0 \quad \text{(by (4) and (5))}$$

$$\geq 0 \quad \text{(by (3) and (6))}.$$

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THEOREM 3. $\left(z^0, \overline{z^0}\right)$ *is an optimal solution for Problem P iff* z^0 *is an optimal solution of the following problem:*

PROBLEM P1

$$\text{minimize} \quad H(z) = \operatorname{Re}\left[\left[\nabla_1 f\left(z^0, \overline{z^0}\right)\right]^t z + \left[\nabla_2 f\left(z^0, \overline{z^0}\right)\right]^H z\right] + \max\left\{\operatorname{Re} k^H z \mid k \in K\right\}$$

$$\text{subject to} \quad Az - b + m \in S \quad \textit{for some} \quad m \in M \,, \qquad z \in T \,.$$

Proof. The proof is similar to that of a theorem proved by Mond ([11], Theorem 4, p. 481).

(i) P $\Rightarrow$ P1. Suppose $\left(z^0, \overline{z^0}\right)$ is an optimal solution for Problem P, but there exists some feasible z^1 such that $H\left(z^1\right) < H\left(z^0\right)$; that is,

$$\begin{aligned}(7)\quad H\left(z^1\right) - H\left(z^0\right) &= \operatorname{Re}\left[\left[\nabla_1 f\left(z^0, \overline{z^0}\right)\right]^t \left(z^1 - z^0\right) + \left[\nabla_2 f\left(z^0, \overline{z^0}\right)\right]^H \left(z^1 - z^0\right)\right] \\ &\qquad + \max\left\{\operatorname{Re} k^H z^1 \mid k \in K\right\} - \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\} \\ &= \operatorname{Re}\left[\left(z^1 - z^0\right)^t \nabla_1 f\left(z^0, \overline{z^0}\right) + \left(z^1 - z^0\right)^H \nabla_2 f\left(z^0, \overline{z^0}\right)\right] \\ &\qquad + \max\left\{\operatorname{Re} k^H z^1 \mid k \in K\right\} - \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\} \\ &< 0 \,.\end{aligned}$$

Since $\left(z^1, \overline{z^1}\right)$ and $\left(z^0, \overline{z^0}\right)$ are feasible solutions for Problem P it follows that for

$$z^2 = \lambda z^1 + (1-\lambda) z^0 \,, \quad 0 \le \lambda \le 1 \,,$$

$\left(z^2, \overline{z^2}\right)$ is also feasible for Problem P.

Now consider

$$\begin{aligned}(8)\quad F\left(z^2, \overline{z^2}\right) - F\left(z^0, \overline{z^0}\right) = \operatorname{Re}\left[f\left(z^2, \overline{z^2}\right) - f\left(z^0, \overline{z^0}\right)\right] &+ \max\left\{\operatorname{Re} k^H z^2 \mid k \in K\right\} \\ &- \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\} \,.\end{aligned}$$

Expanding in a Taylor series, with R_{N+1} denoting the appropriate remainder, we have

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(9) $\operatorname{Re}\left[f\left(z^2, \overline{z^2}\right)-f\left(z^0, \overline{z^0}\right)\right]$

$$= \operatorname{Re}\left\{\sum_{k_1+k_2+\ldots+k_{2n}=1}^{N} \frac{1}{k_1!k_2!\ldots!k_{2n}!} \frac{\partial^{k_1+k_2+\ldots+k_{2n}} f\left(z^0,\overline{z^0}\right)}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}\partial\bar{z}_1^{k_{n+1}}\ldots\partial\bar{z}_n^{k_{2n}}} \times \left(z_1^2-z_1^0\right)^{k_1}\ldots\left(z_n^2-z_n^0\right)^{k_n}\left(\overline{z_1^2-z_1^0}\right)^{k_{n+1}}\ldots\left(\overline{z_n^2-z_n^0}\right)^{k_{2n}} + R_{N+1}\right\}$$

$$= \operatorname{Re}\left[\left(z^2-z^0\right)^t\nabla_1 f\left(z^0, \overline{z^0}\right)+\left(z^2-z^0\right)^H\nabla_2 f\left(z^0, \overline{z^0}\right)\right]$$

$$+ \operatorname{Re}\left\{\sum_{k_1+k_2+\ldots+k_{2n}=2}^{N} \frac{1}{k_1!k_2!\ldots!k_{2n}!} \frac{\partial^{k_1+k_2+\ldots+k_{2n}} f\left(z^0,\overline{z^0}\right)}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}\partial\bar{z}_1^{k_{n+1}}\ldots\partial\bar{z}_n^{k_{2n}}} \times \left(z_1^2-z_1^0\right)^{k_1}\ldots\left(z_n^2-z_n^0\right)^{k_n}\left(\overline{z_1^2-z_1^0}\right)^{k_{n+1}}\ldots\left(\overline{z_n^2-z_n^0}\right)^{k_{2n}} + R_{N+1}\right\}$$

$$= \lambda \operatorname{Re}\left[\left(z^1-z^0\right)^t\nabla_1 f\left(z^0, \overline{z^0}\right)+\left(z^1-z^0\right)^H\nabla_2 f\left(z^0, \overline{z^0}\right)\right]$$

$$+ \operatorname{Re}\left\{\sum_{k_1+k_2+\ldots+k_{2n}=2}^{N} \frac{\lambda^{k_1+k_2+\ldots+k_{2n}}}{k_1!k_2!\ldots!k_{2n}!} \frac{\partial^{k_1+\ldots+k_{2n}} f\left(z^0,\overline{z^0}\right)}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}\partial\bar{z}_1^{k_{n+1}}\ldots\partial\bar{z}_n^{k_{2n}}} \times \left(z_1^1-z_1^0\right)^{k_1}\ldots\left(z_n^1-z_n^0\right)^{k_n}\left(\overline{z_1^1-z_1^0}\right)^{k_{n+1}}\ldots\left(\overline{z_n^1-z_n^0}\right)^{k_{2n}} + R_{N+1}\right\},$$

where k_i are non-negative integers.

Also

(10) $\max\left\{\operatorname{Re} k^H z^2 \mid k \in K\right\} - \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\}$

$$= \max\left\{\operatorname{Re} k^H\left(\lambda z^1+(1-\lambda)z^0\right) \mid k \in K\right\} - \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\}$$

$$\leq \lambda \max\left\{\operatorname{Re} k^H z^1 \mid k \in K\right\} + (1-\lambda)\max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\} - \max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\}$$

$$= \lambda\left[\max\left\{\operatorname{Re} k^H z^1 \mid k \in K\right\}-\max\left\{\operatorname{Re} k^H z^0 \mid k \in K\right\}\right] .$$

From (8), (9), and (10) we have

$$F\left(z^2, \overline{z^2}\right) - F\left(z^0, \overline{z^0}\right) = (9) + (10) .$$

Now, since $R_{N+1} \to 0$ as $N \to \infty$, by choosing $\lambda > 0$ sufficiently small, $F\left(z^2, \overline{z^2}\right) - F\left(z^0, \overline{z^0}\right)$ will have the sign of

$$\operatorname{Re}\left[\left(z^1-z^0\right)^t \nabla_1 f\left(z^0, \overline{z^0}\right) + \left(z^1-z^0\right)^H \nabla_2 f\left(z^0, \overline{z^0}\right)\right] + \max\{\operatorname{Re} k^H z^1 \mid k \in K\} - \max\{\operatorname{Re} k^H z^0 \mid k \in K\}$$

which is negative by (7).

Hence, we have $F\left(z^2, \overline{z^2}\right) - F\left(z^0, \overline{z^0}\right) < 0$, which contradicts the assumption that $\left(z^0, \overline{z^0}\right)$ is an optimal solution of Primal Problem P.

Hence z^0 is an optimal solution for Problem P1.

(ii) P1 ⇒ P. Let z^0 be an optimal solution of Problem P1. Then for any feasible solution z we have

$$\text{(11)} \quad H(z) - H\left(z^0\right) = \operatorname{Re}\left[\left(z-z^0\right)^t \nabla_1 f\left(z^0, \overline{z^0}\right) + \left(z-z^0\right)^H \nabla_2 f\left(z^0, \overline{z^0}\right)\right] + \max\{\operatorname{Re} k^H z \mid k \in K\} - \max\{\operatorname{Re} k^H z^0 \mid k \in K\} \geq 0 .$$

Now

$$\begin{aligned} F(z, \bar{z}) - F\left(z^0, \overline{z^0}\right) &= \operatorname{Re}\left[f(z, \bar{z}) - f\left(z^0, \overline{z^0}\right)\right] + \max\{\operatorname{Re} k^H z \mid k \in K\} - \max\{\operatorname{Re} k^H z^0 \mid k \in K\} \\ &\geq \operatorname{Re}\left[\left(z-z^0\right)^t \nabla_1 f\left(z^0, \overline{z^0}\right) + \left(z-z^0\right)^H \nabla_2 f\left(z^0, \overline{z^0}\right)\right] \\ &\quad + \max\{\operatorname{Re} k^H z \mid k \in K\} - \max\{\operatorname{Re} k^H z^0 \mid k \in K\} \quad \text{(by (2))} \\ &\geq 0 \quad \text{(by (11))}. \end{aligned}$$

Thus $\left(z^0, \overline{z^0}\right)$ is an optimal solution of Problem P.

REMARK. In what follows, we will use only the first part of the theorem; namely, P ⇒ P1.

THEOREM 4. *If* $\left(z^0, \overline{z^0}\right)$ *is an optimal solution of Primal Problem P then there exists a* v^0 *such that* $\left(z^0, \overline{z^0}, v^0\right)$ *is an optimal solution for Dual Problem D and the extreme values of the two objective functions are equal, if the following hypothesis is satisfied:*

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for all $v \in D_{z^0, \overline{z^0}, v}$, $\max\{\text{Re}(v, m) \mid m \in M\}$ *is attained at a point* $m^0 \in P_M$ *, where* $D_{z^0, \overline{z^0}, v}$ *denotes the set of all* v *which satisfy the dual constraint with* $u = z^0$.

Proof. By Theorem 3, part (i), z^0 is an optimal solution for Problem P1. By Theorem 1, the dual of Problem P1 is the following problem, denoted by Problem D1.

PROBLEM D1

$$\text{maximize} \quad G(v) = \text{Re}(v, b) - \max\{\text{Re}(v, m) \mid m \in M\}$$

$$\text{(12)} \quad \text{subject to} \quad -A^H v + \nabla_2 f(z^0, \overline{z^0}) + \overline{\nabla_1 f(z^0, \overline{z^0})} + k \in T^*$$

$$\text{for some} \quad k \in K ,$$

$$\text{(13)} \qquad v \in S^* .$$

By Theorem 1, there exists v^0 optimal for Problem D1 and such that $H(z^0) = G(v^0)$; that is,

$$\text{(14)} \quad \text{Re}\left[\left(\nabla_1 f(z^0, \overline{z^0})\right)^t z^0 + \left(\nabla_2 f(z^0, \overline{z^0})\right)^H z^0\right] + \max\{\text{Re } k^H z^0 \mid k \in K\}$$
$$= \text{Re}(v^0, b) - \max\{\text{Re}(v^0, m) \mid m \in M\} .$$

From (12) and (13), $(z^0, \overline{z^0}, v^0)$ is a feasible solution for Problem D. Now

$$g(z^0, \overline{z^0}, v^0)$$
$$= \text{Re}\left[f(z^0, \overline{z^0}) - z^{0^t} \nabla_1 f(z^0, \overline{z^0}) - z^{0^H} \nabla_2 f(z^0, \overline{z^0}) + b^H v^0\right] - \max\{\text{Re}(m, v^0) \mid m \in M\}$$
$$= \text{Re } f(z^0, \overline{z^0}) + \text{Re } b^H v^0 - \max\{\text{Re}(m, v^0) \mid m \in M\}$$
$$+ \max\{\text{Re } k^H z^0 \mid k \in K\} - \text{Re}(v^0, b) + \max\{\text{Re}(v^0, m) \mid m \in M\} \quad \text{(by (14))}$$
$$= \text{Re } f(z^0, \overline{z^0}) + \max\{\text{Re } k^H z^0 \mid k \in K\}$$
$$= F(z^0, \overline{z^0}) .$$

Thus we have a feasible solution $(z^0, \overline{z^0}, v^0)$ for Dual Problem D which further satisfies

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$$g\left(z^0, \overline{z^0}, v^0\right) = F\left(z^0, \overline{z^0}\right) .$$

Thus, by Theorem 2, it follows that $\left(z^0, \overline{z^0}, v^0\right)$ is an optimal solution for Problem D.

Special cases

If $M = \{0\}$, we observe that Problems P and D reduce to those considered by Mond [11]. We further note that in such a case the hypothesis in Theorem 4 is automatically satisfied and hence, Mond's Theorem ([11], Theorem 5) turns out as a special case of Theorem 4 proved above.

If $f(z, \overline{z}) \equiv c^H z$, Problems P and D reduce to those considered by Mahajan and Vartak [8].

If in addition to $M = \{0\}$, $f(z, \overline{z}) \equiv c^H z$, we take $S = \{0\}$ then Problems P and D reduce to those considered by Smiley [15]. If $M = \{0\}$ and

(15) $$K \equiv \sum_{i=1}^{r} Q^i U^i \quad \text{with} \quad U^i = \left\{u \in C^n \mid u^H Q^i u \leq 1\right\}$$

where $Q^i \in C^{n\times n}$, $i = 1, \ldots, r$ are positive semidefinite hermitian, then it can be shown as in Smiley [15] that Problems P and D reduce to those considered by Mond [10].

If $M = \{0\}$ and

(16) $$f(z, \overline{z}) \equiv \tfrac{1}{2} z^H B z + p^H z ,$$

where B is hermitian positive semidefinite, then Problems P and D reduce to those considered by Mond [12].

If $M = \{0\}$, K is defined by (15), and $f(z, \overline{z})$ is given by (16), then Problems P and D reduce to those considered by Rani [13]. If also

(17) $$S = \left\{z \in C^m \mid |\arg z| \leq \alpha\right\} ,$$

(18) $$T = \left\{w \in C^n \mid |\arg w| \leq \beta\right\} ,$$

for given $\alpha \in R_+^m$, $\beta \in R_+^n$, $\alpha_i \leq \pi/2$, $i = 1, \ldots, m$, $\beta_i \leq \pi/2$,

$i = 1, \ldots, n$, then the problems considered by Rani and Kaul [14] are obtained.

If $M = \{0\}$, K is defined by (15) with $r = 1$ and $f(z, \bar{z}) = p^H z$, Problems P and D reduce to those considered by Mond [9]. If also S and T are defined by (17) and (18), the problems of Bhatia and Kaul [4] are obtained.

If $M = \{0\}$, $K = \{0\}$, and S and T are defined by (17) and (18) the problems considered by Hanson and Mond [6] are obtained.

If $M = \{0\}$, $K = \{0\}$, and $f(z, \bar{z})$ is given by (16) the complex quadratic programming problems of Abrams and Ben-Israel [2] are obtained. If also S and T are given by (17) and (18), Problems P and D reduce to those of Hanson and Mond [5].

If $M = \{0\}$, $K = \{0\}$, $f(z, \bar{z}) \equiv p^H z$, the complex linear programming problems of Ben-Israel [3] are obtained. If also S and T are given by (17) and (18), we obtain the problems of Levinson [7].

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