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MAXIMALITY IN FUNCTION ALGEBRAS

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In this paper we prove that the proper Dirichlet subalgebras of the disc algebra discovered by Browder and Wermer [**1**] are maximal subalgebras of the disc algebra (Theorem 2). We also give an extension to general function algebras of a theorem of Rudin [**4**] on the existence of maximal subalgebras of $C(X)$. Theorem 1 implies that every function algebra defined on an uncountable metric space has a maximal subalgebra.

A function algebra A on X is a uniformly closed, point-separating subalgebra of $C(X)$, containing the constants, where X is a compact Hausdorff space. If A and B are function algebras on X, $A \subset B$, $A \neq B$, we say A is a maximal subalgebra of B if whenever C is a function algebra on X with $A \subset C \subset B$, either $C = A$ or $C = B$.

$C(X)^*$, the dual space of $C(X)$, is identified with the space of all complex, regular, Borel measures on X. If A is a subspace of $C(X)$, $A^\perp$ is the space of all $\mu \in C(X)^*$ such that $\int f\, d\mu = 0$ for each $f \in A$. Let E be a Borel subset of X, and let M and N be subsets of $C(X)^*$. We write μ_E for the measure defined by $\mu_E(S) = \mu(E \cap S)$, M_E for the set $\{\mu_E: \mu \in M\}$, and $M \perp N$ if $\mu \perp \nu$ for every $\mu \in M$, $\nu \in N$.

If A and B are function algebras on X, the function algebra which they generate is denoted by $[A, B]$.

LEMMA 1. *Let A and B be function algebras on X with $A^\perp \perp B^\perp$. Then:*

(a) *$A \cap B$ is a function algebra on X;*

(b) *If C is a function algebra on X with $A \cap B \subset C$, then*

$$C = [A, C] \cap [B, C].$$

Proof. It follows as in [**1**, Theorem 1] that $(A \cap B)^\perp = A^\perp + B^\perp$. To prove (a) we have to show that $A \cap B$ is point-separating. Thus let $x_1, x_2 \in X$, $x_1 \neq x_2$, and let δ_i be the point mass at x_i, $i = 1, 2$. If $A \cap B$ does not separate x_1 and x_2, then $\delta_1 - \delta_2 \in (A \cap B)^\perp$; hence $\delta_1 - \delta_2 = \mu + \nu$, $\mu \in A^\perp$, $\nu \in B^\perp$. Using the fact that $\mu \perp \nu$, one now easily deduces a contradiction.

For (b), observe first that $[A, C]^\perp + [B, C]^\perp$ is weak-* dense in $([A, C] \cap [B, C])^\perp$. We shall show that $[A, C]^\perp + [B, C]^\perp = C^\perp$. It follows that $([A, C] \cap [B, C])^\perp = C^\perp$, which implies (b).

It remains to show that $C^\perp \subset [A, C]^\perp + [B, C]^\perp$. Thus let $\lambda \in C^\perp$, so that $\lambda = \mu + \nu$, $\mu \in A^\perp$, $\nu \in B^\perp$. We will show that $\mu \in [A, C]^\perp$ and $\nu \in [B, C]^\perp$.

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To do so it suffices to prove that if f is an arbitrary element of C, then $f\mu \in A^\perp$ and $f\nu \in B^\perp$. Now for $f \in C$, $f\lambda \in C^\perp$; hence $f\lambda = \mu_1 + \nu_1$, $\mu_1 \in A^\perp$, $\nu_1 \in B^\perp$. If we let $M = \{f\mu, \mu_1\}$ and $N = \{f\nu, \nu_1\}$, then $M \perp N$; thus there exists a Borel subset E of X such that $M_E = M$ and $N_{X-E} = N$. Then

$$f\mu = (f\mu + f\nu)_E = (\mu_1 + \nu_1)_E = \mu_1 \in A^\perp.$$

Similarly $f\nu = \nu_1 \in B^\perp$, completing the proof.

When A is a maximal subalgebra of $C(X)$ and C is not contained in A, then $[A, C] = C(X)$. Hence we have the following result.

LEMMA 2. *Let A and B be function algebras on X, $A^\perp \perp B^\perp$.*

(a) *If B is maximal in $C(X)$ and A is not contained in B, then $A \cap B$ is maximal in A.*

(b) *If A and B are maximal in $C(X)$ and $A \neq B$, then the only function algebras which properly contain $A \cap B$ are A, B, and $C(X)$.*

THEOREM 1. *Let A be a function algebra on X, and suppose that X contains a closed subset T homeomorphic to the Cantor set such that $(A^\perp)_T = 0$. Then A contains a maximal subalgebra A'. In particular, the conclusion is valid if X is an uncountable metric space.*

Proof. Let T be as in the statement of the theorem. Pełczyński has proved that such a set will always exist when X is an uncountable metric space (see [**3**, the proof of Theorem 1]). We apply Rudin's theorem [**4**] and obtain a maximal subalgebra B of $C(X)$ where, as the proof of Rudin's theorem shows, $B^\perp = (B^\perp)_T$. Since $(A^\perp)_T = 0$, $A^\perp \perp B^\perp$. Hence $A' = A \cap B$ is a maximal subalgebra of A, by Lemma 2(a).

Let Γ be the unit circle $|z| = 1$. The disc algebra A_0 is the subalgebra of $C(\Gamma)$ of all functions which admit continuous extensions to $|z| \leqq 1$, analytic in $|z| < 1$.

The subalgebras of A_0 constructed by Browder and Wermer in [**1**] are obtained by a technique we now describe. If q is a homeomorphism of Γ on itself, set $A_0{}^q = \{f \in C(\Gamma): f \circ q \in A_0\}$. q is called *singular* if for some Borel subset E of Γ of Lebesgue measure 2π, $q^{-1}(E)$ has Lebesgue measure zero. When q is singular, $A_0{}^\perp \perp (A_0{}^q)^\perp$, by the F. and M. Riesz theorem [**2**, p. 47]. Hence by Lemma 2(b), which is applicable by virtue of the Wermer maximality theorem [**2**, p. 93], we have the following result.

THEOREM 2. *If q is singular, then $A_0 \cap A_0{}^q$ is a maximal subalgebra of A_0, and the only other function algebras on Γ which properly contain it are $A_0{}^q$ and $C(\Gamma)$.*

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