Bull. Aust. Math. Soc. **109** (2024), 388–392
doi:10.1017/S0004972723000321

THE RESTRICTED CONNECTED HULL: FILLING THE HOLE

SONJA MOUTON and ROBIN HARTE

(Received 18 February 2023; accepted 13 March 2023; first published online 20 April 2023)

Abstract

By defining and applying the restricted topology, we have investigated certain connections between the boundary spectrum, the exponential spectrum, the topological boundary of the spectrum and the connected hull of the spectrum (see Mouton and Harte ['Linking the boundary and exponential spectra via the restricted topology', *J. Math. Anal. Appl.* **454** (2017), 730–745]). We now solve a remaining problem regarding the restricted connected hull.

2020 *Mathematics subject classification*: primary 46H10; secondary 47A10.

Keywords and phrases: exponential and boundary spectra, boundaries and hulls, restricted topology.

1. Introduction and preliminaries

For a complex Banach algebra A with unit 1, let A^{-1} denote the set of all invertible elements. We will indicate elements of the form $\lambda 1$ in A by λ. We denote the *spectrum* $\{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\}$ of an element a in A by $\sigma(a)$ (or by $\sigma(a, A)$, if necessary, to avoid confusion). The symbols $\partial\sigma(a)$ and $\eta\sigma(a)$ will denote the boundary and the connected hull, respectively, of $\sigma(a)$. (As usual, the *connected hull* of a set K in $\mathbb{C}$ is the union of K with its 'holes', where a *hole* of K is a bounded component of $\mathbb{C}\backslash K$.)

With $\partial_A(A\backslash A^{-1})$ denoting the topological (norm) boundary of $A\backslash A^{-1}$ in A and $\mathrm{Exp}(A)$ the set of all (finite) products of exponentials of elements in A, we will consider the *boundary spectrum* $S_\partial(a) := \{\lambda \in \mathbb{C} : a - \lambda \in \partial_A(A\backslash A^{-1})\}$ (see [6]) and the *exponential spectrum* $\varepsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \mathrm{Exp}(A)\}$ (see [4]) of a in A. Both these spectra are nonempty compact subsets of the complex plane and, for every $a \in A$,

$$\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a) \subseteq \varepsilon(a) \subseteq \eta\sigma(a). \tag{1.1}$$

Applications of the boundary and exponential spectra can be found in [7,4], respectively.

The first author acknowledges, with thanks, financial support provided by the National Research Foundation (NRF) of South Africa (Grant Number 132669).

Throughout this note, A will be a complex Banach algebra with unit 1 and B a closed subalgebra of A such that $1 \in B$.

In [8, Definition 3.2 and Theorem 3.5], the *restricted topology* (or the *B-topology*) is defined via the *restricted closure* $\mathrm{cl}^B(K)$ of an arbitrary subset K of A, where $\mathrm{cl}^B(K)$ is the set of all elements $a \in A$ with the property that $a - U$ and K have nonempty intersection, for every neighbourhood U of 0 in B in the relative (norm) topology. The set $K = A \backslash A^{-1}$ being of particular interest in this context, the restricted boundary of $A \backslash A^{-1}$ and the component of A^{-1} containing 1 in the B-topology are denoted by $\partial^B(A \backslash A^{-1})$ and $\mathrm{Comp}^B(1, A^{-1})$, respectively. The *restricted connected hull* $\eta^B(A \backslash A^{-1})$ of $A \backslash A^{-1}$ in A is given by

$$\eta^B(A \backslash A^{-1}) := A \backslash \mathrm{Comp}^B(1, A^{-1}). \tag{1.2}$$

In addition, for an *element* $a \in A$, the *restricted boundary* and the *restricted connected hull* are defined in [8] as

$$\partial_B(a) := \partial_B(a, A) = \{\lambda \in \mathbb{C} : a - \lambda \in \partial^B(A \backslash A^{-1})\} \tag{1.3}$$

and

$$\eta_B(a) := \eta_B(a, A) = \{\lambda \in \mathbb{C} : a - \lambda \in \eta^B(A \backslash A^{-1})\}, \tag{1.4}$$

respectively.

We now recall the following results from [8].

Theorem 1.1 [8, Corollary 6.4]. *For any $a \in A$, $\partial_{\mathbb{C}}(a) \subseteq \partial_B(a) \subseteq \partial_A(a)$, with $\partial_{\mathbb{C}}(a) = \partial\sigma(a)$ and $\partial_A(a) = S_\partial(a)$.*

Theorem 1.2 [8, Corollary 6.9]. *For any $a \in A$, $\eta_A(a) \subseteq \eta_B(a) \subseteq \eta_{\mathbb{C}}(a)$, with $\eta_A(a) = \varepsilon(a)$. In addition, if $a \notin \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \mathbb{C}$, while if $a \in \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \eta\sigma(a)$.*

We note that there is a type of duality between the boundary and exponential spectra:

$$S_\partial(a) = \partial_A(a) \quad \text{and} \quad \varepsilon(a) = \eta_A(a).$$

The last equation, together with (1.1), implies that $\eta_A(a) \subseteq \eta\sigma(a)$, and since $\partial_{\mathbb{C}}(a) = \partial\sigma(a)$, we have $\partial\sigma(a) \subseteq \partial_B(a)$. However, it is clear from the last part of Theorem 1.2 that we do not, in general, have $\eta_{\mathbb{C}}(a) = \eta\sigma(a)$ or that the inclusion $\eta_B(a) \subseteq \eta\sigma(a)$ holds (as in the case $B = A$). In Section 2, we 'fill the hole' in this theory by establishing exactly how far the inclusion $\eta_A(a) \subseteq \eta\sigma(a)$ can be generalised (see Theorems 2.2 and 2.4).

Applying the results in Section 2, we then establish certain mapping properties of η_B in Section 3.

2. The restricted connected hull relative to the connected hull of the spectrum

We first observe the following result.

Lemma 2.1. $\mathrm{Exp}(B) \subseteq \mathrm{Comp}^B(1, A^{-1})$.

This follows trivially, since $\mathrm{Exp}(B) = \mathrm{Comp}^B(1, B^{-1})$, but can alternatively be obtained by considering a map similar to that in the proof of Lemma 3.2 and applying [8, Proposition 5.11].

The next result follows easily.

THEOREM 2.2. *If $a \in B$, then $\eta_B(a) \subseteq \eta\sigma(a)$.*

PROOF. Let $\lambda \notin \eta\sigma(a) := \eta\sigma(a, A)$. Then $a - \lambda \in B$ and, since $\eta\sigma(b, A) = \eta\sigma(b, B)$ for all $b \in B$, we have $0 \notin \eta\sigma(a - \lambda, B)$. It follows from [1, Theorem 3.3.6] that $a - \lambda \in \mathrm{Exp}(B)$, and hence $a - \lambda \in \mathrm{Comp}^B(1, A^{-1})$, by Lemma 2.1. Therefore, (1.2) implies that $a - \lambda \notin \eta^B(A \backslash A^{-1})$, so that $\lambda \notin \eta_B(a)$, by (1.4). □

Although it is possible to have equality in Theorem 2.2 (see [8, Example 6.11]), the inclusion is, in general, proper. This is shown in the following example, where $\mathbb{T}$ indicates the unit circle and $\mathbb{D}$ the open unit disk in $\mathbb{C}$.

EXAMPLE 2.3. Let $l^2(\mathbb{Z})$ be the Hilbert space of all bilateral square-summable sequences, A the Banach algebra $\mathcal{L}(l^2(\mathbb{Z}))$ of all bounded linear operators on $l^2(\mathbb{Z})$ and $a \in A$ the bilateral shift. Then $\eta_A(a) \subsetneq \eta\sigma(a)$.

PROOF. By [2, Corollary 5.30], $A^{-1} = \mathrm{Exp}(A)$, so that $\sigma(b) = \varepsilon(b)$ for all $b \in A$. Since $\sigma(a) = \mathbb{T}$ (see [3, Problem 84]), it follows that $\varepsilon(a) = \mathbb{T}$ and $\eta\sigma(a) = \overline{\mathbb{D}}$. However, $\varepsilon(a) = \eta_A(a)$, by Theorem 1.2, and hence the result follows. □

Finally, we have the following result.

THEOREM 2.4. *If $a \notin B$, then $\eta_B(a) = \mathbb{C}$.*

PROOF. By Theorem 1.2, $\varepsilon(a) = \eta_A(a) \subseteq \eta_B(a)$ for any $a \in A$, and hence it suffices to prove that $\mathbb{C} \backslash \varepsilon(a) \subseteq \eta_B(a)$ whenever $a \notin B$. So suppose that $a \notin B$ and let $\lambda_0 \in \mathbb{C} \backslash \varepsilon(a)$. Then $\lambda_0 \notin \sigma(a)$. If $G = \mathrm{Comp}^B(a - \lambda_0, A^{-1})$, then $a - G = \mathrm{Comp}^B(\lambda_0, B \backslash \sigma(a))$ by [8, Proposition 5.10], and since $a \notin B$, it follows that $1 \notin G$. Therefore, $G \neq \mathrm{Comp}^B(1, A^{-1})$, so that $a - \lambda_0 \notin \mathrm{Comp}^B(1, A^{-1})$. By (1.2), we then have $a - \lambda_0 \in \eta^B(A \backslash A^{-1})$, so that $\lambda_0 \in \eta_B(a)$, by (1.4). □

3. Mapping properties of the restricted connected hull

Let $K(\mathbb{C})$ denote the set of all nonempty, compact subsets of $\mathbb{C}$. A mapping $\omega : A \to K(\mathbb{C})$ is said to be a *Mobius spectrum on A* (see [5]) if $\omega(f(a)) = f(\omega(a))$:

(a) for all $a \in A$ and for all functions f of the form $f(\lambda) = \alpha\lambda + \beta$ ($\alpha, \beta \in \mathbb{C}$); and
(b) for all $a \in A$ and $f(\lambda) = 1/\lambda$, such that f is well defined on $\omega(a) \cup \sigma(a)$.

We immediately have the following result.

PROPOSITION 3.1. *η_B is a Mobius spectrum on B.*

PROOF. Since ε is a Mobius spectrum on any Banach algebra (see [5]), we have $\varepsilon(f(a), B) = f(\varepsilon(a, B))$ for all $a \in B$ and for all functions f of the form $f(\lambda) = \alpha\lambda + \beta$

https://doi.org/10.1017/S0004972723000321 Published online by Cambridge University Press

($\alpha, \beta \in \mathbb{C}$), and that $\varepsilon(a^{-1}, B) = (\varepsilon(a, B))^{-1}$ for all $a \in B$ such that $0 \notin \varepsilon(a, B) \cup \sigma(a, B)$. Then the result follows since $\eta_B(x, B) = \{\lambda \in \mathbb{C} : x - \lambda \in \eta^B(B \backslash B^{-1})\} = \varepsilon(x, B)$ for any $x \in B$. □

Returning to $\eta_B(a) := \eta_B(a, A)$, we now develop some more mapping properties, starting with the following lemma.

LEMMA 3.2. *If $a \in \mathrm{Exp}(B)$ and $b \in B \cap \mathrm{Comp}^B(1, A^{-1})$, then $ab \in \mathrm{Comp}^B(1, A^{-1})$.*

PROOF. Let $a = e^{b_1} e^{b_2} \cdots e^{b_n}$ with $b_1, b_2, \ldots, b_n \in B$, and define $f : [0, 1] \to A$ by $f(t) = e^{tb_1} e^{tb_2} \cdots e^{tb_n} b$. Then $f(0) = b$, $f(1) = ab$ and f is continuous in the norm topology of A. Since $f([0, 1]) \subseteq B$, it follows from [8, Proposition 5.11] that f is continuous in the B-topology. Therefore, we have a continuous function f in the B-topology from $[0, 1]$ to A^{-1} joining b and ab, and so $ab \in \mathrm{Comp}^B(1, A^{-1})$. □

COROLLARY 3.3. *If $a \in B \cap \mathrm{Comp}^B(1, A^{-1})$ and $0 \neq \lambda \in \mathbb{C}$, then $\lambda a \in \mathrm{Comp}^B(1, A^{-1})$.*

It follows from Corollary 3.3 that, for $a \in B$, (1.4) may equivalently be written as

$$\eta_B(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \eta^B(A \backslash A^{-1})\}.$$

THEOREM 3.4. *Let $a \in A$ and $f(\lambda) = \alpha\lambda + \beta$ with $\alpha, \beta \in \mathbb{C}$. Then $\eta_B(f(a)) = f(\eta_B(a))$.*

PROOF. Let $a \in A$. If $\alpha = 0$ (that is, f is constant), then since

$$\eta_B(f(a)) = \{\lambda \in \mathbb{C} : \beta - \lambda \notin \mathrm{Comp}^B(1, A^{-1})\}$$

and $f(\eta_B(a)) = \{\beta\}$, it is clear that $f(\eta_B(a)) \subseteq \eta_B(f(a))$. Now let $\lambda \in \eta_B(f(a))$, that is, $\beta - \lambda \notin \mathrm{Comp}^B(1, A^{-1})$. By Corollary 3.3, $\mathrm{Comp}^B(1, A^{-1})$ contains all nonzero complex scalar multiples of 1, and so $\lambda = \beta \in f(\eta_B(a))$. Hence, $\eta_B(f(a)) \subseteq f(\eta_B(a))$.

If $\alpha \neq 0$ and $a \notin B$, then $f(a) = \alpha a + \beta \notin B$, and so $\eta_B(a) = \mathbb{C} = \eta_B(f(a))$, by Theorem 2.4. Since $f(\eta_B(a)) = f(\mathbb{C}) = \mathbb{C}$, we have $\eta_B(f(a)) = f(\eta_B(a))$. For the case $\alpha \neq 0$ and $a \in B$, let $\lambda \in \eta_B(f(a))$, so that $\alpha a + \beta - \lambda \notin \mathrm{Comp}^B(1, A^{-1})$. If $\mu = \alpha^{-1}(\lambda - \beta)$, then $\lambda = f(\mu)$ and $a - \mu = \alpha^{-1}(\alpha a + \beta - \lambda) \notin \mathrm{Comp}^B(1, A^{-1})$ by Corollary 3.3, since $\alpha a + \beta - \lambda \in B$. Therefore, $\mu \in \eta_B(a)$, so that $\lambda \in f(\eta_B(a))$. The other inclusion is obtained similarly. □

Turning to the inverse function, we first observe the following result.

PROPOSITION 3.5. *Let $a \in B$ with $0 \in \sigma(a, B) \backslash \sigma(a, A)$. If $f(\lambda) = 1/\lambda$, then $f(\eta_B(a)) \subseteq \eta_B(f(a))$.*

PROOF. This is obvious, since $\eta_B(a^{-1}) = \mathbb{C}$, by Theorem 2.4. □

We can now prove the following mapping property of η_B.

THEOREM 3.6. *Let $a \in B$ such that $0 \notin \eta\sigma(a)$. If $f(\lambda) = 1/\lambda$, then $\eta_B(f(a)) = f(\eta_B(a))$.*

PROOF. Let $\lambda \in \eta_B(f(a)) = \eta_B(a^{-1})$, so that $a^{-1} - \lambda \notin \mathrm{Comp}^B(1, A^{-1})$. If $0 \notin \eta\sigma(a) := \eta\sigma(a, A)$, then since $\varepsilon(a, B) \subseteq \eta\sigma(a, B) = \eta\sigma(a, A)$, it follows that $0 \notin \varepsilon(a, B)$. Therefore, $a \in \mathrm{Exp}(B)$, so that $a^{-1} \in \mathrm{Exp}(B)$. By Lemma 2.1, $a^{-1} \in \mathrm{Comp}^B(1, A^{-1})$,

https://doi.org/10.1017/S0004972723000321 Published online by Cambridge University Press

so that $\lambda \neq 0$. If $a - \lambda^{-1} \in \mathrm{Comp}^B(1, A^{-1})$, then $(\lambda^{-1} - a)\lambda \in \mathrm{Comp}^B(1, A^{-1})$, by Corollary 3.3. Since $a^{-1} \in \mathrm{Exp}(B)$, it follows from Lemma 3.2 that $a^{-1} - \lambda = a^{-1}(\lambda^{-1} - a)\lambda \in \mathrm{Comp}^B(1, A^{-1})$; which is a contradiction. Hence, $a - \lambda^{-1} \notin \mathrm{Comp}^B(1, A^{-1})$, so that $\lambda^{-1} \in \eta_B(a)$, and hence $\lambda \in f(\eta_B(a))$. The other inclusion is obtained similarly. □

In the following example, let $\mathbb{T}$ indicate the unit circle and $\mathbb{D}$ the open unit disk in $\mathbb{C}$, as before.

Example 3.7. Let $A = C(\mathbb{T})$ and $B = \mathcal{A}(\overline{\mathbb{D}})$, as in [8, Example 6.11]. Let $a \in B$ denote the identity function $f(\lambda) = \lambda$ on $\overline{\mathbb{D}}$. Then, $\sigma(a, A) = \mathbb{T}$, $\sigma(a, B) = \overline{\mathbb{D}} = \eta_B(a)$ and $\eta_B(a^{-1}) = \mathbb{C}$.

Proof. The first two statements are given by [9, Problem 9, page 399] and it was shown in [8, Example 6.11] that $\eta_B(a) = \overline{\mathbb{D}}$. Since $0 \in \sigma(a, B) \backslash \sigma(a, A)$, we have $a^{-1} \in A \backslash B$, and hence $\eta_B(a^{-1}) = \mathbb{C}$, by Theorem 2.4. □

Example 3.7 shows that we do not, in general, have equality in Proposition 3.5, and also that the condition $0 \notin \eta\sigma(a)$ in Theorem 3.6 cannot be omitted.

References

[1] B. Aupetit, *A Primer on Spectral Theory* (Springer-Verlag, New York, 1991).
[2] R. G. Douglas, *Banach Algebra Techniques in Operator Theory* (Springer, New York, 1998).
[3] P. R. Halmos, *A Hilbert Space Problem Book* (Springer-Verlag, New York, 1982).
[4] R. E. Harte, 'The exponential spectrum in Banach algebras', *Proc. Amer. Math. Soc.* **58** (1976), 114–118.
[5] R. E. Harte and A. W. Wickstead, 'Boundaries, hulls and spectral mapping theorems', *Proc. Roy. Irish Acad.* **81A** (1981), 201–208.
[6] S. Mouton, 'On the boundary spectrum in Banach algebras', *Bull. Aust. Math. Soc.* **74** (2006), 239–246.
[7] S. Mouton, 'A condition for spectral continuity of positive elements', *Proc. Amer. Math. Soc.* **137** (2009), 1777–1782.
[8] S. Mouton and R. E. Harte, 'Linking the boundary and exponential spectra via the restricted topology', *J. Math. Anal. Appl.* **454** (2017), 730–745.
[9] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis* (John Wiley and Sons, New York, 1958).

SONJA MOUTON, Department of Mathematical Sciences,
Stellenbosch University, Stellenbosch, South Africa
e-mail: smo@sun.ac.za

ROBIN HARTE, School of Mathematics,
Trinity College, Dublin, Ireland
e-mail: rharte@maths.tcd.ie

https://doi.org/10.1017/S0004972723000321 Published online by Cambridge University Press