ON SOME RELATIONS BETWEEN PARTIAL AND ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Dr. A. WALTHER on his 60th birthday.

1. Introduction. The theory of solutions of partial differential equations

$$\Delta u + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u = 0 \tag{1.1}$$

with analytic coefficients can be based upon the theory of analytic functions of a complex variable; the basic tool in this approach is integral operators which map the set of solutions of (1.1) onto the algebra of analytic functions. For certain classes of operators this mapping which is first defined in the small, can be continued to the large, cf. Bergman **(3)**. In this way theorems on analytic functions give rise to theorems on (real and complex) solutions of (1.1). Some of the operators possess a remarkable property: they generate solutions of certain partial differential equations (1.1) which also satisfy ordinary linear differential equations in x or y. This was first observed by Bergman **(1;2)** in the special case of the equation $\Delta u + u = 0$. This property is of interest since it permits the investigation of such solutions of (1.1) by means of the theory of ordinary differential equations. The present paper is concerned with a class of partial differential equations (1.1) which possess solutions of that type. We shall derive an infinite set of independent particular solutions and obtain relations between singularities of the coefficients of (1.1) and those of the corresponding ordinary differential equations; cf. §§ 3–5. These results will enable us to characterize some basic properties of those solutions of (1.1); cf. § 6.

2. Partial differential equations of class $\mathfrak{C}$. If we introduce the variables $z = x + iy$, $z^* = x - iy$, the equation (1.1) takes the form

$$u_{zz^*} + a(z, z^*)u_z + b(z, z^*)u_{z^*} + c(z, z^*)u = 0 \tag{2.1}$$

where

$$u_{zz^*} = \tfrac{1}{4}\Delta u, \quad u_z = \tfrac{1}{2}(u_x - iu_y), \quad u_{z^*} = \tfrac{1}{2}(u_x + iu_y),$$
$$a = \tfrac{1}{4}(\alpha + i\beta), \quad b = \tfrac{1}{4}(\alpha - i\beta), \quad c = \tfrac{1}{4}\gamma.$$

If we set

$$u = U \exp\left(-\int_0^{z^*} a(z, t)dt\right)$$

we obtain from (2.1)

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$$L(U) \equiv U_{zz^*} + B(z, z^*)U_{z^*} + C(z, z^*)U = 0 \tag{2.2}$$

where

$$B = b - \int_0^{z^*} a_z(z, t)dt, \quad C = c - a_z - ab.$$

We note that for complex values of x and y the variables z and z^* are independent.

Definition 1. *An operator of the form*

$$U(z, z^*) \equiv P(f) = \int_{-1}^{1} E(z, z^*, t)f(\tfrac{1}{2}z(1 - t^2))(1 - t^2)^{-\frac{1}{2}}dt \tag{2.3}$$

is called a Bergman operator. In (2.3) *the "associated function" $f(z)$ of $U(z,z^*)$ is an analytic function of a complex variable regular at the origin. The "generating function" $E(z,z^*,t)$ is independent of the special choice of $f(z)$.*

In order that $U(z,z^*)$ be a solution of (2.2) the function $E(z,z^*,t)$ must satisfy the equation

$$(1 - t^2)E_{z^*t} - t^{-1}E_{z^*} + 2zt\, L(E) = 0, \tag{2.4}$$

as can be seen by inserting (2.3) into (2.2).

Definition 2. *A partial differential equation* (2.2) *is said to be of the class* $\mathfrak{E}$ *if its solutions can be generated in the form* (2.3) *with a generating function of the type*

$$E(z, z^*, t) = \exp Q(z, z^*, t), \qquad Q(z, z^*, t) = \sum_{\mu=1}^{m} q_\mu(z, z^*)t^\mu. \tag{2.5}$$

Necessary and sufficient conditions have been obtained for the coefficients of (2.2) in order that (2.2) should be of the class $\mathfrak{E}$; cf. Kreyszig **(4)**.

3. Existence of ordinary differential equations satisfied by solutions of partial differential equations of the class $\mathfrak{E}$. If in (2.3), $f(z) = z^n$, $n = 0, 1, \ldots$, the corresponding solutions of the partial differential equations of the class $\mathfrak{E}$ satisfy a linear ordinary differential equation; cf. Kreyszig **(5)**. It was conjectured that the (more important) solutions with meromorphic associated functions have a similar property. However, the method used in **(5)** fails in this case. In order to treat this problem in a systematic way we first consider solutions $U(z,z^*)$ which correspond to associated functions

$$f_n(z) = (z - \zeta)^{-n}, \qquad \zeta \neq 0, \qquad n = 1, 2, \ldots. \tag{3.1}$$

In order to derive ordinary differential equations satisfied by $U(z,z^*)$ we have to consider this function in certain planes of the (real four-dimensional) zz^*-space. The form of these equations will depend on the choice of these planes. We take the planes $y = y_0 =$ const. Then we have the advantage htat $U(z,z^*) \equiv \tilde{U}(x,y)$ is an analytic function of x.

THEOREM 1. *Each solution* $\tilde{U}(x,y) = U(z,z^*)$ *of a partial differential equation* (2.2) *of the class* $\mathfrak{E}$ *with an associated function* (3.1) *satisfies an ordinary linear differential equation*

$$N(U) \equiv \tilde{N}(\tilde{U}) = \sum_{\rho=0}^{r} G_\rho(x, y_0) \frac{d^\rho \tilde{U}}{dx^\rho} = 0, \qquad G_r = 1, y = y_0 = \text{const}, \tag{3.2}$$

of order

$$r \leqslant m + 3. \tag{3.3}$$

The coefficients $G_\rho(x,y) \equiv g_\rho(z,z^*)$ *are rational functions of* $q_\mu(z,z^*)$, $\mu = 0,1,\ldots,m$. *The order r is independent of n.*

Proof. In consequence of (2.5) and (3.1) the integrand of (2.3) takes the form

$$J(x, y, t) \equiv j(z, z^*, t) = \exp Q(z, z^*, t)\, s(z, t)^{-n}(1 - t^2)^{-\frac{1}{2}}, \tag{3.4}$$
$$S(x, y, t) \equiv s(z, t) = \tfrac{1}{2}z(1 - t^2) - \zeta.$$

It suffices to prove that J satisfies the non-homogeneous equation

$$\tilde{N}(J) = R, \quad R(x, y, t) = \frac{d}{dt}[(1 - t^2)H(x, y, t)] \tag{3.5}$$

where H is a regular function of t for $|t| \leqslant 1$. If we integrate both sides of this equation with respect to t from -1 to 1 we obtain (3.2). We choose

$$H = P\, S^{-r+1} J \tag{3.6}$$

where

$$P(x, y, t) \equiv p(z, z^*, t) = \sum_{\lambda=0}^{l} p_\lambda(z, z^*)\, t^\lambda; \tag{3.7}$$

the degree l and the coefficients $p_\lambda(z,z^*)$ will be suitably determined, see below. We have

$$J_t \equiv \frac{\partial J}{\partial t} = (Q_t + (1 - t^2)^{-1}t + s^{-1}nzt)\, J, \tag{3.8}$$

$$J^{(\alpha)} \equiv \frac{\partial^\alpha J}{\partial x^\alpha} = T_\alpha\, J \tag{3.9}$$

where

$$T_1 = \frac{\partial Q}{\partial x} - n(1 - t^2)(2S)^{-1} \tag{3.10}$$

and

$$T_\alpha = |A_{\alpha 1}\, A_{\alpha 2} \ldots A_{\alpha\alpha}|\,, \qquad \alpha = 2, 3, \ldots, \tag{3.11}$$

is a determinant with the column vectors

$$A_{\alpha,\beta+1} = \left(\binom{\beta}{\beta}T_1^{(\beta)}, \binom{\beta}{\beta - 1}T_1^{(\beta-1)}, \ldots, \binom{\beta}{0}T_1, -1, 0, 0, \ldots\right),$$
$$T_1^{(\beta)} \equiv \frac{\partial^\beta T_1}{\partial x^\beta}, \qquad \beta = 0, 1, \ldots, \alpha - 1;$$

(the number of zeros decreases with increasing β; in $A_{\alpha,\alpha-1}$ there are no more zeros left, and in $A_{\alpha\alpha}$ the term T_1 is the last one). This can easily be proved by induction. From (3.5) – (3.8) we find

$$R = \{ -tP + (1-t^2)(P_t + P[S^{-1}(r+n-1)tz + Q_t])\} \, J \, S^{-r+1}. \tag{3.12}$$

If we insert (3.9) and (3.12) into (3.5), omit the common factor J and multiply each term by S^r, each side of the resulting equation becomes a polynomial in t. If we choose

$$l = (m+2)r - m - 3, \tag{3.13}$$

cf. (3.7), these two polynomials have the same degree, namely $(m+2)r$. In the equation thus obtained the coefficients of each power of t must be the same on both sides. Hence we obtain a system of $(m+2)r+1$ linear equations. If we choose

$$r = m + 3 \tag{3.14}$$

the number of equations equals the total number of the coefficients $G_0, \ldots, G_{r-1}$ of (3.2) and of the coefficients $p_0, \ldots, p_l$ of (3.7). In order to be able to determine these functions G_ρ and p_λ it suffices that the determinant $D(z,z^*)$ of the coefficients of the system does not vanish identically, since every neighbourhood of a point of a zero surface of D contains always points at which $D(z,z^*) \not\equiv 0$. Furthermore, it can readily be seen that the rank of D is always different from zero. Hence if $D(z,z^*) \equiv 0$ there exists a subdeterminant of D which does not vanish identically. In the case $D \equiv 0$ the order r of (3.2) reduces to values smaller than $m+3$; cf. (2.5), and the coefficients of (3.2) can be determined in a similar manner. This completes the proof.

This result may be extended to the case of solutions with arbitrary rational associated functions as follows.

THEOREM 2. *Each solution $\tilde{U}(x,y) \equiv U(z,z^*)$ of a partial differential equation* (2.2) *of the class $\mathfrak{E}$ with a rational associated function $f(z)$ satisfies an ordinary linear differential equation in x whose coefficients are rational functions of $q_0, \ldots, q_m$, cf.* (2.5). *If $f(z)$ has poles of orders β_κ at $z = z_\kappa$, $\kappa = 1,2,\ldots,k$, the equation has the order*

$$r \leqslant (\alpha+\beta+1)m + \alpha + 3\beta \tag{3.15}$$

where $\beta = \beta_1 + \beta_2 + \ldots + \beta_k$ and α is the degree of the polynomial $F_1(z)$ in the representation of $f(z)$ as a sum of $F_1(z)$ and a proper rational function $F_2(z)$; m is defined by (2.5).

Proof. The polynomial $F_1(z)$ is a sum of at most $\alpha+1$ terms. To each of these terms and to each partial fraction of $F_2(z)$ there corresponds a particular solution $\tilde{U}_\delta(x,y)$ of (2.2). We thus have

$$\tilde{U}(x,y) = \sum_{\delta=1}^{d} \tilde{U}_\delta(x,y), \qquad d \leqslant \alpha+\beta+1. \tag{3.16}$$

Each of the functions $\tilde{U}_\delta(x,y)$ corresponding to $F_1(z)$ satisfies an ordinary linear differential equation of the order $r^* \leqslant m + 1$, cf. (**5**, Theorem 2), while each of the other functions satisfies such an equation of the order $r^{**} \leqslant m + 3$, cf. Theorem 1 of this paper. Thus, we have a system (S) of ordinary linear differential equations whose coefficients are rational functions of $q_0, \ldots, q_m$, cf. (2.5). We differentiate each of these differential equations and also the equation (3.16) r times and eliminate all the functions $\tilde{U}_\delta(x,y)$ and their derivatives from the enlarged system (S^*) thus obtained. In order to be able to do so we have to choose r so that the number of equations of (S^*) equals the number of functions to be eliminated. It can easily be seen that r cannot be greater than $(\alpha + \beta + 1)\, m + \alpha + 3\beta$. Since we differentiated (3.16) r times the r^{th} derivative of $\tilde{U}(x,y)$ is the highest one which occurs in (S^*). This completes the proof.

4. Subclasses of the class $\mathfrak{E}$. The coefficients $B(z,z^*)$ and $C(z,z^*)$ of the partial differential equations (2.2) of the class $\mathfrak{E}$ are related to the coefficients $q_\mu(z,z^*)$ of the generating function (2.5) as follows (**4**, Theorem 1).

(I) *If* $q_1(z,z^*) \not\equiv 0$ *then*

$$B = -\frac{\partial q_0}{\partial z} - \frac{q_2}{z}, \quad C = -\frac{q_1}{2z}\frac{\partial q_1}{\partial z^*}. \tag{4.1}$$

(II) *If* $q_1 \equiv 0$ *then also* $q_3 \equiv 0,\ q_5 \equiv 0, \ldots,$ *and*

$$B = -\frac{\partial q_0}{\partial z} - \frac{q_2}{z}, \quad C = -\frac{1}{2z}\frac{\partial q_2}{\partial z^*}; \tag{4.2}$$

q_0 depends only on z and can have singularities. In case (I) q_1 depends on z and z^* and can have singularities, considered as a function of z^* for any finite constant value of z. In case (I), q_2 is regular while in case (II) q_2, considered as a function of z^* for any finite constant value of z, can have singularities.

Hence the class $\mathfrak{E}$ consists of two subclasses $\mathfrak{E}_{\text{I}}$ and $\mathfrak{E}_{\text{II}}$ corresponding to the two cases (I) and (II).

In case (II) the function $Q(z,z^*,t)$, defined by (2.5), is an even function of t. Hence, in this case, the functions T_α, cf. (3.10), (3.11), are also even functions of t. Let $P(x,y,t)$ be an odd function of t; then $R\,S^r J^{-1}$ is an even function of t; cf. (3.4)–(3.7). Hence, in this case the polynomials considered in the proof of Theorem 1 are even functions of t and have the degree $(m + 2)r$. The function $P(x,y,t)$ has now only $\frac{1}{2}(l + 1)$ coefficients $p_\lambda(z,z^*)$ where l is defined by (3.13). The total number $\frac{1}{2}(l + 1) + r$ of the functions G_ρ and p_λ must equal the number of powers occurring in the above-mentioned polynomials. We thus obtain the result that each solution of a partial differential equation (2.2) of the subclass $\mathfrak{E}_{\text{II}}$ with an associated function (3.1) satisfies an ordinary linear differential equation of the order

$$r = \leqslant \tfrac{1}{2} m + 2, \qquad (m \text{ even}). \tag{4.3}$$

It can be similarly proved that such a solution with an associated function $f_n(z) = z^n$, $n = 0,1,\ldots,$ satisfies an ordinary linear differential equation of the order

(4.4) $$r \leqslant \tfrac{1}{2} m + 1, \qquad (m \text{ even}).$$

Applying to these results the idea of the proof of Theorem 2 we obtain the following

COROLLARY. *Each solution of a partial differential equation* (2.2) *of the subclass* $\mathfrak{E}_{II}$ *with a rational associated function satisfies an ordinary differential equation in x of the order*

(4.5) $$r \leqslant m + \beta + (\tfrac{1}{2}m + 1)(\alpha + \beta), \qquad (m \text{ even}),$$

where α *and* β *are defined as in Theorem 2. The coefficients of this equation are rational functions of* $q_0, \ldots, q_m$, cf. (2.5).

Partial differential equations of the subclass $\mathfrak{E}_{II}$ thus have the remarkable property that the corresponding ordinary differential equations have a smaller order than those corresponding to partial differential equations of the subclass $\mathfrak{E}_{I}$.

5. Relations between singularities of the partial differential equation (2.2) **and those of the corresponding ordinary differential equation** (3.2)**.** The relations between $q_0, \ldots, q_m$ (cf. (2.5)) and the coefficients B, C of (2.2) on the one hand, and between $q_0, \ldots, q_m$ and the coefficients G_ρ of (3.2) on the other hand, enable us to obtain direct relations between the singularities of the given partial differential equation (2.2) and the ordinary differential equation (3.2) which we have derived. Since the procedure of obtaining such relations is similar to that developed in **(5)** we omit details and state the result only. We find

THEOREM 3. *The singularities of the ordinary differential equation* (3.2) *and those of the corresponding partial differential equation* (2.2) *of the class* $\mathfrak{E}$ *are related as follows.*

(*i*) *If B, considered as a function of z for any finite value* $z^* = const$, *has a pole of the order s at a point* $z = a$, *the coefficient* G_ρ *of* (3.2), *considered as a function of z, has a pole of the order*

(5.1) $$s_1(s,\rho) = sw_\rho, \qquad w_\rho = m + 3 - \rho,$$

at $z = a$. *If* $s = 1$ *then* (3.2) *is of Fuchsian type at* $z = a$.

(*ii*) *If* $q_1(z,z^*) \not\equiv 0$ *and C, considered as a function of* z^* *for any finite value* $z = const$, *has a pole of the order* $2s-1$, $s > 1$, *at a point* $z^* = a^*$, *the coefficient* G_ρ *of* (3.2), *considered as a function of* z^*, *has a pole of the order*

(5.2) $$s_2(s,\rho) = \begin{cases} s\, w_\rho - (s-1)\epsilon_\rho & (m = 1) \\ s\, w_\rho + (s-1)\epsilon_{\rho+1} & (m \geqslant 2) \end{cases}$$

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at $z^* = a^*$, *where*

$$\epsilon_\alpha = \begin{cases} 0 & (\alpha \text{ even}) \\ 1 & (\alpha \text{ odd}). \end{cases}$$

It should be noted that, for a fixed value of m, these relations are the same for all solutions of (2.2) with the associated functions (3.1).

6. Final remark. Let us finally state some remarks about the characterization of solutions of (2.2) by means of the preceding results.

(a) The solutions $U(z,z^*) \equiv \tilde{U}(x,y)$ of partial differential equations (2.2) of the class $\mathfrak{E}$ with rational associated functions also satisfy an ordinary linear differential equation, *considered as functions of* y *for any finite value* $x = x_0 = const$, as can be proved by using the preceding methods. This result and the results obtained in §§ 3–5 enable us to investigate these (single or multi-valued) solutions of (2.2) outside of the domain of validity of the integral representation (2.3). An appropriate theory of this kind **(2)** leads to a characterization of the behaviour of the solutions in the neighbourhood of branch surfaces and some other basic properties; the theory can immediately be applied to the class of equations (2.2) under consideration, but we should stress the fact that for this purpose we need the detailed information about the ordinary differential equations which is given by the preceding theorems.

(b) The coefficients of the ordinary differential equations satisfied by $U(z,z^*) \equiv \tilde{U}(x,y)$ are rational functions of $q_0, \ldots, q_m$. In the special case of partial differential equations (2.2) with rational coefficients the coefficients of the ordinary differential equations are *rational functions of x and y*, respectively. Hence, in this case, the singularities of the solutions of (2.2) with rational associated functions lie on two-dimensional algebraic manifolds in the real four-dimensional space.

(c) So far we have obtained conditions on the associated functions of the solutions $U(z,z^*) \equiv \tilde{U}(x,y)$ of (2.2) in order that $U(x,y)$ satisfies ordinary differential equations. These conditions may be replaced by conditions on the coefficients $a_{\kappa\lambda}$ of the development

$$U(z,z^*) = \sum_{\kappa,\lambda=0}^{\infty} a_{\kappa\lambda} z^\kappa z^{*\lambda}. \tag{6.1}$$

Let the associated function $f(z)$ of $U(z,z^*)$ be represented in the form

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \tag{6.2}$$

and the generating function (2.5) of the operator (2.3) in the form

$$E(z, z^*, t) = \exp Q(z, z^*, t) = \sum_{\mu,\sigma=0}^{\infty} b_{\mu\sigma}(t)\, z^\mu z^{*\sigma}. \tag{6.3}$$

Then, by (2.3),

$$U(z, 0) = \sum_{\kappa=0}^{\infty} a_{\kappa 0} z^{\kappa} = \sum_{\mu,\nu=0}^{\infty} c_\nu A_{\mu\nu} z^{\mu+\nu}$$

where

$$A_{\mu\nu} = 2^{-\nu} \int_{-1}^{1} b_{\mu 0}(t)(1 - t^2)^{\nu - \frac{1}{2}} dt.$$

By comparing the coefficients of corresponding powers of z on both sides we obtain

$$a_{\kappa 0} = \sum_{\nu=0}^{\kappa} c_\nu A_{\kappa-\nu,\nu}, \qquad \kappa = 0, 1, \ldots . \tag{6.4}$$

The solution of this system yields representations of the coefficients c_ν of the associated function in terms of the coefficients $a_{\kappa 0}$ of the development (6.1). Using these representations and theorems by Hadamard **(6)** we obtain information on the nature and location of the singularities of the associated function of $U(z,z^*)$ from the sequence $\{a_{\kappa 0}\}$ of the coefficients in (6.1). This yields sufficient conditions on the coefficients $a_{\kappa 0}$ in order that $U(z,z^*) \equiv \tilde{U}(x,y)$ satisfy ordinary linear differential equations. In this connection the important problem arises as to what extent similar conclusions can be drawn if other subsequences, say $\{a_{\kappa\lambda}\}$, $\lambda > 0$ and fixed, of the coefficients in (6.1) are known. This question will be considered in another paper.

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