Canad. Math. Bull. Vol. 25 (1), 1982

ON QUASISIMILARITY FOR SUBNORMAL OPERATORS, II

BY
JOHN B. CONWAY*

ABSTRACT. Let S be a subnormal operator and let $\mathcal{A}(S)$ be the weak-star closed algebra generated by S and 1. An example of an irreducible cyclic subnormal operator S is found such that there is a T in $\mathcal{A}(S)$ with S and T quasisimilar but not unitarily equivalent. However, if S is the unilateral shift, $T \in \mathcal{A}(S)$, and S and T are quasisimilar, then $S \cong T$.

This paper will show that any analytic Toeplitz operator that is quasisimilar to the unilateral shift must be unitarily equivalent to it.

Unlike normal operators, two similar subnormal operators need not be unitarily equivalent. The first example of this unfortunate phenomenon seems to be due to Donald Sarason ([5], Solution 156). Many examples of this are now known. In fact, the subnormal operators that are similar to the unilateral shift have been characterized [2], and these include many operators that are not shifts.

With such examples one might wonder if any interesting questions remain to be asked. Indeed, there are. One class of such questions is obtained by insisting that the similar subnormal operators have additional properties. For example in [4] the author and R. F. Olin asked the following question. If S is a subnormal operator and $T \in \mathcal{A}(S)$, the weak * (*viz.*, ultraweakly) closed algebra generated by S, and if S and T are similar, must S and T be unitarily equivalent? Warren Wogen [8] gave an example which showed the answer to this question to be "No".

However Wogen's example of S had the property that $\mathcal{A}(S)$ could be split into the direct sum of two algebras, and this property was vital for establishing that the operator T in $\mathcal{A}(S)$ was not unitarily equivalent to S.

This led the author and Olin to ask if the question has an affirmative answer if it is also assumed that S is irreducible.

In this note an example of an irreducible subnormal operator S is given such that S and $-S$ are similar but not unitarily equivalent. On the positive side, it is

Received by the editors February 22, 1980 and in revised form, May 23, 1980

AMS subject classification numbers: Primary, 47B20; Secondary, 47B35.

Key words and phrases: subnormal operator, Toeplitz operator, quasisimilarity.

* The author received partial support from the National Science Foundation (USA) during the preparation of this paper.

shown that if S is the unilateral shift, $T \in \mathcal{A}(S)(=\{T_\phi : \phi \in H^\infty\})$, and S and T are quasisimilar, then S and T must be unitarily equivalent.

This paper is a continuation of [3] and some of the same notation and terminology will be used.

For any compactly supported measure μ on $\mathbb{C}$, let S_μ be multiplication by z on $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$. If N_μ denotes multiplication by z on $L^2(\mu)$, then N_μ is the minimal normal extension of S_μ.

Let m be normalized arc length measure on $\partial\mathbb{D}$; so S_m is the unilateral shift. Put $\mu = m + \delta(\frac{1}{2})$, where $\delta(\frac{1}{2})$ denotes the unit point mass at $\frac{1}{2}$. Let $S = S_\mu$ and $T = -S$. It follows that S and S_m are similar. Indeed, it is easy to show that there is a constant C such that

$$C\int |p|^2\, d\mu \le \int |p|^2\, dm \le \int |p|^2\, d\mu$$

for every polynomial p, whence the similarity is immediate.

Thus T is similar to $(\approx)$ $-S_m$; but $S_m \cong -S_m$, so $T \approx S_m \approx S$. On the other hand, the minimal normal extension of T is $-N_\mu$ and $-N_\mu$ and N_μ are not unitarily equivalent. Hence, S and T cannot be unitarily equivalent. Finally, because $S \approx S_m$ and S_m has no idempotents in its commutant, S is irreducible. (The author would like to thank the referee for this simplification of his original example.)

Recall that if S and T are operators, S and T are said to be *quasisimilar* if there are operators X and Y such that $\ker X = (0)$, X and Y have dense range, and $XS = TX$, $SY = YT$.

THEOREM 1. *If S is the unilateral shift of multiplicity one, $T \in \mathcal{A}(S)$, and S and T are quasisimilar, then S and T are unitarily equivalent.*

Before proving the theorem, recall that for $S = S_m$, $\mathcal{A}(S) = \{S\}' = \{T_\phi : \phi \in H^\infty\}$. If $\phi \in H^\infty$ and T_ϕ is unitarily equivalent to S, then T is an isometry; hence ϕ is an inner function. Moreover, $H^2 \ominus \phi H^2 = (\text{range } T_\phi)^\perp$ is one-dimensional. Thus, T_ϕ is unitarily equivalent to S iff

$$\phi(z) = e^{i\beta}\frac{z-\alpha}{1-\bar{\alpha}z},$$

where $0 \le \beta \le 2\pi$ and $|\alpha| < 1$.

To prove the theorem a result from [2] is needed. Since the contents of this thesis have not been published, a proof of the result will be included.

THEOREM 2 (Clary [3]). *If T is a subnormal operator and T is quasisimilar to the unilateral shift, then $\sigma_e(T) = \sigma_{ap}(T) = \partial\mathbb{D}$.*

Here σ_e and σ_{ap} denote the essential spectrum and the approximate point spectrum, respectively.

To prove Theorem 2, two lemmas are necessary.

LEMMA 1. *If μ is a measure on $\partial\mathbb{D}$ that is equivalent to m and $\log(d\mu/dm)\in L^1(m)$, then for every compact subset K of $\mathbb{D}$ there is a constant C such that for every polynomial p and every point a in K,*

$$|p(a)|^2 \le C^2 \int |p|^2\, d\mu.$$

Proof. This lemma is known, though a reference is lacking. In any case, the proof is easy. The hypothesis implies there is an outer function f in $H^2(=H^2(m))$ such that $|f|^2 = d\mu/dm$ ([7], p. 53). If $U: L^2(\mu)\to L^2(m)$ is defined by $Ug = gf$, then U is a unitary and $UH^2(\mu)=H^2$.

For $|a|<1$, let $k_a(z)=(1-\bar{a}z)^{-1}$. If $h\in H^2$, then $\langle h, k_a\rangle_{H^2}=h(a)$. Also $\|k_a\|^2=(1-|a|^2)^{-1}$. For any polynomial p,

$$\begin{aligned}|f(a)p(a)|^2 &= |\langle Up, k_a\rangle_{H^2}|^2 = |\langle p, U^{-1}k_a\rangle_{H^2(\mu)}|^2\\ &\le \left(\int |p|^2\, d\mu\right)\left(\int |U^{-1}k_a|^2\, d\mu\right)\\ &= (1-|a|^2)^{-1}\int |p|^2\, d\mu.\end{aligned}$$

Let $r=\max\{|a|: a\in K\}$; so $r<1$. Also, because f is outer, $\min\{|f(a)|: a\in K\} = M>0$. It follows from the above inequality that

$$|p(a)|^2 \le [M^2(1-r^2)]^{-1}\int |p|^2\, d\mu$$

whenever $a\in K$. ■

LEMMA 2. *If μ is a measure on $\mathbb{D}^-$, $\mu_0=\mu\,|\,\partial\mathbb{D}$, and μ_0 is equivalent to m with* $\log(d\mu_0/dm)$ *in* $L^1(m)$, *then if $\nu=\mu\,|\,(\mathbb{D}^-\backslash K)$ for some compact subset K of $\mathbb{D}$, $S_\nu\approx S_\mu$.*

Proof. Let C be the constant obtained in Lemma 1 when the measure in Lemma 1 is replaced by μ_0. Hence,

$$\begin{aligned}\int |p|^2\, d\nu \le \int |p|^2\, d\mu = \int_K |p|^2\, d\mu + \int |p|^2\, d\nu \le C\mu(K)\int |p|^2\, d\mu_0 + \int |p|^2\, d\nu\\ \le [1+C\mu(K)]\int |p|^2\, d\nu.\end{aligned}$$

Hence $S_\nu\approx S_\mu$. ■

Proof of Theorem 2. It follows from [1] that $\sigma(T)=\mathbb{D}^-$; so $\partial\mathbb{D}\subseteq\sigma_{ap}(T)$. Because T is pure ([3], Proposition 2.3), T has no eigenvalues. Hence $\sigma_{ap}(T)\subseteq \sigma_e(T)$. Because T is cyclic, $\dim[\operatorname{ran}(T-\lambda)]^\perp\le 1$ for all scalars λ. Thus, $\sigma_e(T) = \sigma_{ap}(T)$. It remains to show that $\sigma_{ap}(T)\subseteq\partial\mathbb{D}$.

Because $T\sim S_m$, $T\cong S_\mu$ where μ is a measure supported on $\mathbb{D}^-$ and if $\mu_0=\mu\,|\,\partial\mathbb{D}$, then μ_0 is equivalent to m and $\log(d\mu 0/dm)\in L^1(m)$. (This is from [2]; the condition is also sufficient for T to be similar to S_m. Another proof of necessity is to be found in Theorem 4.2 of [3]. Also see [6].) If $0<r<1$, let

https://doi.org/10.4153/CMB-1982-004-2 Published online by Cambridge University Press

$A_r = \{z : r \leq |z| \leq 1\}$ and put $\nu_r = \mu \mid A_r$. By Lemma 2, $S_\mu \approx S_{\nu_r}$ for $0 < r < 1$. Thus $\sigma_{ap}(S_\mu) = \sigma_{ap}(S_{\nu_r}) \subseteq \sigma_{ap}(N_{\nu_r}) \subseteq$ support $\nu_r \subseteq A_r$. Since r was arbitrary, $\sigma_{ap}(S_\mu) \subseteq \partial\mathbb{D}$. ∎

Proof of Theorem 1. Since $T \in \mathcal{A}(S)$, $T = T_\phi = \phi(S)$ for some ϕ in H^∞. Note that to show that $T \cong S$, it suffices to show that ϕ is inner. Indeed, if ϕ is inner and $T_\phi \sim S$, then T_ϕ is a pure cyclic isometry and, hence, $\dim(H^2 \ominus \phi H^2) = 1$. It follows that ϕ is a single Blaschke factor and so $T_\phi \cong S$.

To show that ϕ is inner, it suffices to show that $\phi(\mathbb{D}) = \mathbb{D}$ and ϕ is one-to-one. But from [1], $\sigma(T) = \sigma(S) = \mathbb{D}^-$; hence $\phi(\mathbb{D})^- = \mathbb{D}^-$. Since $\phi(\mathbb{D})$ is open, $\phi(\mathbb{D}) \subseteq \mathbb{D}$. If $\alpha \in \mathbb{D}$, then Theorem 2 implies that $\alpha \notin \sigma_{ap}(T)$. Thus $\operatorname{ran}(T - \alpha)$ is closed. Because T is cyclic and $\alpha \in \sigma(T)$, $\operatorname{ran}(T - \alpha)$ has codimension 1. But $\operatorname{ran}(T - \alpha)$ is invariant for S. It follows that $\operatorname{ran}(T - \alpha) = (\phi - \alpha)H^2 = BH^2$, where B is a single Blaschke factor. Therefore $\phi - \alpha$ must have a zero of multiplicity one in $\mathbb{D}$. This shows that $\phi(\mathbb{D}) = \mathbb{D}$ and ϕ is one-to-one. ∎

Bibliography

1. W. S. Clary, *Equality of spectra of quasisimilar hyponormal operators*, Proc. Amer. Math. Soc., **53** (1975), 88–90.
2. W. S. Clary, *Quasisimilarity and subnormal operators*, Ph.D. thesis, University of Michigan, 1973.
3. J. B. Conway, *On quasisimilarity for subnormal operators*, Illinois, J. Math., 24 (1980), 689–702.
4. J. B. Conway and R. F. Olin, *A functional calculus for subnormal operators*, II, Memoirs Amer. Math. Soc., vol. 184, 1977.
5. P. R. Halmos, *A Hilbert space problem book*, D. Van Nostrand Co., Princeton, 1967.
6. W. W. Hastings, *Subnormal operators quasisimilar to an isometry*, Trans. Amer. Math. Soc., **256** (1979), 145–161.
7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J. (1962).
8. W. R. Wogen, *On some operators with cyclic vectors*, Indiana Univ. Math. J., **27** (1978), 163–171.

Indiana University
Bloomington, Indiana 47405