Let (m, n) be a pair of positive integers satisfying $(*)$. If $m = n$, then $m = n = 1$. Suppose $m > n$ and let $t = m - n$. Then $t \geqslant 1$ and $m = t + n$. Substituting in $(*)$ gives:

$$\begin{aligned} & m^2 - n^2 = mn \pm 1 \\ \Leftrightarrow\ & (t+n)^2 - n^2 = (t+n)n \pm 1 \\ \Leftrightarrow\ & n^2 - t^2 = tn - (\pm 1) \\ \Leftrightarrow\ & n^2 - t^2 = nt \pm 1. \end{aligned}$$

So if (m, n) satisfy $(*)$, then so do (n, t). Furthermore (n, t) is a lower pair than (m, n). (For if $m^2 - n^2 = mn \pm 1$ as above, then $m = \frac{1}{2}(n + \sqrt{(5n^2 \pm 4)})$ and so $m \leqslant 2n$ and $t = m - n \leqslant n$.)

By replacing (m, n) by (n, t), this process can be repeated producing smaller pairs of integers satisfying $(*)$ until the pair $(1, 1)$ is reached. Reversing the process, the pair (m, n) must be one of the sequence $(1, 1)$, $(2, 1)$, $(3, 2)$, $(5, 3)$, $(8, 5)$, $(13, 8)$, ... Hence the original pair of integers satisfying $(*)$ must be two consecutive terms from the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ..."

Correspondence

Looking for patterns

DEAR EDITOR,

Recent *Gazette* articles refer to the problem of how to avoid producing the result

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

like a rabbit from a conjuror's hat. Having always tried to encourage my students to look for patterns, I have found the following method simple but effective:

n	1	2	3	4	5	6	7	...
$\sum_{r=1}^{n} r$	1	3	6	10	15	21	28	...
$\sum r^2$	1	5	14	30	55	91	140	...
$\sum r^2/\sum r$	1	$\frac{5}{3}$	$\frac{7}{3}$	3	$\frac{11}{3}$	$\frac{13}{3}$	5	...,
i.e.	$\frac{3}{3}$	$\frac{5}{3}$	$\frac{7}{3}$	$\frac{9}{3}$	$\frac{11}{3}$	$\frac{13}{3}$	$\frac{15}{3}$	...

This suggests that

$$\sum r^2/\sum r = \frac{2n+1}{3} \quad \text{or} \quad \sum r^2 = \frac{n(n+1)}{2} \cdot \frac{2n+1}{3},$$

and it then seems quite natural to attempt to prove the result by induction.

Yours sincerely,

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A counter-example

DEAR EDITOR,

In answer to Robert Eastaway's question at the end of note **65.26**, Lander and Parkin discovered in 1966 that

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$