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ON THE STRUCTURE OF FINITE $T_0 + T_5$ SPACES

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The object of this paper is to study some structural aspects of finite $T_0 + T_4$ and $T_0 + T_5$ spaces in order to establish certain recursion relations that can be used to obtain the number of (labelled as well as unlabelled) $T_0 + T_5$ topologies on a finite set. Here, as in [**2**], a topology $\mathscr{T}$ is a $T_4(T_5)$ space provided for any pair of disjoint closed sets A and B (separated sets A and $B \equiv A \cap$ closure $B = B \cap$ closure $A = \emptyset$) there exist disjoint open sets O_A and O_B of $\mathscr{T}$ such that $A \subseteq O_A$ and $B \subseteq O_B$. An almost immediate consequence of these investigations is that the inherent simplicity of the connected $T_0 + T_5$ topologies ensures that they are reconstructable.

This article assumes a complete familiarity with the material developed in [**1**]. The spaces in this paper are always T_0 and are defined on a finite point set N. Let $\mathscr{T}$ be a topology on N and A a subset of N. Then $A^*(\mathscr{T})$, or more simply A^* when there is no risk of confusion, will denote the minimal open set of $\mathscr{T}$ that contains A. That is

$$A^*(\mathscr{T}) = \cap \{O | A \subseteq O \in \mathscr{T}\}.$$

A consequence of the T_0 property is that $\alpha \neq \beta$ and $\alpha \in \beta^*(\mathscr{T})$ implies $\beta \notin \alpha^*(\mathscr{T})$. A point α is a *maximal point* of $\mathscr{T}$ provided $\alpha \notin \beta^*(\mathscr{T})$ for all $\beta \neq \alpha$. For any set A, $|A|$ will denote the cardinality of A. The single element set $\{\alpha\}, \alpha \in N$, will be written simply as α. The union of α with a set A is written $\alpha + A$, and the relative difference of two sets A and B as $A - B$.

Let $\mathscr{T}$ be a topology on N. Let $\mathbf{C} = [\alpha_1, \ldots, \alpha_m]$ be a sequence of m distinct elements, $m \geqslant 1$, of N. $\mathbf{C}$ is called a *chain* of $\mathscr{T}$ of length m provided:

(1) if $\alpha_1^* - \alpha_1 = \beta^*$ for some $\beta \in N$, then there exists a $\gamma \in N$ such that $\gamma \neq \alpha_1$ and $\gamma^* - \gamma = \beta^*$;

(2) if $\beta^* - \beta = \alpha_m^*$ for some $\beta \in N$, then there exists a $\gamma \in N$ such that $\gamma \neq \beta$ and $\gamma^* - \gamma = \alpha_m^*$,

and if $m > 1$ and $1 \leqq i < m$, then

(3) $\alpha_{i+1}^* - \alpha_{i+1} = \alpha_i^*$;

(4) $\beta^* - \beta = \alpha_i^*$ for some $\beta \in N$ implies that $\beta = \alpha_{i+1}$.

The length of the chain $\mathbf{C}$ will be denoted by $L(\mathbf{C})$. The supporting open set of $\mathbf{C}$, written as $^*\mathbf{C}(\mathscr{T})$, or more simply as $^*\mathbf{C}$, when there is no risk of confusion, is defined to be the open set $\alpha_1^* - \alpha_1$ of $\mathscr{T}$. The notation $\{\mathbf{C} : i\}$, for $1 \leqq i \leqq m$, will be used to indicate the subset consisting of the first i terms of the sequence $\mathbf{C}$, and $\{\mathbf{C} : 0\} = \emptyset$. $\mathbf{C}$ will be used to denote both the sequence

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$[\alpha_1, \ldots, \alpha_m]$ and the unordered set $\{\alpha_1, \ldots, \alpha_m\}$. The meaning of $\mathbf{C}$ will always be clear from the context in which it will be used.

It is shown in [**1**] that the collection of chains of a topology partitions N. Moreover, under a homeomorphism between two topologies, the elements of a chain are always mapped, in the prescribed order, onto the elements of a chain of equal length.

An equivalence relation $\approx(\mathscr{T})$ may be defined on the set of chains of a topology $\mathscr{T}$ by requiring that if $\mathbf{C}, \mathbf{D}$ are chains of $\mathscr{T}$, then $\mathbf{C} \approx (\mathscr{T})\mathbf{D}$ if and only if $*\mathbf{C} = *\mathbf{D}$. A collection $\mathscr{C}$ of r distinct chains of a topology $\mathscr{T}$ is a *r chain cell*, or more simply a *cell*, of $\mathscr{T}$ if and only if $\mathscr{C}$ is an equivalence class of the equivalence relation $\approx(\mathscr{T})$. The *supporting open set* of $\mathscr{C}$, denoted by $*\mathscr{C}(\mathscr{T})$, or more simply by $*\mathscr{C}$ when there is no risk of confusion, is the (uniquely defined) supporting open set of any chain of $\mathscr{C}$. If $\mathscr{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_r\}$ is a cell, then $\mathscr{C}$ will denote both the collection of its constituent chains as well as the subset $\mathbf{C}_1 \cup \ldots \cup \mathbf{C}_r$. The meaning of $\mathscr{C}$ will be clear from the context in which it will be used. Like chains, the cells of a topology also behave like complete units under homeomorphisms.

1.

LEMMA 1. *Suppose that O is an open set of a topology $\mathscr{T}$ on N. Let $\{\mathbf{C}_1, \ldots, \mathbf{C}_i, \ldots, \mathbf{C}_p\}$ be the collection of chains of $\mathscr{T}$ that have non-void intersections with O. If $|\mathbf{C}_i \cap O| \leqq t_i \leqq L(\mathbf{C}_i)$ for $i = 1, \ldots, p$ then the set*

$$\bigcup_{i=1}^{p} \{\mathbf{C}_i : t_i\}$$

is an open set of $\mathscr{T}$.

Proof. The case $t_i = |\mathbf{C}_i \cap O|$ for $i = 1, \ldots, p$ is Lemma 7-(2) of [**1**]. Assume therefore that $|\mathbf{C}_i \cap O| < t_i$ for at least one i, so that $O \subset \bigcup \{\mathbf{C}_i : t_i\}$. If $\alpha \in \bigcup \{\mathbf{C}_i : t_i\}$ then $\alpha \in \mathbf{C}_j$ for some $\mathbf{C}_j \in \{\mathbf{C}_1, \ldots, \mathbf{C}_p\}$. Now $\alpha^* = *\mathbf{C}_j \cup \{\mathbf{C}_j : k\}$ for some $k \leqq t_j$. Since $O \cap \mathbf{C}_j \neq \emptyset$, therefore

$$*\mathbf{C}_j \subset O \subset \bigcup \{\mathbf{C}_i : t_i\}$$

and since $\{\mathbf{C}_j : k\} \subseteq \{\mathbf{C}_j : t_j\}$, therefore $\alpha^* \subseteq \bigcup \{\mathbf{C}_i : t_i\}$ and so $\bigcup \{\mathbf{C}_i : t_i\}$ is open.

LEMMA 2. *Let $\mathscr{C}$ be a cell of a topology $\mathscr{T}$ on N, such that $*\mathscr{C} \neq \emptyset$. If A_1 and A_2 are two subsets of N such that $A_1 \cap \mathscr{C} \neq \emptyset$ and $A_2 \cap \mathscr{C} \neq \emptyset$, then there do not exist disjoint open sets O_1, O_2 of $\mathscr{T}$ such that $A_1 \subseteq O_1$ and $A_2 \subseteq O_2$.*

Proof. The result follows immediately from Lemma 9-3(c) of [**1**].

LEMMA 3. *Let $\mathbf{C}$ be a chain of a topology $\mathscr{T}$ on N. If A_1 and A_2 are two open (closed) sets of $\mathscr{T}$ such that $A_1 \cap \mathbf{C} \neq \emptyset$ and $A_2 \cap \mathbf{C} \neq \emptyset$, then $A_1 \cap A_2 \neq \emptyset$.*

Proof. Let $\mathbf{C} = [\alpha_1, \ldots, \alpha_m]$. If A_1 and A_2 are open, then $\alpha_1^* \subseteq A_1 \cap A_2$. If A_1 and A_2 are closed, then $A_1 = N - O_1$ and $A_2 = N - O_2$ for some $O_1, O_2 \in \mathscr{T}$. The assumption $A_1 \cap \mathbf{C} \neq \emptyset \neq A_2 \cap \mathbf{C}$ implies that there exist i,

$j \leqq m$ such that $\alpha_i \notin O_1$ and $\alpha_j \notin O_2$. Since O_1 and O_2 are open, this in turn implies that $\alpha_m \notin (O_1 \cup O_2)$ so that $\alpha_m \in (A_1 \cap A_2)$.

LEMMA 4. *For any topology $\mathscr{T}$ on N, there exists a cell $\mathscr{F}$, called the first cell of $\mathscr{T}$, with the property that $^*\mathscr{F} = \emptyset$. This first cell is uniquely defined in the sense that if $\mathscr{C}$ is a cell of $\mathscr{T}$ and $^*\mathscr{C} = \emptyset$, then $\mathscr{C} = \mathscr{F}$.*

Proof. The result is Lemma 10 of [**1**].

It is now necessary to introduce a partial order on the collection of cells of a topology. If $\mathscr{C}$ and $\mathscr{D}$ are two cells of a multi-cell topology $\mathscr{T}$ on N, let $\mathscr{C} \lhd \mathscr{D}$ indicate that $\mathscr{C} \neq \mathscr{D}$ and there exists an $\alpha \in \mathscr{D}$ such that $\alpha^* \cap \mathscr{C} \neq \emptyset$. Then it is easily shown that $\mathscr{C} \lhd \mathscr{D}$ and $\mathscr{D} \lhd \mathscr{C}$ cannot be simultaneously true. Also, if $\mathscr{T}$ is a multi-cell topology and (1) if $\mathscr{F}$ is the first cell of $\mathscr{T}$, then $\mathscr{F} \lhd \mathscr{C}$ for any cell $\mathscr{C} \neq \mathscr{F}$, and (2) if $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$ are distinct cells of $\mathscr{T}$, then $\mathscr{C} \lhd \mathscr{D}$ and $\mathscr{D} \lhd \mathscr{E}$ implies $\mathscr{C} \lhd \mathscr{E}$ so that $\lhd$ defines a partial ordering on the collection of cells of $\mathscr{T}$.

LEMMA 5. *Let $\mathscr{T}$ be a multi-cell topology. Then there exists at least one cell $\mathscr{D}$ of $\mathscr{T}$ such that the relation $\mathscr{D} \lhd \mathscr{C}$ does not hold for nay cell $\mathscr{C}$.*

Comment. Such a cell will be termed a *maximal cell* of $\mathscr{T}$. If $\mathscr{T}$ is a single cell topology, then this cell is both the maximal and the first cell of $\mathscr{T}$.

Proof. Let $\mathscr{C}_1$ be an arbitrary cell and suppose $\mathscr{C}_1$ is not maximal. A sequence $\mathscr{C}_1 \lhd \ldots \lhd \mathscr{C}_i \lhd \mathscr{C}_{i+1} \ldots$ may be built up by searching for a cell $\mathscr{C}_{i+1}$, such that $\mathscr{C}_i \lhd \mathscr{C}_{i+1}$, if $\mathscr{C}_i$ is not maximal. Clearly, since all the cells of this sequence are distinct and any two cells are disjoint subsets of N, therefore any such sequence of cells must terminate at a term $\mathscr{C}_j$ such that $\mathscr{C}_j$ is a maximal cell. For otherwise, the finiteness of N is contradicted.

In general, a topology may have more than one maximal cell. However, if a multi-cell topology satisfies the T_4 or the T_5 separation property and is connected, then it has precisely one maximal cell. In other words, if $\mathscr{D}$ is a maximal cell of a connected T_4 or a T_5 topology and the cell $\mathscr{C} \neq \mathscr{D}$, then $\mathscr{C} \lhd \mathscr{D}$. This is demonstrated by the following sequence of Lemmas.

LEMMA 6. *Let $\mathscr{M}$ be a maximal cell of a topology $\mathscr{T}$ on N.*

(1) *The set $S_1 = \bigcup \{\mathbf{C} | \mathbf{C} \cap {}^*\mathscr{M} = \emptyset\}$ is a closed set of $\mathscr{T}$. If $\mathscr{T}$ is a multi-cell topology, then the set $S_2 = \bigcup \{\mathbf{C} | \mathbf{C} \notin \mathscr{M}$ and $\mathbf{C} \cap {}^*\mathscr{M} = \emptyset\}$ is also a closed set of $\mathscr{T}$.*

(2) *$\mathscr{M}$ is a closed set of $\mathscr{T}$. If $\mathscr{T}$ is a multi-cell topology then the set S_2, defined in* (1) *above, and $\mathscr{M}$ are disjoint closed sets.*

(3) *If $\mathscr{T}$ is a multi-cell topology and if $\mathscr{M}$ is a multi-chain cell, then $\mathscr{T}$ does not satisfy the T_4 axiom.*

Proof. Let $\{\mathbf{C}_1, \ldots, C_i, \ldots, \mathbf{C}_p\}$ be the collection of chains of $\mathscr{T}$ having non-void intersections with $^*\mathscr{M}$.

(1) Let $O_1 = \bigcup_{i=1}^{p} \mathbf{C}_i$. By Lemma 1, O_1 is an open set of $\mathscr{T}$. Clearly $S_1 = N - O_1$ and so S_1 is a closed set of $\mathscr{T}$. Now let $O_2 = O_1 \cup \mathscr{M}$. Obviously,

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$^*\mathscr{M} \subseteq O_1$ so that $O_2 = O_1 \cup (^*\mathscr{M} \cup \mathscr{M})$ and since $^*\mathscr{M} \cup \mathscr{M} \in \mathscr{T}$, therefore $O_2 \in \mathscr{T}$ and so $S_2 = N - O_2$ is closed.

(2) If $\mathscr{T}$ has only one cell, then $\mathscr{M} = N$. If $\mathscr{T}$ is a multi-cell topology, let $O_3 = \cup \{\mathscr{C} | \mathscr{C} \neq \mathscr{M}\}$. Since $\mathscr{M}$ is maximal, therefore if $\mathscr{C} \neq \mathscr{M}$ then $^*\mathscr{C} \cap \mathscr{M} = \emptyset$ which implies that $^*\mathscr{C} \subset O_3$ and so $O_3 \in \mathscr{T}$. Therefore $\mathscr{M} = N - O_3$ is closed. Since $\mathscr{M} \subseteq O_2$, therefore $\mathscr{M}$ and S_2 are disjoint.

(3) Now suppose that $\mathscr{T}$ is a multi-cell topology and that $\mathbf{M}_1$ and $\mathbf{M}_2$ are two chains of $\mathscr{M}$. Let $P_i = O_3 \cup (\mathscr{M} - \mathbf{M}_i)$, $i = 1, 2$. Since $\mathscr{M} \cap {}^*\mathscr{M} = \emptyset$, therefore $^*\mathscr{M} \subseteq O_3$. If $\alpha \in (\mathscr{M} - \mathbf{M}_i)$, then $\alpha \in \mathbf{M}$ where $\mathbf{M}$ is some chain of $\mathscr{M}$ different from $\mathbf{M}_i$. Therefore $\alpha^* \subseteq {}^*\mathbf{M} \cup \mathbf{M} = {}^*\mathscr{M} \cup \mathbf{M} \subseteq P_i$. Therefore P_1 and P_2 are open. Since $\mathbf{M}_1 = N - P_1$ and $\mathbf{M}_2 = N - P_2$, and since distinct chains are disjoint, therefore $\mathbf{M}_1$ and $\mathbf{M}_2$ are disjoint closed subsets. Since $\mathscr{T}$ is a multi-cell topology and $\mathscr{M}$ is maximal, therefore $\mathscr{M}$ cannot be the first cell of $\mathscr{T}$. Hence $^*\mathscr{M} \neq \emptyset$. If $Q_1, Q_2 \in \mathscr{T}$ and $\mathbf{M}_1 \subseteq Q_1$ and $\mathbf{M}_2 \subseteq Q_2$, then $^*\mathscr{M} \subseteq Q_1 \cap Q_2$ so that $\mathbf{M}_1$ and $\mathbf{M}_2$ cannot be separated by disjoint open sets of $\mathscr{T}$.

LEMMA 7. *Let $\mathscr{T}$ be a multi-cell T_4 topology on N and $\mathscr{M}$ a maximal cell of $\mathscr{T}$.*

(1) *If a chain of $\mathscr{T}$ intersects $^*\mathscr{M}$, then that chain is a subset of $^*\mathscr{M}$.*

(2) *If $\mathscr{C}$ is a multi-chain cell, other than the first cell, and if some chain of $\mathscr{C}$ intersects $^*\mathscr{M}$, then every chain of $\mathscr{C}$ is a subset of $^*\mathscr{M}$, that is $\mathscr{C} \subseteq {}^*\mathscr{M}$.*

(3) *If $\mathscr{C} \neq \mathscr{M}$ is a cell such that $^*\mathscr{C} \cap {}^*\mathscr{M} \neq \emptyset$, then every chain of $\mathscr{C}$ is a subset of $^*\mathscr{M}$, that is $\mathscr{C} \subseteq {}^*\mathscr{M}$.*

Proof. Let $\{\mathbf{C}_1, \ldots, \mathbf{C}_i, \ldots, \mathbf{C}_p\}$ be the collection of chains of $\mathscr{T}$ having non-void intersections with $^*\mathscr{M}$.

(1) Let $|\mathbf{C}_i \cap {}^*\mathscr{M}| = t_i \leqq L(\mathbf{C}_i)$. Suppose that the chain $\mathbf{C}_i = [\alpha_1, \ldots, \alpha_q]$ is not a subset of $^*\mathscr{M}$, that is $1 \leqq t_i < q$. Since

$$^*\mathscr{M} = \bigcup_{i=1}^{p} \{\mathbf{C}_i : t_i\},$$

therefore $\alpha_q \notin {}^*\mathscr{M}$. Further, since $^*\mathscr{M} \cap \mathscr{M} = \emptyset$ and $\mathbf{C}_i \cap {}^*\mathscr{M} \neq \emptyset$, therefore $\mathbf{C}_i$ is not a chain of $\mathscr{M}$ and so $\alpha_q \notin \mathscr{M}$. Therefore $\alpha_q \notin O = {}^*\mathscr{M} \cup \mathscr{M} \in \mathscr{T}$, and so $\alpha_q \in N - O$. Now let O_1 and O_2 be two open sets of $\mathscr{T}$ such that $N - O \subseteq O_1$ and $\mathscr{M} \subseteq O_2$. Then $^*\mathscr{M} \subseteq O_2$ which implies $\alpha_1 \in O_2$, and $\mathbf{C}_i \subseteq \alpha_q^* \subseteq O_1$ so that $\alpha_1 \in O_1 \cap O_2$. Thus the disjoint closed sets $N - O$ and $\mathscr{M}$ cannot be separated by disjoint open sets and this contradicts the hypothesis that $\mathscr{T}$ is T_4.

(2) Suppose that $\mathbf{C}$ is a chain of $\mathscr{C}$ and $\mathbf{C} \cap {}^*\mathscr{M} \neq \emptyset$. Because of result (1) above, it will be sufficient to show that if $\mathbf{D}$ is a chain of $\mathscr{C}$, then $^*\mathscr{M} \cap \mathbf{D} = \emptyset$ implies that $\mathscr{T}$ is not T_4. Let S be the union of all chains of $\mathscr{T}$ that neither belong to the cell $\mathscr{M}$ nor intersect $^*\mathscr{M}$. Then the assumption $^*\mathscr{M} \cap \mathbf{D} = \emptyset$ implies $\mathbf{D} \subseteq S$. Suppose that $S \subseteq O_1$ and $\mathscr{M} \subseteq O_2$, for some $O_1, O_2 \in \mathscr{T}$. Then since $^*\mathscr{M} \cap \mathbf{C} \neq \emptyset$, therefore $^*\mathbf{D} = {}^*\mathbf{C} = {}^*\mathscr{C} \subseteq O_1 \cap O_2$. Since $\mathscr{C}$ is not the first cell, therefore $^*\mathscr{C} \neq \emptyset$. Consequently S and $\mathscr{M}$, which are disjoint closed sets by Lemma 6-(2), cannot be separated by disjoint open sets of $\mathscr{T}$.

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(3) By results (1) and (2) above, it follows that either $\mathbf{C} \cap {}^*\mathscr{M} = \emptyset$ or $\mathbf{C} \subseteq {}^*\mathscr{M}$ for every chain $\mathbf{C} \in \mathscr{C}$. Suppose that $\mathbf{C} \cap {}^*\mathscr{M} = \emptyset$ for every chain $\mathbf{C} \in \mathscr{C}$. Then $\mathscr{C} \subseteq S$, where S is the closed set defined in (2) above. Now suppose that the disjoint closed sets S and $\mathscr{M}$ are contained in the open sets O_1 and O_2 respectively. Then ${}^*\mathscr{C} \cap {}^*\mathscr{M} \subseteq O_1 \cap O_2$ and, since, by hypothesis, ${}^*\mathscr{C} \cap {}^*\mathscr{M} \neq \emptyset$ therefore $\mathscr{T}$ is not T_4.

COROLLARY. ${}^*\mathscr{M} = \bigcup \{\mathbf{C} | \mathbf{C} \cap {}^*\mathscr{M} \neq \emptyset\}$.

LEMMA 8. *Let $\mathscr{T}$ be a multi-cell T_4 topology on N and let $\mathscr{M}$ be a maximal cell of $\mathscr{T}$. Then a necessary condition for $\mathscr{T}$ to be connected is that $\mathbf{C} \subseteq {}^*\mathscr{M}$ for every chain $\mathbf{C} \notin \mathscr{M}$.*

Proof. Assume the contrary and let $\{\mathbf{C}_1, \ldots, \mathbf{C}_i, \ldots, \mathbf{C}_p\}$ be the non-void collection of chains of $\mathscr{T}$ that do not belong to $\mathscr{M}$ and are not subsets of ${}^*\mathscr{M}$. By Lemma 7-(1), $\mathbf{C}_i \cap {}^*\mathscr{M} = \emptyset$. Let $G = \mathbf{C}_1 \cup \ldots \cup \mathbf{C}_p$. Clearly G and the open set ${}^*\mathscr{M} = \bigcup \{\mathbf{D} | \mathbf{D} \cap {}^*\mathscr{M} \neq \emptyset\}$ are disjoint sets. It will now be demonstrated that G is open. For this purpose it is sufficient to show that ${}^*\mathbf{C}_i \subseteq G$ for all i, $1 \leqq i \leqq p$. Let $\mathbf{C}_i \in \mathscr{C} \neq \mathscr{M}$. Since $\mathbf{C}_i \cap {}^*\mathscr{M} = \emptyset$, therefore Lemma 7-(3) implies that ${}^*\mathscr{C} \cap {}^*\mathscr{M} = \emptyset$. Since $\mathscr{M}$ is maximal, therefore ${}^*\mathscr{C} \cap \mathscr{M} = \emptyset$. It is obvious that the pairwise disjoint sets $G, \mathscr{M}$ and ${}^*\mathscr{M}$ form a partition of N. Therefore ${}^*\mathbf{C}_i = {}^*\mathscr{C} \subseteq G$ and so $G \in \mathscr{T}$. Thus G and $(\mathscr{M} \cup {}^*\mathscr{M})$ is an open partition of N and so $\mathscr{T}$ is disconnected.

An immediate consequence of Lemmas 6-(3) and 8 is:

LEMMA 9. *If $\mathscr{T}$ is a connected multi-cell T_4 topology on N, then*
(1) *there exists one, and only one, maximal cell of $\mathscr{T}$,*
(2) *this uniquely defined cell $\mathscr{M}$ is a single chain cell, and*
(3) ${}^*\mathscr{M} = N - \mathscr{M}$.

The demonstrations used in Lemmas 6, 7 and 8 indicate these results to be somewhat "negative" in the sense that they investigate conditions under which the T_4 property is not violated. Let a *trivially* T_4 space be one in which $A \cap B = \emptyset$ and both A, B closed imply that one of A, B is the void set. Trivially T_4 spaces are clearly connected and T_4. Conversely, if a connected multi-cell space does not violate T_4, then it is trivially T_4. This is an immediate consequence of Lemma 9 and the next result.

LEMMA 10. *If $\mathscr{T}$ is a T_0 topology possessing a single maximal point, then $\mathscr{T}$ is connected and is trivially* T_4.

Proof. If α is the only maximal point of $\mathscr{T}$, then $\alpha^* = N$. $\mathscr{T}$ is therefore connected. Also, any non-void closed set contains α. Thus T_4 is trivially satisfied as it is impossible to obtain a pair of disjoint closed subsets both of which are non-void.

However, there exist "non-trivial" connected $T_0 + T_5$ spaces in the sense that they contain pairs of non-void separated sets.

LEMMA 11. (1) *Let $\mathscr{T}$ be a single cell topology on the n point set N.*
(a) *If this cell is a single chain cell, then $\mathscr{T}$ is connected and is trivially* T_4.
(b) *If this cell is a multi-chain cell, then $\mathscr{T}$ is disconnected* T_4.
(2) *Every single cell topology has the* T_5 *property.*

Proof. (1a) In this case $\mathscr{T}$ is of the form

$$\mathscr{T} = \{\emptyset, \{\alpha_1\}, \ldots, \{\alpha_1, \ldots, \alpha_i\}, \ldots, N\}$$

so that $\alpha_n{}^* = N$. $\mathscr{T}$ is clearly connected.

(1b) In this case let $\mathscr{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_i, \ldots, \mathbf{C}_p\}$ be the only cell of $\mathscr{T}$. Then $^*\mathscr{C} = {}^*\mathbf{C}_i = \emptyset$ and so the union of an arbitrary collection of chains is an open set. The required result follows from the observation that if A and B are disjoint closed sets, then by Lemma 3, $A \subseteq \bigcup \{\mathbf{C}_i | i \in I\}$ and $B \subseteq \bigcup \{\mathbf{C}_j | j \in J\}$ where I and J are disjoint subsets of $\{1, \ldots, p\}$.

(2) This result follows immediately from (1) and the fact [**2**, p. 92] that the T_5 property is hereditary and moreover, a space is T_5 if and only if it is hereditarily T_4.

LEMMA 12. *Let $\mathscr{T}$ be a connected multi-cell* T_4 *topology on N and let $\mathscr{M}$ be the uniquely defined single chain and maximal cell of $\mathscr{T}$. Then the subspace $\mathscr{U}$ induced on the open set $N' = N - \mathscr{M}$ by $\mathscr{T}$ is either non-*T_4 *or disconnected* T_4.

Proof. Let $\mathscr{M} = \{\mathbf{M}\}$. By the maximality of $\mathscr{M}$, it follows that if $\alpha \in N'$ then $\alpha^*(\mathscr{T}) \cap \mathscr{M} = \emptyset$, so that $N' \in \mathscr{T}$ and $\alpha^*(\mathscr{T}) = \alpha^*(\mathscr{U})$. Therefore, apart from $\mathscr{M}$ and $\mathbf{M}$, the collection of chains and cells of $\mathscr{T}$ and $\mathscr{U}$ are identical. Now suppose that $\mathscr{U}$ is a connected T_4 space. Then by Lemmas 9 and 11-(1), there exists a chain $\mathbf{S}$ such that $^*\mathbf{S} = N' - \mathbf{S}$. Let $\mathbf{S} = [\alpha_1, \ldots, \alpha_p]$ and $\mathbf{M} = [\beta_1, \ldots, \beta_q]$. Then $\alpha_p{}^* = {}^*\mathbf{S} \cup \mathbf{S} = N'$ and so $\beta_1{}^* - \beta_1 = {}^*\mathbf{M} = N' = \alpha_p{}^*$. Condition (2) in the definition of a chain now asserts that there exists a $\gamma \neq \beta_1$ such that $\gamma^* - \gamma = {}^*\mathbf{M}$. At the same time, the fact that $\mathscr{M}$ is a single chain cell implies that there does not exist any γ with these properties. This is a contradiction, and so $\mathscr{U}$ is either disconnected T_4 or non-T_4.

Let $[\alpha_1, \ldots, \alpha_m]$ be a sequence of m, $m < n$, distinct elements of N and let $\mathscr{U}$ be a topology on $P = N - \{\alpha_1, \ldots, \alpha_m\}$. Then $\mathscr{U} + [\alpha_1, \ldots, \alpha_m]$ will denote that topology $\mathscr{T}$ on N defined by the statement: O is an open set of $\mathscr{T}$ if and only if either $O \in \mathscr{U}$ or $O = P \cup \{\alpha_1, \ldots, \alpha_i\}$ for $1 \leqq i \leqq m$. The results of the previous Lemmas are now strengthened and summarised into Theorem 1. No further proofs are needed except the observation that in Theorem 1-(2) use is made of the hereditary T_4 property.

THEOREM 1. (1) *A multi-cell topology $\mathscr{T}$ on N is a connected* T_4 *space if and only if there exists a sequence $[\alpha_1, \ldots, \alpha_m]$, $m < n$, of m distinct elements of N, and a topology $\mathscr{U}$ on $N - \{\alpha_1, \ldots, \alpha_m\}$ such that $\mathscr{U}$ is either non-*T_4 *or disconnected* T_4 *and $\mathscr{T} = \mathscr{U} + [\alpha_1, \ldots, \alpha_m]$. A connected* T_4 *space is always trivially* T_4.

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(2) *If a* T_5 *space has more than one cell, then the first cell (that is, the cell having* $\emptyset$ *as the supporting open set) is a multiple chain cell. All other cells are single chain cells. Further, a multi-cell topology* $\mathscr{T}$ *on* N *is a connected* T_5 *space if and only if there exists a sequence* $[\alpha_1, \ldots, \alpha_m]$, $m < n$, *and a disconnected* T_5 *space* $\mathscr{U}$ *on* $N - \{\alpha_1, \ldots, \alpha_m\}$ *such that* $\mathscr{T} = \mathscr{U} + [\alpha_1, \ldots, \alpha_m]$.

(3) *Let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be topologies on* N *such that* $\mathscr{T}_1 = \mathscr{U}_1 + [\alpha_1, \ldots, \alpha_p]$ *and* $\mathscr{T}_2 = \mathscr{U}_2 + [\beta_1, \ldots, \beta_q]$ *where* $\mathscr{U}_1$ *and* $\mathscr{U}_2$ *are either non-*T_4 *or disconnected* T_4 *spaces on* $N - \{\alpha_1, \ldots, \alpha_p\}$ *and* $N - \{\beta_1, \ldots, \beta_q\}$ *respectively. Then*

(a) $\mathscr{T}_1 = \mathscr{T}_2$ *if and only if* $p = q$, $\alpha_i = \beta_i$ *for* $i \leqq p$ *and* $\mathscr{U}_1 = \mathscr{U}_2$.

(b) $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *are homeomorphic if and only if* $p = q$ *and* $\mathscr{U}_1$ *and* $\mathscr{U}_2$ *are homeomorphic.*

Theorem 1-(3) states that the space $\mathscr{U}$ and the sequence $[\alpha_1, \ldots, \alpha_m]$ in Theorem 1-(1) and (2) are uniquely determined. In fact $[\alpha_1, \ldots, \alpha_m]$ appears as a chain of $\mathscr{T}$ with $N - \{\alpha_1, \ldots, \alpha_m\}$ as its supporting open set. This chain will be referred to as the *covering chain* of the connected T_4 space $\mathscr{T}$ and $\mathscr{U}$ as the *base topology* of $\mathscr{T}$.

Let $T^c_{0+4}(n)$ = number of distinct connected $T_0 + T_4$ spaces, $T^c_{0+5}(n)$ = number of distinct connected $T_0 + T_5$ spaces, $H^c_{0+4}(n)$ = number of homeomorphism classes of connected $T_0 + T_4$ spaces and $H^c_{0+5}(n)$ = number of homeomorphism classes of connected $T_0 + T_5$ spaces. The corresponding quantities for the disconnected case are denoted by $T^d_{0+4}(n)$, $T^d_{0+5}(n)$, $H^d_{0+4}(n)$ and $H^d_{0+5}(n)$. $T_0(n)$ and $H_0(n)$ will represent, respectively, the number of distinct T_0 spaces and the number of homeomorphism classes of T_0 spaces. The argument n in all these quantities denotes the cardinality of the set on which the topologies are defined.

THEOREM 2.

(1)

(a) $T^c_{0+4}(1) = T^c_{0+5}(1) = H^c_{0+4}(1) = H^c_{0+5}(1) = 1.$

(b) $T^c_{0+4}(2) = T^c_{0+5}(2) = 2.$

(c) $H^c_{0+4}(2) = H^c_{0+5}(2) = 1.$

(2)

(a) $T^d_{0+4}(1) = T^d_{0+5}(1) = H^d_{0+4}(1) = H^d_{0+5}(1) = 0.$

(b) $T^d_{0+4}(2) = T^d_{0+5}(2) = H^d_{0+4}(2) = H^d_{0+5}(2) = 1.$

For (3) *to* (10), *assume* $n \geqq 3$.

(3) $$T^c_{0+4}(n) = n! + \sum_{m=1}^{n-2} m! \binom{n}{m} \{T_0(n-m) - T^c_{0+4}(n-m)\}.$$

(4) $$T^c_{0+5}(n) = n! + \sum_{m=1}^{n-2} \left\{T^d_{0+5}(n-m) \binom{n}{m} m!\right\}.$$

(5) $$H^c_{0+4}(n) = 1 + \sum_{m=1}^{n-2} \{H_0(n-m) - H^c_{0+4}(n-m)\}.$$

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(6) $H^c_{0+5}(n) = 1 + \sum_{m=1}^{n-2} H^d_{0+5}(n-m)$.

(7) $T^c_{0+4}(n+1) = (n+1)T_0(n)$.

(8) $H^c_{0+4}(n+1) = H_0(n)$.

(9) $T^d_{0+m}(n) = \sum \frac{n!}{(n_1!)^{r_1}r_1! \ldots (n_u!)^{r_u}r_u!} \prod_{i=1}^{u} \{T^c_{0+m}(n_i)\}^{r_i}, m = 4, 5.$

(10) $H^d_{0+m}(n) = \sum \prod_{i=1}^{u} \binom{H^c_{0+m}(n_i) + r_i - 1}{r_i}, m = 4, 5.$

where in (9) *and* (10), *the* $\sum$ *extends over all possible partitions* $\sum_{i=1}^{u} r_i n_i$ *of* n *with* $0 < n_1 \ldots < n_u$ *and* $r_i \geqq 1$.

Proof. (3) is an immediate consequence of Theorem 1-(1) and -(3) and the fact that it is possible to select exactly $\binom{n}{m} m!$ distinct m-term sequences from a set of n elements. Similarly (4) follows from Theorem 1-(2). The $n!$ term in (3) and (4) is present to take into account the $n!$ distinct single chain (and therefore single cell) n point topologies that are all, by Lemma 10, both T_4 and T_5. The fact that these single chain spaces are all homeomorphic explains the '1' term in (5) and (6). Let $\psi(k) = T_0(k) - T^c_{0+4}(k)$. Then from (3) it follows that:

$$T^c_{0+4}(n+1) = (n+1)\left[T_0(n) - T^c_{0+4}(n) + \left\{n! + \sum_{m=2}^{n-1} \psi(n+1-m) \binom{n}{m-1} (m-1)!\right\}\right]$$
$$= (n+1)\left[T_0(n) - T^c_{0+4}(n) + \left\{n! + \sum_{j=1}^{n-2} \psi(n-j) \binom{n}{j} j!\right\}\right],$$

from which (7) is obvious. Similarly (8) is a consequence of (5). The rest of Theorem 2 is elementary.

Using these recursion relations it is easily verified that $T_{0+4}(4) = T^c_{0+4}(4) + T^d_{0+4}(4) = 76 + 61 = 137$, and $T_{0+5}(4) = T^c_{0+5}(4) + T^d_{0+5}(4) = 64 + 61 = 125$, so that 12 distinct, non-T_5 and $T_0 + T_4$ topologies can be defined on a four point set. They are all connected and trivially T_4. There do not exist spaces that are $T_0 + T_4$ but not $T_0 + T_5$ on sets with fewer than four points. It should be observed that the expressions (3) and (7) for T^c_{0+4} are not "pure" recursion relations, in the sense that they involve $T_0(n)$. However, the expressions for T_5 spaces do not suffer from any such difficulties. The reason for this is the fact that the T_5 property is hereditary.

2. For any topology $\mathscr{T}$ on an n point set N, let $\mathscr{T}_v$ denote the $(n-1)$ point subspace induced by $\mathscr{T}$ on the point set $N - v$. Then the reconstruction conjecture for finite topologies can be stated precisely as follows:

Reconstruction conjecture: Let $\mathscr{T}$ be a topology on N. Suppose that each $\mathscr{T}_v$, $v \in N$, is known *to within homeomorphism*. Then $\mathscr{T}$ is itself determined to within homeomorphism by the collection of the subspaces $\mathscr{T}_v$.

The reconstruction conjecture breaks down for certain three point topologies. For example, let

$$\mathscr{S} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$$

and

$$\mathscr{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

Then $\mathscr{S}$ and $\mathscr{T}$ are not homeomorphic. However, both possess the same collection of two point subspaces: $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$, $\{\emptyset, \{x\}, \{x, y\}\}$ and $\{\emptyset, \{x\}, \{x, y\}\}$. In the rest of this paper, all topologies will therefore be considered to have been defined on sets with at least four points.

THEOREM 3. *Let $\mathscr{T}$ be a topology on N. Suppose that each subspace T_v, $v \in N$, is known to within homeomorphism. Then these subspaces determine whether or not $\mathscr{T}$ is simultaneously connected,* T_0 *and* T_5. *In the event that it is connected* $T_0 + T_5$, *$\mathscr{T}$ is itself determined to within homeomorphism by the $\mathscr{T}_v$.*

Proof. An Algorithm to reconstruct $\mathscr{T}$ is outlined below. Lemmas 13 to 16, which follow later on, ensure a correct output. The word "collection" stands for "collection of subspaces $\mathscr{T}_v$", $v \in N$. The collection is said to satisfy condition

(+1) if each of at least $n - 1$ of the subspaces $\mathscr{T}_v$ have only one maximal point;

(+2) if each of the n subspaces $\mathscr{T}_v$ are single chain spaces;

(+3) if only one of the subspaces $\mathscr{T}_v$ is disconnected.

START 1. Test whether each $\mathscr{T}_v$, $v \in N$, is T_0. If not, then go to 9. If yes, then $\mathscr{T}$ is T_0. Now go to 2.

2. Test whether the collection satisfies (+1). If not, then go to 9. If yes, then $\mathscr{T}$ is connected and T_4. Now go to 3.

3. Test whether each $\mathscr{T}_v$, $v \in N$, is T_4. If not, then go to 9. If yes, then $\mathscr{T}$ is connected $T_0 + T_5$. Now go to 4.

4. Test whether the collection satisfies (+2). If yes, then go to 8. If not, then go to 5.

5. Test whether the collection satisfies (+3). If yes, then go to 7. If not, then go to 6.

END 6. Choose a $\mathscr{T}_v$ such that the length k of its covering chain is not greater than the length of the covering chain of any other $\mathscr{T}_w$, $w \neq v$. Let $m = k + 1$. Now arbitrarily label, from the collection $N - \{\alpha_1, \ldots, \alpha_m\}$, the points of the base topology of $\mathscr{T}_v$ and call this labelled space $\mathscr{U}$. Then $\mathscr{T}$ is homeomorphic to $\mathscr{U} + [\alpha_1, \ldots, \alpha_m]$.

END 7. Arbitrarily label the points of the only disconnected $n - 1$ point subspace from the collection $\{\alpha_2, \ldots, \alpha_m\}$ and call this labelled space $\mathscr{U}$. Then $\mathscr{T}$ is homeomorphic to $\mathscr{U} + [\alpha_1]$.

END 8. $\mathscr{T}$ is homeomorphic to $\{\emptyset, \{\alpha_1\}, \ldots, \{\alpha_1, \ldots, \alpha_i\}, \ldots, N\}$.
END 9. $\mathscr{T}$ is not simultaneously connected, T_0 and T_5.

The supporting Lemmas now follow.

LEMMA 13. *A* T_0 *topology* $\mathscr{T}$ *on* N *has only one maximal point if and only if condition* (+1) *is satisfied.*

Proof. If α is the only maximal point of $\mathscr{T}$, then $\alpha^*(\mathscr{T}) = N$ and so $\alpha^*(\mathscr{T}_v) = N - v$ for all $v \neq \alpha$. Hence α is the only maximal point for each $\mathscr{T}_v, v \neq \alpha$. The condition is therefore necessary. Now suppose that $\mathscr{T}$ has two maximal points. Since $n \geqq 4$, therefore there exist at least two non-maximal points γ, δ. Then both $\mathscr{T}_\gamma$ and $\mathscr{T}_\delta$ have two maximal points. Another possibility is that $\mathscr{T}$ has more than 2 maximal points. If α, β are maximal, then both $\mathscr{T}_\alpha$ and $\mathscr{T}_\beta$ have more than one maximal point. This establishes the sufficiency.

LEMMA 14. *A* T_0 *topology* $\mathscr{T}$ *on* N *is a single chain space if and only if condition* (+2) *is satisfied.*

The proof is elementary and is omitted.

LEMMA 15. *Let* $\mathscr{T}$ *be a connected* $T_0 + T_5$ *topology. Then the covering chain of* $\mathscr{T}$ *has length* = 1 *if and only if condition* (+3) *is satisfied.*

Proof. If $\mathscr{T} = \mathscr{U} + [\alpha]$, then $\mathscr{T}_\alpha = \mathscr{U}$ which is disconnected by Theorem 1. If $\beta \neq \alpha$, then $\alpha^*(\mathscr{T}_\beta) = N - \beta$ and so $\mathscr{T}_\beta$ is connected. The condition is therefore necessary. Now suppose that $\mathscr{T} = \mathscr{U} + [\alpha_1, \ldots, \alpha_m]$ where $m \geqq 2$. Then each $\mathscr{T}_v$ is connected. This is because α_m is the only maximal point of $\mathscr{T}_v$ if $v \neq \alpha_m$ and $\mathscr{T}_{\alpha_m}$ has α_{m-1} as the only maximal point. The condition is therefore sufficient.

Comment. The proof clearly demonstrates that there can exist at most one disconnected $n - 1$ point subspace of a n point connected $T_0 + T_5$ topology.

LEMMA 16. *Let* $\mathscr{T}$ *be a connected* $T_0 + T_5$ *topology, other than a single chain space. Suppose that the covering chain of* $\mathscr{T}$ *has length* m, $m \geqq 2$. *Then*
(1) *There exists a subspace* $\mathscr{T}_v$ *whose covering chain has length* $m - 1$.
(2) *The length of the covering chain of any subspace* $\mathscr{T}_v$ *is at least* $m - 1$.
(3) *If the subspace* $\mathscr{T}_v$ *has a covering chain with length* $m - 1$, *then the base topologies of* $\mathscr{T}$ *and* $\mathscr{T}_v$ *are identical.*

Proof. Let $\mathscr{T} = \mathscr{U} + [\alpha_1, \ldots, \alpha_m]$. $\mathscr{U}$ has at least two maximal points. Therefore if $v \in \{\alpha_1, \ldots, \alpha_m\}$, then $\mathscr{T}_v$ has a covering chain of length $m - 1$ and the base topologies of $\mathscr{T}$ and $\mathscr{T}_v$ are identical (to $\mathscr{U}$). If $v \in N' = N - \{\alpha_1, \ldots, \alpha_m\}$, then $\alpha_1^*(\mathscr{T}_v) = N' - v$ so that $[\alpha_1, \ldots, \alpha_m]$ is either the covering chain or a part of the covering chain of $\mathscr{T}_v$. Therefore in this case the length of the covering chain of $\mathscr{T}_v$ is at least m. This proves (1) and (2). (3) follows as it is now clear that $\mathscr{T}_v$ has a covering chain of length $m - 1$ if and only if $v \in \{\alpha_1, \ldots, \alpha_m\}$.

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Comment. If $\mathscr{T} = \{\emptyset, \{\alpha_1\{, \{\alpha_2\}, \{\alpha_1, \alpha_2\}\} + [\alpha_3, \ldots, \alpha_m]$, then it is clear that $\mathscr{T}_{\alpha_1}$ and $\mathscr{T}_{\alpha_2}$ are both single chains of length $n - 1$. Therefore, in instruction (6) of the reconstruction Algorithm, if a $\mathscr{T}_w$ consists only of a single chain then this chain should be considered as the covering chain of $\mathscr{T}_w$.

Further discussion on the reconstruction problem for finite topologies will appear elsewhere.

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