Proceedings of the Edinburgh Mathematical Society (1998) 41, 277–287 ©

TAME AUTOMORPHISMS OF FINITELY GENERATED ABELIAN GROUPS

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(Received 17th June 1996)

We characterize tame automorphisms of finitely generated abelian groups via a simple determinant condition.

1991 *Mathematics subject classification:* 20E05,20F55.

0. Introduction

An automorphism of a finitely generated group is *tame* relative to a particular generating set if it lifts to an automorphism of the free group. Tameness of an automorphism depends on the presentation map (which is equivalent to the choice of a finite generating set) – indeed we show in Proposition 1 that every automorphism is tame relative to some presentation map. The issue becomes interesting when a particular presentation map is specified. For example, a classical theorem of Nielsen says that all automorphisms of a surface group are tame relative to the standard presentation (see, e.g., [4]). It is elementary to show that all automorphisms of $\mathbb{Z}^r$ are tame relative to the standard presentation map (Proposition 2 below) and that the same statement is false when torsion is present.

In this note we characterize automorphisms of finitely generated abelian groups that are tame relative to the standard presentations. In particular, we prove the following theorems (notation explained in Section 2 and Section 3).

Theorem 2. *If $\alpha : T_p \to T_p$ is an automorphism of the abelian p-group*

$$T_p \cong (\mathbb{Z}_{p^{k_1}})^{r_1} \oplus (\mathbb{Z}_{p^{k_2}})^{r_2} \oplus \ldots \oplus (\mathbb{Z}_{p^{k_n}})^{r_n}$$

then α is tame with respect to the standard presentation map ε_p if and only if $\det((M_\alpha)_{p^{k_1}}) = \pm 1 \pmod{p^{k_1}}$.

Theorem 3. *If $G \cong T \oplus F$ is a finitely generated abelian group (T the torsion subgroup, F free abelian) and $\alpha : G \to G$ is an automorphism, then α is tame with respect to the standard presentation ε if and only if α_T, the restriction of α to T, is tame with respect to ε_T.*

1. Preliminaries

Definition 1. Let G be a finitely generated group and let $F_r = F(a_1, \ldots, a_r)$ denote the free group on r generators. Suppose that $\varepsilon : F_r \to G$ is a presentation map for the group G and that α is an automorphism of G. A *lift* of α is an endomorphism $\beta : F_r \to F_r$ which makes the following diagram commute:

$$\begin{tikzcd} F_r \arrow[r, "\beta"] \arrow[d, "\varepsilon"'] & F_r \arrow[d, "\varepsilon"] \\ G \arrow[r, "\alpha"] & G \end{tikzcd}$$

We say that β *induces* α. If there exists an automorphism β that is a lift of α then α is said to be *tame with respect to* ε – otherwise α is said to be *wild*.

The following example shows that an automorphism of G may be tame with respect to one presentation but wild with respect to another (and indicates how to produce non-tame automorphisms when torsion is present).

Example 1. Let $\alpha : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $\alpha(1) = 2$ and consider the presentation maps $\varepsilon_1 : F(a) \to \mathbb{Z}_5$ and $\varepsilon_2 : F(a, b) \to \mathbb{Z}_5$ by $\varepsilon_1(a) = \varepsilon_2(a) = 1$, $\varepsilon_2(b) = 3$. Then α is clearly not tame with respect to ε_1 but with respect to ε_2, α lifts to the automorphism $\beta(a) = ba^{-1}$, $\beta(b) = a$.

The following proposition generalizes this example. The result was first proven by Rapaport [**3**, Theorem 2] – we include a proof for completeness.

Proposition 1. *Let α be an automorphism of G and $\varepsilon : F_r \to G$ be a presentation map. Then there exists a presentation map $\varepsilon' : F_{2r} \to G$ relative to which α is tame.*

Proof. Choose words w_i, v_i in $F_r = F(a_1, \ldots, a_r)$ such that $\varepsilon(w_i) = \alpha(\varepsilon(a_i))$ and $\varepsilon(v_i) = \alpha^{-1}(\varepsilon(a_i))$. Consider the presentation map $\varepsilon' : F(a_1, \ldots, a_r, b_1, \ldots, b_r) \to G$ defined by $\varepsilon'(a_i) = \varepsilon(a_i)$ and $\varepsilon'(b_i) = \varepsilon(v_i)$ and define $\beta : F(a_1, \ldots, a_r, b_1, \ldots, b_r) \to F(a_1, \ldots, a_r, b_1, \ldots, b_r)$ by

$$a_i \mapsto b_i v_i^{-1} w_i \text{ and } b_i \mapsto a_i.$$

It is easy to check that $\{\beta(a_1), \ldots, \beta(a_r), \beta(b_1), \ldots, \beta(b_r)\}$ is Nielsen equivalent to $\{a_1, \ldots, a_r, b_1, \ldots, b_r\}$. (Nielsen equivalence of sets of a given cardinality in a group is generated by the elementary equivalences of permutations, inversions and the multiplication of one element by another. In a free group automorphisms are characterized by the property that a basis is Nielsen equivalent to its image (see, e.g., Theorem 3.2 of [**2**] and the preceding discussion). Thus β is an automorphism. Furthermore, β is a lift of α relative to ε'. □

If A is an abelian group, it is more convenient to work with an *abelian presentation map* $\eta : \mathbb{Z}^r \to A$. Any abelian presentation map η determines a presentation map ε in the obvious way compatible with the map $F(a_1, \ldots, a_r) \to \mathbb{Z}^r$ given by $a_i \mapsto \delta_i$, (the standard generators of $\mathbb{Z}^r$ being $\delta_1, \ldots, \delta_r$). The following elementary result implies that for an abelian group A, tame automorphisms are precisely those that lift to automorphisms of $\mathbb{Z}^r$.

Proposition 2. *Any automorphism of F_r induces an automorphism of $\mathbb{Z}^r$. Conversely, any automorphism of $\mathbb{Z}^r$ is induced by an automorphism of F_r.*

Proof. Since $\mathbb{Z}^r \cong F_r/[F_r, F_r]$ and $[F_r, F_r]$ is characteristic the first claim is clear. Now suppose we are given an automorphism β of $\mathbb{Z}^r$. Given a basis of $\mathbb{Z}^r$ we can represent β by an integral matrix, M_β. Since M_β is invertible, it is a product of elementary matrices, each of which is induced by a Nielsen automorphism of F_r. Let $\bar{\beta}$ denote the product of these Nielsen automorphisms. Then $\bar{\beta}$ is an automorphism of F_r which induces β. □

For the remainder of this note, all lifts of endomorphisms of an abelian group will be with respect to the appropriate abelian presentation map unless otherwise stated.

2. Tame automorphisms of finite abelian p-groups

Definition 2. We denote by T_p a finite abelian p-group (p a prime) of rank r with representation

$$T_p \cong (\mathbb{Z}_{p^{k_1}})^{r_1} \oplus (\mathbb{Z}_{p^{k_2}})^{r_2} \oplus \ldots \oplus (\mathbb{Z}_{p^{k_n}})^{r_n} \qquad 1 \leq k_1 < k_2 < \ldots < k_n, r = \sum_{i=1}^{n} r_i.$$

The standard generators $\delta_1, \ldots, \delta_r$ of $\mathbb{Z}^r$ are mapped to the generators $x_1, \ldots, x_r$ of the cyclic factors of T_p under the *standard abelian presentation map* η_p which corresponds to the *standard presentation map* $\varepsilon_p : F(a_1, \ldots, a_r) \to T_p$.

Suppose α is an endomorphism of T_p: for each j there are integers a_{ij} (not unique) such that

$$\alpha(x_j) = a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{rj}x_r.$$

The matrix $M_\beta = (a_{ij})$ represents a lift of α to $\mathbb{Z}^r$ given by $\beta : \mathbb{Z}^r \to \mathbb{Z}^r$ via

$$\beta(\delta_j) = a_{1j}\delta_j + a_{2j}\delta_2 + \ldots + a_{rj}\delta_r.$$

Definition 3. Let $M = (a_{ij})$ and $M' = (a'_{ij})$ be $r \times r$ integral matrices and p^k be a power of the prime p. We say $M \sim_{p^k} M'$ if $a_{ij} = a'_{ij} \pmod{p^k}$ for $k \geq 1$.

If β and β' induce the same endomorphism of T_p then $M_\beta \sim_{p^i} M_{\beta'}$ $(1 \le i \le k_1)$. This motivates the following definition.

Definition 4. Let α be an endomorphism of T_p and let β be any lift of α. The matrix $(M_\alpha)_{p^i}$ is obtained from M_β by reducing (mod p^i) $(1 \le i \le k_1)$. In general, $(M_\alpha)_{p^i}$ will not represent a lift of α.

Lemma 1. *Let α be an endomorphism of $T_p \cong (\mathbb{Z}_{p^{k_1}})^{r_1} \oplus (\mathbb{Z}_{p^{k_2}})^{r_2} \oplus \ldots \oplus (\mathbb{Z}_{p^{k_n}})^{r_n}$, where $r = \sum_{i=1}^n r_i$. Then there are $r_i \times r_i$ integral matrices A_i such that $(M_\alpha)_p$ has the form*

$$(M_\alpha)_p = \begin{pmatrix} A_1 & * & * & \ldots & * \\ 0 & A_2 & * & \ldots & * \\ 0 & 0 & A_3 & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_n \end{pmatrix}.$$

Proof. Let $\beta : \mathbb{Z}^r \to \mathbb{Z}^r$ be a lift of α. For a fixed j consider the generator δ_j of $\mathbb{Z}^r$ and choose m such that $s_{m-1} < j \le s_m$ $(s_m = \sum_{i=1}^m r_i)$. The r-tuple $\beta(\delta_j) = (a_{1j}, \ldots, a_{rj})$ represents the j^{th} column of M_β. To prove the lemma we need only show that $a_{ij} = 0$ (mod p) for $i > s_m$. Since β is a lift of α, $\eta_p(\beta(\delta_j)) = \alpha(x_j)$, where x_j is a generator of T_p of order p^{k_m} defined above and $\alpha(x_j) = (\bar{a}_{1j}, \ldots, \bar{a}_{rj})$ where $\bar{a}_{ij} = a_{ij}$ (mod p^{k_t}), $s_{t-1} < i \le s_t$. If $i > s_m$ then $\bar{a}_{ij}$ represents an element of $\mathbb{Z}_{p^{k_t}}$ $(t > m)$ of order dividing p^{k_m} : thus $p^{k_t - k_m} \mid a_{ij}$. $\square$

Theorem 1. *An endomorphism α of T_p is an automorphism if and only if* $\det((M\alpha)_p) \ne 0$ (mod p).

Proof. $(\Rightarrow)$ Suppose α is an automorphism. Let $\pi : T_p \to T_p/pT_p$ and let $\bar{\alpha}$ denote the automorphism induced by α on $T_p/pT_p \cong (\mathbb{Z}_p)^r$. If $\beta : \mathbb{Z}^r \to \mathbb{Z}^r$ is any lift of α then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^r & \xrightarrow{\beta} & \mathbb{Z}^r \\ \eta\big\downarrow & & \big\downarrow\eta \\ T_p & \xrightarrow{\alpha} & T_p \\ \pi\big\downarrow & & \big\downarrow\pi \\ (\mathbb{Z}_p)^r & \xrightarrow{\bar{\alpha}} & (\mathbb{Z}_p)^r \end{array}.$$

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It follows that the matrix obtained by reducing the entries of M_β (mod p) represents the automorphism $\bar{\alpha}$ relative to the standard basis of the $\mathbb{Z}_p$-vector space $(\mathbb{Z}_p)^r$. But any matrix representing an automorphism of $(\mathbb{Z}_p)^r$ must be invertible over $\mathbb{Z}_p$ and so must have a non-zero determinant in $\mathbb{Z}_p$ [1].

($\Leftarrow$) Let $\beta:\mathbb{Z}^r\to\mathbb{Z}^r$ be a lift of α and let $t_i=\sum_{j=i}^n r_j$. Denote the lower right $t_i\times t_i$ submatrix of M_β by M_{β_i} and let $\beta_i:\mathbb{Z}^{t_i}\to\mathbb{Z}^{t_i}$ be the corresponding map. Then β_i induces an endomorphism $\bar{\beta}_i$ on $(\mathbb{Z}_{p^{k_i}})^{t_i}$ obtained by reducing each entry of M_{β_i} mod p^{k_i}. Let $M_{\bar{\beta}_i}$ denote this matrix. It represents $\bar{\beta}_i$ with respect to the standard basis of $(\mathbb{Z}_{p^{k_i}})^{t_i}$.

Claim. *For each i, $\bar{\beta}_i$ is an automorphism.*

We shall prove that $\det(M_{\bar{\beta}_i})$ is a unit in $\mathbb{Z}_{p^{k_i}}$: invertibility then follows from the classical adjoint formula for the inverse. It suffices to show that $\det(M_{\beta_i})$ (mod p) is a unit in $\mathbb{Z}_p$. Taking the determinant of an integral matrix and reducing it modulo p is equivalent to first reducing the entries of the matrix modulo p and then taking the determinant over $\mathbb{Z}_p$. We therefore need only show that the determinant of the lower right $t_i\times t_i$ submatrix of $(M_\alpha)_p$ represents a unit of $\mathbb{Z}_p$. By Lemma 1, $(M_\alpha)_p$ has the form

$$(M_\alpha)_p=\begin{pmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_n \end{pmatrix}.$$

Each A_i is an $r_i\times r_i$ matrix so the lower right $t_i\times t_i$ submatrix is given by

$$\begin{pmatrix} A_i & * & * & \dots & * \\ 0 & A_{i+1} & * & \dots & * \\ 0 & 0 & A_{i+2} & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_n \end{pmatrix}.$$

Since $\prod_{i=1}^n \det(A_i)\neq 0$ (mod p) by Lemma 1, the claim is established.

Now let $(w_1,\dots,w_r)\in\mathbb{Z}^r$ and suppose that $\alpha(\eta_p((w_1,\dots,w_r)))=0$. If we denote $\beta((w_1,\dots,w_r))=(z_1,\dots,z_r)$ then $\eta_p((z_1,\dots,z_r))=0$ since β is a lift of α. Thus $z_i=0$ (mod p^{k_m}), $s_{m-1}<i\le s_m$. Let $(w_i)_{p^{k_m}}$ represent w_i (mod p^{k_m}). By construction of $\bar{\beta}_1$

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we have

$$\bar{\beta}_1(((w_1)_{p^{k_1}},\ldots,(w_r)_{p^{k_1}}))=((z_1)_{p^{k_1}},\ldots,(z_r)_{p^{k_1}})=0.$$

Since $\bar{\beta}_1$ is an automorphism we must have $(w_i)_{p^{k_1}}=0$ in $\mathbb{Z}_{p^{k_1}}$ and in particular $w_1=\ldots=w_{r_1}=0\ (\mathrm{mod}\ p^{k_1})$. Applying $\bar{\beta}_2$ gives the equation

$$\bar{\beta}_2(((w_{r_1}+1)_{p^{k_2}},\ldots,(w_r)_{p^{k_2}}))=((z_{r_1+1})_{p^{k_2}},\ldots,(z_r)_{p^{k_2}})=0.$$

Thus $w_{r_1+1}=\ldots=w_{r_1+r_2}=0\ (\mathrm{mod}\ p^{k_2})$. Repeating this process shows that $w_i=0\ (\mathrm{mod}\ p^{k_m})$, $s_{m-1}<i\leq s_m$ and so $\eta_p((w_1,\ldots,w_r))=0$. We conclude that α is injective. □

The proposition above is extended to finitely generated abelian groups in Section 3. We now turn to the issue of classifying tame automorphisms of T_p.

Lemma 2. *Suppose M is an $r\times r$ integral matrix. Then* $\det(M)=\pm1\ (\mathrm{mod}\ p^k)$ *if and only if there exists an $r\times r$ integral matrix M' with $M\sim_{p^k}M'$ such that* $\det(M')=\pm1$.

Proof. ($\Leftarrow$) This is clear since $\det(M)=\det(M')\ (\mathrm{mod}\ p^k)$.

($\Rightarrow$) Suppose $\det(M)=\pm1\ (\mathrm{mod}\ p^k)$. It is straightforward to show that if a new matrix is obtained from M by a single elementary row (column) operation over $\mathbb{Z}$ then the claim holds for this new matrix if and only if it holds for M (the new matrix will still have determinant $\pm1\ (\mathrm{mod}\ p^k)$). Since $\mathbb{Z}$ is a Euclidean domain, M is row and column equivalent to a diagonal matrix [1] and so we may assume that

$$M=\begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0\\ 0 & a_{22} & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & a_{rr}\end{pmatrix}.$$

Consider the following matrices with integral variables τ_i:

$$A=\begin{pmatrix} \tau_1 & \tau_2 & \tau_3 & \ldots & \tau_r\\ p^k & a_{22} & 0 & \ldots & 0\\ 0 & p^k & a_{33} & \ldots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \ldots & p^k & a_{rr}\end{pmatrix}$$

and

$$M' = \begin{pmatrix} a_{11} + p^k\tau_1 & p^k\tau_2 & p^k\tau_3 & \dots & p^k\tau_r \\ p^k & a_{22} & 0 & \dots & 0 \\ 0 & p^k & a_{33} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & p^k & a_{rr} \end{pmatrix}.$$

We will solve for τ_i such that $\det(M') = \pm 1$. By assumption $\det(M) = \pm 1 + p^k m$ for some integer m. Since $\det(M') = \det(M) + p^k \det(A) = \pm 1 + p^k m + p^k \det(A)$ we need only solve for τ_i such that $\det(A) = -m$. We proceed by induction on the size of A with the result clear if A is a 2×2 integral matrix (since $gcd(a_{22}, p^k) = 1$ by assumption). Then $\det(A)$ is

$$a_{rr} \det \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 & \dots & \tau_{r-1} \\ p^k & a_{22} & 0 & \dots & 0 \\ 0 & p^k & a_{33} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & p^k & a_{r-1r-1} \end{pmatrix} \pm \tau_r \det \begin{pmatrix} p^k & a_{22} & 0 & \dots & 0 \\ 0 & p^k & a_{33} & \dots & 0 \\ 0 & 0 & p^k & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & p^k \end{pmatrix}.$$

But $gcd(a_{rr}, p^{k(r-1)}) = 1$ so there are integers ℓ_1 and ℓ_2 such that $a_{rr}\ell_1 \pm \ell_2 p^{k(r-1)} = -m$. Applying the induction hypothesis gives integers $\tau_1, \dots, \tau_{r-1}$ such that the determinant of the first matrix in the above equation is ℓ_1. Let $\tau_r = \ell_2$. $\square$

Theorem 2. *Let α be an automorphism of $T_p \cong (\mathbb{Z}_{p^{k_1}})^{r_1} \oplus (\mathbb{Z}_{p^{k_2}})^{r_2} \oplus \dots \oplus (\mathbb{Z}_{p^{k_n}})^{r_n}$. Then α is tame with respect to ε_p if and only if* $\det((M_\alpha)_{p^{k_1}}) = \pm 1 \pmod{p^{k_1}}$.

Proof. ($\Rightarrow$) If $\tilde{\alpha}$ is an automorphic lift of α then $\det(M_{\tilde{\alpha}}) = \pm 1$. Recall that the matrix $(M_\alpha)_{p^{k_1}}$ is derived by reducing the matrix of any lift modulo p^{k_1}. Hence $M_{\tilde{\alpha}} \sim_{p^{k_1}} (M_\alpha)_{p^{k_1}}$. The implication follows from Lemma 2.

($\Leftarrow$) Recall that the rank of T_p is $r = \sum_{i=1}^{n} r_i$ and that $s_m = \sum_{i=1}^{m} r_i$ for $1 \le m \le n$. For $r \times r$ integral matrices write $A \sim B$ if $a_{ij} = b_{ij} \pmod{p^{k_m}}$, $s_{m-1} < i \le s_m$. Both A and B represents lifts of the same automorphism of T_p if and only if $A \sim B$. We say that A *satisfies property P* if there exists A' with $\det(A') = \pm 1$ such that $A \sim A'$. As before it can be shown that if B is obtained from A by a single elementary column operation over $\mathbb{Z}$ then A satisfies property P if and only if B does. Observe that elementary row operations do not necessarily preserve property P since the relation $\sim$ is a condition on the rows of a matrix. Now suppose that β is a lift of α and that M_β is its representative matrix. We need only show M_β satisfies property P. Since $(M_\alpha)_{p^{k_1}}$ is obtained from M_β

by reducing each entry modulo p^{k_1} and $\det((M_\alpha)_{p^{k_1}}) = \pm 1 \pmod{p^{k_1}}$ it follows that $\det(M_\beta) = \pm 1 \pmod{p^{k_1}}$. From the remarks above we may assume, after some column operations if necessary, that M_β has the form

$$M_\beta = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{r1} & a_{r2} & a_{r3} & \dots & a_{rr} \end{pmatrix}.$$

For any integer τ consider the matrix $A_r \sim M_\beta$

$$A_\tau = \begin{pmatrix} a_{11} + p^{k_1}\tau & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{r1} & a_{r2} & a_{r3} & \dots & a_{rr} \end{pmatrix}.$$

It suffices to solve for τ so that $\det(A_\tau) = \pm 1 \pmod{p^{k_n}}$ for then we may alter A_τ modulo p^{k_n} by Lemma 2 to obtain a matrix A'_τ whose determinant is ± 1. This shows that M_β satisfies property P. Let $d_{1r} = \prod_{i=1}^{r} a_{ii}$ and $d_{2r} = \prod_{i=2}^{r} a_{ii}$. Then

$$\begin{aligned} \det(A) &= d_{1r} + p^{k_1}\tau d_{2r} \\ &= \pm 1 + p^{k_1} m + p^{k_1}\tau d_{2r}, \quad \text{for some } m \\ &= \pm 1 + p^{k_1}(m + \tau d_{2r}). \end{aligned}$$

If $n = 1$ we are done. Otherwise, $(d_{2r}, p^{k_n - k_1}) = 1$ so there exists an τ such that $\tau d_{2r} = -m \pmod{p^{k_n - k_1}}$. This completes the proof. □

3. Tame automorphisms of finitely generated abelian groups

Let T denote a finite abelian group of rank r. T has a standard decomposition

$$T \cong T_{p_1} \oplus \dots \oplus T_{p_n}.$$

Each T_{p_i} is the p_i-torsion subgroup of rank r_i, $r = \sum_{i=1}^{n} r_i$. Let $\eta_{p_i} : \mathbb{Z}^{r_i} \to T_{p_i}$ be the standard abelian presentation map of T_{p_i} as defined above. Then the map $\eta_T = \eta_{p_1} \oplus \dots \oplus \eta_{p_n} : \mathbb{Z}^r \to T$ is called the *standard abelian presentation map* of T.

If α is an endomorphism of T then $\alpha = \alpha_1 \oplus \dots \oplus \alpha_n$ where α_i is the restriction of α to T_{p_i}, and α is an automorphism if and only if each α_i is an automorphism. In

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considering lifts β of α with respect to η_T, we can restrict ourselves to those that respect this direct sum representation, so that $\beta = \beta_1 \oplus \ldots \oplus \beta_n$ where β_i is a lift of α_i with respect to η_{p_i}. We may represent β by an integral matrix of the form

$$M_\beta = \begin{pmatrix} M_\beta & 0 & \ldots & 0 \\ 0 & M_{\beta_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{\beta_n} \end{pmatrix}.$$

M_{β_i} is a matrix representing β_i and $(M_{\alpha_i})_{p_i}$ is obtained from this matrix by reducing each entry mod p_i. The α is an automorphism if and only if $\det((M_{\alpha_i})_{p_i}) \neq 0 \pmod{p_i}$ for each i.

One would expect that an automorphism of T is tame if and only if it restricts to tame automorphisms of each of the p-subgroups. The following proposition shows that this is indeed the case.

Proposition 3. *Let $T \cong T_{p_1} \oplus \ldots \oplus T_{p_n}$ be a finite abelian group where each T_{p_i} is the p_i-torsion subgroup. Then an automorphism α is tame with respect to ε_T if and only if α_i, the restriction of α to T_{p_i}, is tame with respect to ε_{p_i}.*

Proof. $(\Rightarrow)$ This is immediate since the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\pi_i"'] & T \arrow[d, "\pi_i"] \\ T_{p_i} \arrow[r, "\alpha_i"] & T_{p_i} \end{tikzcd} .$$

$(\Leftarrow)$ If $\tilde{\alpha}_i$ is a lift of α_i with respect to ε_{p_i} then $\tilde{\alpha}_1 \oplus \ldots \oplus \tilde{\alpha}_n$ is a lift of $\alpha_1 \oplus \ldots \oplus \alpha_n$ with respect to ε_T. Also, the following diagram commutes:

$$\begin{tikzcd} T_{p_1} \oplus \ldots \oplus T_{p_n} \arrow[r, "\alpha_1 \oplus \ldots \oplus \alpha_n"] \arrow[d, "\cong"'] & T_{p_1} \oplus \ldots \oplus T_{p_n} \arrow[d, "\cong"] \\ T \arrow[r, "\alpha"] & T \end{tikzcd} .$$

These two facts yield the desired result. $\square$

Let G denote a finitely generated abelian group of rank r. Then $G \cong T \oplus F$ where T is the torsion subgroup of rank r_1 and F is the free subgroup of rank r_2, $r = r_1 + r_2$.

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Let $\eta_T : \mathbb{Z}^{r_1} \to T$ be an abelian presentation of T as above and let $\eta_F : \mathbb{Z}^{r_2} \to F$ present F in the obvious way. Then $\eta = \eta_T \oplus \eta_F : \mathbb{Z}^r \to G$ is called the *standard abelian presentation map* of G and ε, ε_T and ε_F are the associated presentation maps.

Let α be an endomorphism of G and suppose β is a lift of α with respect to ε. Denote the restriction of α to T by α_T and let α_F denote the composition $F \xrightarrow{\iota F} T \oplus F \xrightarrow{\alpha} T \oplus F \xrightarrow{\pi F} F$. Since T represents the torsion subgroup, $\alpha_T : T \to T$. If we write $\mathbb{Z}^r \cong \mathbb{Z}^{r_1} \oplus \mathbb{Z}^{r_2}$ then M_β has the form

$$M_\beta = \begin{array}{c c} & \begin{array}{cc} \mathbb{Z}^{r_1} & \mathbb{Z}^{r_2} \end{array} \\ \begin{array}{c} \mathbb{Z}^{r_1} \\ \mathbb{Z}^{r_2} \end{array} & \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \end{array}.$$

A is an integral matrix inducing α_T and C is an integral matrix inducing α_F. We show that α is an automorphism of G if and only if A satisfies the determinant condition on finite abelian groups given above and $\det(C) = \pm 1$.

Proposition 4. *Let G be a finitely generated abelian group and let α be an endomorphism of G. Then α is an automorphism if and only if α_T and α_F are automorphisms.*

Proof. ($\Rightarrow$) That α_T must be an automorphism is well known. Since T is an invariant subgroup α induces an isomorphism $\bar{\alpha} : (T \oplus F)/T \to (T \oplus F)/T$. Consider the map $\varphi : F \to (T \oplus F)/T$ given by $\varphi(f) = \overline{(0,f)}$. Then φ is also an isomorphism and it is easy to check that $\alpha_F = \varphi^{-1}\bar{\alpha}\varphi$.

($\Leftarrow$) Define γ to be the composition $F \xrightarrow{\iota F} T \oplus F \xrightarrow{\alpha} T \oplus F \xrightarrow{\pi T} T$. Then α is given by

$$\alpha(t,f) = (\alpha_T(t) + \gamma(f), \alpha_F(f)).$$

If δ denotes the composition $\alpha_T^{-1}\gamma\alpha_F^{-1}$, then the map β given by

$$\beta(t,f) = (\alpha_T^{-1}(t) - \delta(f), \alpha_F^{-1}(f))$$

represents the inverse to α. □

Since any automorphism of a free abelian group is tame, it is not surprising that the issue of tameness of an automorphism of G depends only on its restriction to the torsion subgroup.

Theorem 3. *Let $G \cong T \oplus F$ be a finitely generated abelian group and let α be an automorphism of G. Then α is tame with respect to ε if and only if α_T, the restriction of α to the torsion subgroup T, is tame with respect to ε_T.*

Proof. ($\Rightarrow$) Suppose $\tilde{\alpha}$ is an automorphism lifting α. We have shown that we can represent $\tilde{\alpha}$ by a matrix $M_{\tilde{\alpha}}$:

$$M_{\tilde{\alpha}} = \begin{array}{cc} & \begin{array}{cc} \mathbb{Z}^{r_1} & \mathbb{Z}^{r_2} \end{array} \\ \begin{array}{c} \mathbb{Z}^{r_1} \\ \mathbb{Z}^{r_2} \end{array} & \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \end{array}.$$

Then A is an integral matrix inducing α_T. Since $\tilde{\alpha}$ is an automorphism of a free abelian group, $\det(A)\det(C) = \pm 1$. In particular, A represents an automorphic lift of α_T.

($\Leftarrow$) Let β be a lift of α and suppose that $M_\beta = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ as usual. Since α is an automorphism, $\det(C) = \pm 1$ by Proposition 4. If A' is an integral matrix representing an automorphic lift of α_T then the matrix $\begin{pmatrix} A' & B \\ 0 & C \end{pmatrix}$ represents an automorphic lift of α. $\square$

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