The Riemann surfaces of a function and its fractional integral.

By William Fabian.

1. *Introduction.* For a many-valued function $f(z)$ of the complex variable z, a Riemann surface can be constructed such that, at any point z on the surface, the function has only one value; a function normally multiform, is therefore uniform on a certain Riemann surface.

The operator $D^{-\lambda}$ represents a λ^{th} integral of a function and is defined by [1]

$$D^{-\lambda}(l_a)\, f(z) = \frac{1}{\Gamma(\lambda+\gamma)} \left(\frac{d}{dz}\right)^{\gamma} \int_a^z (z-t)^{\lambda+\gamma-1} f(t)\, dt,$$

where l is a simple curve in the plane of the complex variable, along which the integration is carried out. λ may be real or complex, and γ is the least integer greater than or equal to zero such that $R(\lambda) + \gamma > 0$, $R(\lambda)$ being the real part of λ.

In this note we are concerned with relations between the Riemann surfaces of a function and its fractional integral.

2. *Transformation of Riemann surfaces.*

Theorem 1. *Let $f(z)$ be analytic within a circle with centre at a, and which contains l in its interior. Then a is a branch-point of $D^{-\lambda}(l_a) f(z)$ for non-integral values of λ.*

If λ is a rational fraction r/s expressed in its lowest terms, then a is the vertex of a cycle of s roots.

If λ is irrational or complex, then a is the vertex of an infinite number of roots.

Proof. The Taylor series for $f(z)$ at a within the given circle is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n.$$

Then applying the operator $D^{-\lambda}$ to each term of this series, we easily find that, within the given circle,

$$D^{-\lambda}(l_a) f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(\lambda+n+1)} (z-a)^{\lambda+n}.$$

The conclusion follows immediately.

[1] Fabian, *Phil. Mag.*, 20, 783 (1935).

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Theorem 2. *Let $f(z)$ be analytic in a bounded region E, except for an isolated singularity within E at a point p different from a, at which $f(z)$ can be expanded in a Laurent series.*

Then, for non-integral values of λ, p is a branch-point of $D^{-\lambda}(l_a)f(z)$, with cycles of an infinite number of roots.

Proof. In the t-plane, where l joins the points $t = a$ and $t = z$, let C be a closed contour through the point $t = z$, which lies wholly in E, encloses p, and excludes l. Denote by S_m the curve traced out by a point t which passes along l from a to z and then describes C m times. Then

$$\begin{aligned} D^{-\lambda}(S_m)f(z) &= D^{\gamma}D^{-\lambda-\gamma}(S_m)f(z) \\ &= D^{\gamma}\{D^{-\lambda-\gamma}(l_a)f(z) + m\,D^{-\lambda-\gamma}(C)f(z)\} \\ &= D^{-\lambda}(l_a)f(z) + m\,D^{\gamma}D^{-\lambda-\gamma}(C)f(z) \\ &= D^{-\lambda}(l_a)f(z) + m\,D^{-\lambda-\gamma}(C)f^{(\gamma)}(z), \end{aligned} \tag{1}$$

on integrating $D^{-\lambda-\gamma}(C)f(z)$ by parts γ times.

By a previous theorem [1]

$$D^{-\lambda-\gamma}(C)f^{(\gamma)}(z) = 2\pi i \sum_{\sigma=1}^{\infty} (-1)^{\sigma-1} A_\sigma \frac{(z-p)^{\lambda-\sigma}}{\Gamma(\lambda-\sigma+1).\ (\sigma-1)!},$$

where $\sum_{\sigma=-\infty}^{\infty} A_\sigma (z-p)^{-\sigma}$ is the Laurent series for $f(z)$ at p.

The conclusion now follows from (1).

Theorem 3. *Let $f(z)$ be analytic in a bounded region E on the Riemann surface associated with $f(z)$, except for a branch-point within E at a point p different from a, at which $f(z)$ can be expanded in a Puiseux series. Let the number of roots of $f(z)$ in the cycle* [2] *at p be r.*

If the Puiseux series for $f(z)$ at p does not contain negative integral powers of $(z-p)$, the number of roots of $D^{-\lambda}(l_a)f(z)$, where λ is non-integral, in the corresponding cycle at p does not exceed r. If the series contains negative integral powers of $(z-p)$, the number of roots of $D^{-\lambda}(l_a)f(z)$, where λ is non-integral, in the corresponding cycle at p is infinite.

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[1] Fabian: *Phil. Mag.*, 21, 277 (1936).

[2] If $f(z)$ has M cycles at p, $f(z)$ is to be regarded as having M branch-points at p, and the theorem applies to each of these branch-points separately.

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Proof. On the Riemann surface associated with $f(t)$, let C be a closed contour through the point $t = z$, which lies wholly in E, encloses p and excludes l, where l joins a and z. Denote by S_m the curve traced out by a point t which passes along l from a to z and then describes C m times.

As in the proof of Theorem 2, we have

$$D^{-\lambda}(S_m)\, f(z) = D^{-\lambda}(l_a)\, f(z) + m\, D^{-\lambda-\gamma}(C)\, f^{(\gamma)}(z). \tag{1}$$

By a previous theorem,[1] from which the value of $D^{-\lambda-\gamma}(C)\, f^{(\gamma)}(z)$ can be immediately deduced, it follows that $D^{-\lambda-\gamma}(C)\, f^{(\gamma)}(z)$, for non-integral values of λ, is or is not zero, according as the Puiseux series for $f(z)$ at p does not or does contain negative integral powers of $(z - p)$. The result then follows from (1).

[1] Fabian: *Phil. Mag.*, 21, 276 (1936).

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