Bull. Austral. Math. Soc. 43A15
Vol. 57 (1998) [409–414]

THE PREDUAL OF THE SPACE OF CONVOLUTORS ON A LOCALLY COMPACT GROUP

Michael Cowling

Let $Cv_p(G)$ be the space of convolution operators on the Lebesgue space $L^p(G)$, for an arbitrary locally compact group G. We describe $Cv_p(G)$ as a dual space, whose predual, $\overline{A}_p(G)$, is a Banach algebra of functions on G, under pointwise operations, with maximal ideal space G. This involves a variation of Herz's definition of $A_p(G)$; the benefit of this new definition is that all of $Cv_p(G)$ is obtained as the dual in the nonamenable setting. We also discuss further developments of this idea.

1. Introduction

Let G be a locally compact group, equipped with a left Haar measure, m; we abbreviate $dm(x)$ and $dm(y)$ in integrals to dx and dy, and write $|E|$ for the measure $m(E)$ of the set E. Let $L^p(G)$ denote the usual Lebesgue space relative to this measure. In the following, we assume that $1 < p < \infty$, and denote by p' the conjugate index $p/(p-1)$. We write $\mathcal{K}(G)$ for the set of all compact subsets of G with nonvoid interiors which contain the identity element e of G. For K in $\mathcal{K}(G)$, $L^p(K)$ denotes the subspace of $L^p(G)$ of functions supported in K. For a function on G, $\check{f}$ denotes the reflected version of f, that is, $\check{f}(x) = f(x^{-1})$ for all x in G.

We denote by $Cv_p(G)$ the space of convolutors of $L^p(G)$, that is, the Banach space of all bounded linear operators on $L^p(G)$ which commute with right translations. For Abelian G, Figà-Talamanca [**8**] introduced the Banach space $A_p(G)$, whose elements are bounded continuous functions on G which tend to 0 at infinity, and showed that $Cv_p(G)$ is the dual space of $A_p(G)$; Herz (cited by Eymard [**6**]) extended the definition to nonabelian groups, and generalised the theorem to the case when G is amenable. Herz also showed that, in the general case, the dual space $PM_p(G)$ of $A_p(G)$ is a (possibly proper) subspace of $Cv_p(G)$, which may be characterised as the weak operator topology closure of the set of operators of the form $g \mapsto f * g$, where $f \in L^1(G)$. (See also Figà-Talamanca and Gaudry [**9**], who proved similar results for convolutors from $L^p(G)$ to $L^q(G)$). After Herz's work, I showed that $PM_p(G) = Cv_p(G)$ for some nonamenable groups G [**1**], but the question of the relationship between these two

Received 4th September, 1997

spaces for a general locally compact group G is still undecided. I also showed [**2**] that every element of $Cv_p(G)$ is given by convolution with a quasimeasure, that is, an element of the dual space of $A_c(G)$, the space of compactly supported $A_2(G)$-functions, equipped with the inductive limit topology, and then H.G. Feichtinger showed that every element of $Cv_p(G)$ is given by convolution with an element of $\mathcal{S}(G)'$, where $\mathcal{S}(G)$ is the "minimal Siegel algebra" on G [**7**].

A number of papers have used $A_p(G)$ and its properties in proving results about convolutors in $PM_p(G)$; see, for instance, [**5**]. In order to be able to use these techniques for $Cv_p(G)$, it would be desirable to establish a similar duality for this space. One aim of this paper is to describe a way of doing this.

A second aim of this paper is to refocus attention on the problem of approximability of arbitrary convolutors on locally compact groups by compactly supported convolutors. This problem, which was considered virtually intractable in the early 1970's, now seems to be accessible, as remarked at the end of this paper.

2. Main Results

Given K in $\mathcal{K}(G)$, we define $\breve{A}_{p,K}(G)$ to be the space of all functions u on G, necessarily continuous and compactly supported, which admit representations of the form

$$u = \sum_{n=1}^{\infty} g_n * \check{f}_n,$$

where $f_n \in L^p(K)$, $g_n \in L^{p'}(K)$, and

$$\sum_{n=1}^{\infty} \|f_n\|_p \, \|g_n\|_{p'} < \infty.$$

The norm of u in $\breve{A}_{p,K}(G)$ is defined to be the infimum of all such sums of products of norms, over all such representations of u. In other words, $\breve{A}_{p,K}(G)$ is the image of the projective tensor product $L^p(K) \otimes_\gamma L^{p'}(K)$ under the continuous linear map which sends $f \otimes g$ to $g * \check{f}$. Note that $A_p(G)$ is defined similarly, but with no support restriction. We now define $\breve{A}_p(G)$ to be the union of all the spaces $\breve{A}_{p,K}(G)$ as K varies over $\mathcal{K}(G)$, and endow u in $\breve{A}_p(G)$ with the norm

$$\|u\|_{\breve{A}_p} = \inf\left\{ \|u\|_{\breve{A}_{p,K}} : u \in \breve{A}_{p,K}(G),\ K \in \mathcal{K}(G) \right\}.$$

It is easy to check that $\breve{A}_p(G)$ is a normed linear space. The difficulty might lie in showing that $\|u\|_{\breve{A}_p} > 0$ unless $u = 0$. To do this, we note that u is certainly

https://doi.org/10.1017/S0004972700031828 Published online by Cambridge University Press

in $A_p(G)$, and $\|u\|_{A_p} \leqslant \|u\|_{\breve{A}_p}$, so that if $\|u\|_{\breve{A}_p} = 0$, then $u = 0$ in $\breve{A}_p(G)$ because $u = 0$ in $A_p(G)$. Alternatively, one may merely remark that $\|u\|_\infty \leqslant \|u\|_{\breve{A}_p}$.

It is helpful to prove the following lemma, which complements the trivial inclusion $\breve{A}_p(G) \subseteq A_p(G)$ just mentioned.

LEMMA 1. *Suppose that $K \in \mathcal{K}(G)$. There exists a constant $C(K)$ such that, if $u \in A_p(G)$ and u has support in K, then $u \in \breve{A}_p(G)$, and*

$$\|u\|_{\breve{A}_p} \leqslant C(K)\ \|u\|_{A_p}.$$

PROOF: We take a relatively compact neighbourhood V of the identity e, and observe that $|V|^{-1}\chi_{KV} * \check{\chi}_V$ takes the value 1 on K. Denote the right regular representation on G by ρ. Herz's trick [11] for expressing the product of two $A_p(G)$-functions as another $A_p(G)$-function may be used to show that, if $\sum_{n=1}^{\infty} g_n * \check{f}_n$ is an $A_p(G)$-representation of u, then

$$\begin{aligned}
u &= |V|^{-1}\ \chi_{KV} * \check{\chi}_V\, u \\
&= |V|^{-1}\left(\chi_{KV} * \check{\chi}_V\right)\left(\sum_{n=1}^{\infty} g_n * \check{f}_n\right) \\
&= |V|^{-1} \sum_{n=1}^{\infty} \int_G \int_G g_n(y)\, \chi_{KV}(x)\, \check{f}_n(y^{-1}\cdot)\, \check{\chi}_V(x^{-1}\cdot)\, dy\, dx \\
&= |V|^{-1} \sum_{n=1}^{\infty} \int_G \int_G g_n(xy)\, \chi_{KV}(x)\, \check{f}_n(y^{-1}x^{-1}\cdot)\, \check{\chi}_V(x^{-1}\cdot)\, dy\, dx \\
&= |V|^{-1} \sum_{n=1}^{\infty} \int_G \int_G g_n(xy)\, \chi_{KV}(x)\, \check{f}_n(y^{-1}x^{-1}\cdot)\, \check{\chi}_V(x^{-1}\cdot)\, dx\, dy \\
&= |V|^{-1} \sum_{n=1}^{\infty} \int_G (\rho(y) g_n \cdot \chi_{KV}) * (\rho(y) f_n \cdot \chi_V)\check{}\ dy.
\end{aligned}$$

It follows that $u \in \breve{A}_{p,K_1}(G)$, where $K_1 = KV$, and further that

$$\begin{aligned}
\|u\|_{\breve{A}_{p,K_1}} &\leqslant |V|^{-1} \sum_{n=1}^{\infty} \int_G \|\rho(y) f_n \cdot \chi_V\|_p\ \|\rho(y) g_n \cdot \chi_{KV}\|_{p'}\, dy \\
&\leqslant |V|^{-1} \sum_{n=1}^{\infty} \|f_n\|_p\ \|\chi_V\|_p\ \|g_n\|_{p'}\ \|\chi_{KV}\|_{p'},
\end{aligned}$$

from which the lemma follows, with $C(K)$ equal to $|V|^{-1}\ \|\chi_V\|_p\ \|\chi_{KV}\|_{p'}$. ∎

Note that, if G is amenable, then we may make $C(K)$ arbitrarily close to 1, by taking V big enough. In general, this may not be possible, but we may still identify $\breve{A}_p(G)$ with the set of all compactly supported functions in $A_p(G)$, though the $A_p(G)$ and $\breve{A}_p(G)$-norms might differ.

Now we come to our main result.

THEOREM 2. *The dual of $\breve{A}_p(G)$ may be identified with $Cv_p(G)$, in the following way. For any T in $Cv_p(G)$, the linear functional Φ_T on $\breve{A}_p(G)$ is defined by the formula*

$$\Phi_T(u) = \sum_{n=1}^{\infty} \langle Tf_n, g_n\rangle,$$

for any $\breve{A}_p(G)$-representation $\sum_{n=1}^{\infty} g_n * \check{f}_n$ of u. Then Φ_T is well defined, and every continuous linear functional on $\breve{A}_p(G)$ arises in this way.

PROOF: First we show that Φ_T is well defined. Let $(h_\alpha : \alpha \in A)$ be an approximate identity for convolution of $C_c(G)$-functions. Given T in $Cv_p(G)$,

$$T(h_\alpha * f) = (Th_\alpha) * f = T_\alpha f,$$

say, so T is a limit of convolutions T_α by $L^p(G)$-functions Th_α. Given an $\breve{A}_p(G)$-representation of u, $\sum_{n=1}^{\infty} g_n * \check{f}_n$ say, there is a set K in $\mathcal{K}(G)$ such that all f_n and g_n are supported inside K. Then Fubini's theorem shows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \langle T_\alpha f_n, g_n\rangle &= \sum_{n=1}^{\infty} \langle (Th_\alpha) * f_n, g_n\rangle \\
&= \sum_{n=1}^{\infty} \langle (\chi_{K_1}.Th_\alpha) * f_n, g_n\rangle \\
&= \int_G (\chi_{K_1}.Th_\alpha)(x) \sum_{n=1}^{\infty} g_n * \check{f}_n(x)\, dx \\
&= \int_G (\chi_{K_1}.Th_\alpha)(x)\, u(x)\, dx \\
&= \int_G (Th_\alpha)(x)\, u(x)\, dx,
\end{aligned}$$

where $K_1 = KK^{-1}$. Thus Φ_{T_α} is well defined. Now a limiting argument shows that Φ_T is too. (It may be remarked that this is like the proof of duality for $A_p(G)$, except that one does not need to worry about localising).

https://doi.org/10.1017/S0004972700031828 Published online by Cambridge University Press

Now we show that every linear functional on $\check{A}_p(G)$ arises in this way. Given a linear functional Φ on $\check{A}_p(G)$, and f in $C_c(G)$, define Tf in $L^p(G)$ by the rule $\langle Tf, g\rangle = \Phi(g * \check{f})$ for all g in $C_c(G)$. It is easy to check that the map T defined in this way actually lies in $Cv_p(G)$ and that $\Phi = \Phi_T$. ∎

Let $\overline{A}_p(G)$ be the norm completion of $\check{A}_p(G)$. Then Theorem 2 implies that $Cv_p(G)$ is the dual of $\overline{A}_p(G)$. We conclude by describing $\overline{A}_p(G)$ in more detail.

THEOREM 3. *The space $\overline{A}_p(G)$ is a Banach algebra, whose Gel'fand spectrum is G.*

PROOF: From the argument of Lemma 1, it is obvious that $\overline{A}_p(G)$ is a Banach algebra. If ϕ is a multiplicative linear functional on $\overline{A}_p(G)$, then ϕ does not vanish on some u in some $\check{A}_p(G)$. This u vanishes off a compact set K. Given the equivalence of the $\check{A}_p(G)$ and $A_p(G)$–norms on the set of $A_p(G)$–functions vanishing off K (Lemma 1), it is clear that ϕ is a multiplicative linear functional on $A_p(G)$, and so lies in G, by Eymard's characterisation of the maximal ideal space of $A_p(G)$ [**6**]. ∎

Thus the space $A_p(G)$ is the quotient of $\overline{A}_p(G)$ by its radical.

3. Discussion

One interesting corollary of Theorem 2 is the following: if $PM_p(G) = Cv_p(G)$ for some group G, then the norms on $\check{A}_p(G)$ and $A_p(G)$ must coincide on the dense subspace $\check{A}_p(G)$. Then every compactly supported $A_p(G)$–function has an $A_p(G)$–representation where the sum of the products of the L^p and $L^{p'}$–norms is within ε of the $A_p(G)$–norm, and all the functions in the representation are supported in a fixed compact set. This is hinted at in [**1**]. In particular, when $p = 2$, the Kaplansky density theorem implies that $Cv_2(G) = PM_2(G)$, so that the $A_2(G)$ and $\check{A}_2(G)$–norms always coincide on $A_c(G)$, no matter what G is. See [**3**] for an application of this idea.

A further application in a similar vein is to the weakly amenable groups of Cowling and Haagerup [**4**]. If G is weakly amenable, then it admits an approximate identity for multiplication of $A_c(G)$–functions $(u_\alpha)_{\alpha\in A}$, which are completely bounded multipliers of $A_2(G)$, and hence bounded multipliers of $\check{A}_p(G)$, of norm at most Λ_G. By duality, these multiply elements of $Cv_p(G)$, permitting localisation. It follows that, for weakly amenable groups, $PM_p(G) = Cv_p(G)$, whence the $A_p(G)$ and $\check{A}_p(G)$–norms coincide on $\check{A}_p(G)$. Similarly the arguments of Haagerup and Krause [**10**] indicate that elements of $Cv_p(G)$ can be regularised even when G is not weakly amenable. Thus, for example, if $G = SL(2, \mathbf{Z}) \ltimes \mathbf{Z}^2$, then the $\check{A}_p(G)$ and $A_p(G)$–norms coincide on $\check{A}_p(G)$.

It would be interesting to know how the sizes of the sets K_1 depend on K in Lemma 1. For nonamenable groups for which the $\check{A}_p(G)$ and $A_p(G)$–norms coincide on $\check{A}_p(G)$, it may be possible to obtain, for any compact subset K of G, a single larger set K_1 so that the $A_p(G)$ and $\breve{A}_{p,K_1}(G)$–norms are essentially the same for all $A_p(G)$–functions supported in K.

REFERENCES

[1] M. Cowling, 'La synthèse des convoluteurs de L^p de certains groupes pas moyennables', *Boll. Un. Mat. Ital. A* **14** (1977), 551–555.

[2] M. Cowling, 'Some applications of Grothendieck's theory of topological tensor products in harmonic analysis', *Math. Ann.* **232** (1978), 273–285.

[3] M. Cowling, 'Rigidity for lattices in semisimple Lie groups: von Neumann algebras and ergodic actions', *Rend. Sem. Mat. Univ. Politec. Torino* **47** (1989), 1–37.

[4] M. Cowling and U. Haagerup, 'Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one', *Invent. Math.* **96** (1989), 507–549.

[5] J. Delaporte and A. Derighetti, 'p-pseudomeasures and closed subgroups', *Monatsh. Math.* **119** (1995), 37–47.

[6] P. Eymard, *Algèbres A_p et convoluteurs de L^p*, Lecture Notes in Math. **180**, (Sém. Bourbaki 1969–70 **367**) (Springer–Verlag, Berlin, Heidelberg, New York, 1970).

[7] H.G. Feichtinger, 'Un espace de Banach de distributions temperées sur les groupes localement compacts abeliens', *C.R. Acad. Sci. Paris Sér. A* **290** (1980), 791–794.

[8] A. Figà-Talamanca, 'Translation invariant operators in L^p', *Duke Math. J.* **32** (1965), 495–501.

[9] A. Figà-Talamanca and G.I. Gaudry, 'Density and representation theorems for multipliers of type (p,q)', *J. Austral. Math. Soc.* **7** (1967), 1–6.

[10] U. Haagerup and J. Kraus, 'Approximation properties for group C^*-algebras and group von Neumann algebras', *Trans. Amer. Math. Soc.* **344** (1994), 667–699.

[11] C.S. Herz, 'Remarques sur la Note précédente de M. Varopoulos', *C.R. Acad. Sci. Paris Sér. A* **260** (1965), 6001–6004.

School of Mathematics
University of New South Wales
Sydney NSW 2052
Australia
e-mail: michaelc@maths.unsw.edu.au