Canad. Math. Bull. Vol. **30** (3), 1987

A DECOMPOSITION THEOREM FOR COMPLEX NILMANIFOLDS

BY
JEAN-JACQUES LOEB, KARL OELJEKLAUS
AND WOLFGANG RICHTHOFER

ABSTRACT. A complex nilmanifold X is isomorphic to a product $X \simeq \mathbb{C}^p \times N/\Gamma$, where N is a simply connected nilpotent complex Lie group and Γ is a discrete subgroup of N not contained in a proper connected complex subgroup of N. The pair (N, Γ) is uniquely determined up to holomorphic group isomorphisms.

A complex manifold X is called a *nilmanifold* if a complex nilpotent Lie group G is acting holomorphically and transitively on X, i.e. $X \cong G/H$, where H is a closed complex subgroup of G. We may always assume that G is simply connected and that the G-action on X is almost effective. In this paper we analyse the structure of nilmanifolds extending the results of [3] and [2].

It was shown in [3] that for a generalized Iwasawa manifold $X = G/H$, i.e. G is a complex Heisenberg group and $H \subset G$ a complex subgroup, such that $\mathcal{O}(X) \cong \mathbb{C}$, the pair (G, H) is uniquely determined in the following sense: Let $X = \tilde{G}/\tilde{H}$ be another generalized Iwasawa manifold biholomorphic to X, then there is a holomorphic Lie group isomorphism $\varphi: G \to \tilde{G}$, which maps H onto $\tilde{H}$. It turns out that the condition on the holomorphic functions on X is very strong and makes the proof of the result above very easy (see [1]). However, an analogous theorem in the real category ([5], Thm. 2.11) indicates how to weaken the condition on G/H to a certain maximality assumption on H (see Lemma). A subgroup $H \subset G$ is called *maximal* if it is not contained in a proper connected complex subgroup of G. (Note that $\mathcal{O}(G/H) \cong \mathbb{C}$ implies the maximality of H.) This yields the following decomposition theorem for nilmanifolds:

THEOREM. *A complex nilmanifold $X = G/H$ is biholomorphic to $\mathbb{C}^p \times N/\Gamma$, where Γ is a discrete maximal subgroup of the simply connected complex Lie group N. The decomposition $X = \mathbb{C}^p \times N/\Gamma$ is unique in the following sense: Let $\mathbb{C}^{p'} \times N'/\Gamma'$ be another decomposition with the above properties. Then $p = p'$ and there exists a complex Lie group isomorphism $\rho: N \to N'$ such that $\rho(\Gamma) = \Gamma'$.*

For the proof we need the following

Received by the editors April 22, 1986.
AMS Subject Classification (1980): 32M10, 32M05.
Key word: Complex nilmanifold.

LEMMA. *Let N, M denote simply connected nilpotent complex Lie groups and Γ a discrete maximal subgroup of N. Let $\varphi : N \to M$ be a holomorphic map such that for all $n \in N$, $\gamma \in \Gamma : \varphi(n \cdot \gamma) = \varphi(n) \cdot \varphi(\gamma)$. Then there is a unique holomorphic homomorphism $\tilde{\varphi} : N \to M$ with $\tilde{\varphi} | \Gamma = \varphi | \Gamma$.*

PROOF. The uniqueness follows from Malcev's theorem ([5]), Prop 2.5 and the maximality of Γ.

Existence of $\tilde{\varphi}$. Let J be the complex subgroup of N defined by $J = \{n \in N | f(n) = f(e), \forall f \in \mathcal{O}(N)^{\Gamma}\}$, e = identity element of N. The restriction of φ to J is a holomorphic homomorphism, because for a fixed $n \in N$ the holomorphic map $J \to M$, $x \to \varphi(n \cdot x)\varphi(x)^{-1}\varphi(n)^{-1}$ is Γ-invariant (This function was considered by Ahiezer [1].), hence constant. Denote by N_0 the minimal connected (real) subgroup of N containing Γ. By [5], Thm. 2.11 there is a unique (real) Lie group homomorphism φ' from N_0 to M such that $\varphi' | \Gamma = \varphi | \Gamma$. Let $N_0^{\mathbb{C}}$ be the "complexification" of N_0 in N. Since Γ is maximal we have that $N_0^{\mathbb{C}} = N$. Assume that φ' is holomorphic on the maximal connected complex subgroup C in N_0. Then φ' extends uniquely to a holomorphic homomorphism $\tilde{\varphi} : N \to M (N_0^{\mathbb{C}} = N!)$. Therefore it is enough to prove the holomorphy of φ' on C. The compactness of N_0/Γ implies that C is contained in the identity component J^0 of J. Moreover, the group $J^0 \cdot \Gamma$ is closed in N and as a consequence $J^0 \cap N_0 / J^0 \cap \Gamma$ is compact. In concluding the proof we note that φ and φ' coincide on $J^0 \cap N$, hence on C. □

PROOF OF THE THEOREM.

Existence. Let $X = G/H$, G simply connected (without loss of generality). Denote by V the smallest connected complex Lie group in G containing H. Since the normalizer $N_G(H^0)$ of H^0 in G is connected the identity component H^0 of H is normal in V and $G/V \cong \mathbb{C}^n$ ([4]). Hence, by Grauert's Oka principle, $X = \mathbb{C}^n \times (V/H^0/H/H^0) = \mathbb{C}^n \times N/\Gamma$. By construction Γ is maximal in N.

Uniqueness. Assume that $X \cong \mathbb{C}^{p'} \times N'/\Gamma'$, where N' is simply connected and Γ' is maximal in N'. Let $M = \mathbb{C}^{p'} \times N'$. By passing to the universal covering, we define a map φ from N to M as in the lemma. Then $\tilde{\varphi}$ is a complex isomorphism from N to N'.

1. D. N. Ahiezer, Math. Reviews 81 a: 32025.

2. B. Gilligan and A. T. Huckleberry, *On non-compact complex nilmanifolds*. Math. Ann. **238** (1978), pp. 39–49.

3. A. T. Huckleberry and G. Schumacher, *Holomorphic maps of generalized Iwasawa manifolds*. Manuscripta Math. **30** (1979/80), pp. 107–117.

4. Y. Matsushima, *On discrete subgroups and homogeneous spaces of nilpotent Lie groups*. Nag. Math. J. **2** (1951), pp. 95–110.

5. M. S. Ragunathan, *Discrete subgroups of Lie groups*. Erg. Math. Grenzgeb. 1972, Bd. 68, Springer.

RUHR-UNIVERSITÄT BOCHUM
POSTFACH 102148
4360 BOCHUM, W. GERMANY