GENERALIZED DISCRETE VALUATION RINGS

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Jategaonkar (**5**) has constructed a class of rings which can be used to provide counterexamples to problems concerning unique factorization in non-commutative domains, the left-right symmetry of the global dimension for a right-Noetherian ring and the transfinite powers of the Jacobson radical of a right-Noetherian ring. These rings have the following property:

(W) Every non-empty family of right ideals of the ring R contains exactly one maximal element.

In the present paper we wish to consider rings, with unit element, which satisfy property (W). This property means that the right ideals are inverse well-ordered by inclusion, and it is our aim to describe these rings by their order type. Rings of this kind appear as a generalization of discrete valuation rings in R; see (**1**; **2**).

In the following, R will always denote a ring with unit element satisfying (W). To prove the main result, we need two preliminary lemmas.

LEMMA 1. *Every right ideal of R is a two-sided ideal and a principal right ideal.*

Proof. Let $0 \neq J$ be any right ideal of R and I_0 the maximal element in the family of those right ideals which are properly contained in J. Then if a is an element in J not contained in I_0, it follows that $J = aR$. To prove that all right ideals are two-sided, let a_iR be a maximal right ideal with the following property: it is not a left ideal. Then there exists an element r in R with

(i) $ra_i = a_k$,

where a_k is not contained in a_iR. Hence $a_kR \supset a_iR$ ($\supset$ denotes strict containment) and

(ii) $a_kr_1 = a_i$ for an element r_1 in R with $r_1R \neq R$.

It follows from (i) and (ii) that for some element r' in R,

$$a_k = ra_i = ra_kr_1 = a_kr'r_1$$

since a_kR is a two-sided ideal. If $r'r_1$ is a unit in R, then $r'r_1r_2 = 1$ for some r_2 in R, and therefore $r'(r_1r_2r' - 1) = 0$. However, r_1r_2r' is not a unit since $r_1R \neq R$, hence $r_1r_2r' - 1$ is a unit since R is local. Therefore $r' = 0$, which implies that $a_k = 0$, a contradiction to $a_kR \supset a_iR$. If $r'r_1$ is not a unit in R, then $1 - r'r_1$ is a unit since R is local; therefore $a_k(1 - r'r_1) = 0$ implies $a_k = 0$, which is again a contradiction.

The above lemma shows that the rings considered here are all subcommutative rings. In the following, this fact will be used frequently. Further, there is

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no distinction between right and left units. For, if $uu' = 1$ for u, u' in R, it follows that $u'\bar{u} = 1$ for some $\bar{u}$ in R so that $u = uu'\bar{u} = \bar{u}$, which shows that $u'u = 1$.

LEMMA 2. *Let cR be any right ideal in R and let a and b be elements in R with ab in cR, b not in cR. Then a^n is contained in cR for some integer n.*

Proof. Assume that no power a^n of a is contained in cR. Then we obtain an ascending chain of right ideals B_n with $B_n = \{r \in R;\ a^n r \in cR\}$ and hence $B_n = B_{n+1}$ for some n (Lemma 1). Next, let $a^n r = bs$ be an element in $a^nR \cap bR$, where $r, s \in R$. Then $a^{n+1}r$ is an element in cR, and therefore r is in $B_{n+1} = B_n$, and $a^nR \cap bR \subseteq cR$. However, $a^nR \cap bR = a^nR$ or $a^nR \cap bR = bR$ by the linear ordering of the right ideals of R; and since neither of these ideals is contained in cR, we have a contradiction.

To every ring R with property (W) there belongs the order type T of the chain of right ideals:

$$R = A_0 \supset A_1 \supset \ldots \supset A_\alpha \supset A_{\alpha+1} \supset \ldots \supset A_\tau = 0 \qquad T = \tau + 1.$$

Further, the order type T can be expanded in the form

$$(+) \qquad T = \omega^{I_1} n_1 + \omega^{I_2} n_2 + \ldots + \omega n_{h-1} + n_h + 1$$

in powers of ω with non-negative integers n_i and exponents

$$I_1 > I_2 > \ldots . > 1 > 0$$

which are ordinal numbers; see (**4**, p. 68). Using this expansion of T, we prove the following result.

THEOREM. *Let R be a ring with property* (W) *and with the order type T. Then there exists a family $\{p_j\}$ of elements p_j in R, where j runs through all ordinal numbers $\leqq I_1$, with the following properties:*

(1) *The p_jR are prime ideals for all j and $p_iR \subseteq p_jR$ if and only if $i \geqq j$. In particular, the p_jR are the only proper prime ideals in R;*

(2) *If $i < j$, there exists a unit $\epsilon_{i,j}$ in R with $p_ip_j = p_j\epsilon_{i,j}$;*

(3) *Every element $0 \neq a$ in R has a representation $a = p_{i_1}p_{i_2}\ldots p_{i_n}\epsilon$, where ϵ is a unit and $i_1 \geqq i_2 \geqq i_n$. This representation is unique up to ϵ;*

(4) *R has no proper zero divisors if and only if $T = \omega^{I_1} + 1$. In this case, we have $p_{I_1} = 0$ but $p_i \neq 0$ for all $i < I_1$. In all the other cases, the p_i are all non-zero but ${p_{I_1}}^{n_1}{p_{I_2}}^{n_2}\ldots p_1^{n_{h-1}}p_0^{n_h} = 0$ if T has the form $(+)$ given above.*

Proof. We first define the elements p_i in R. Let p_0 be a generator of the maximal right ideal of R. If i is a limit number, take p_i in R such that $p_iR = \bigcap_{j<i} p_jR$, and if i is not a limit number, choose p_i to be a generator of $\bigcap_{n=1}^{\infty} p_{i-1}^n R = p_iR$. Every p_i is therefore uniquely determined up to right factors which are units.

Now we prove (2) and the first part of (1). These are clearly true for $j = 0$. Thus we may assume that (2) and the first part of (1) have been proved for

$j < \lambda$. If λ is not a limit number, then there exists $\lambda - 1$. Assuming, as we may, that $p_{\lambda-1}^n \neq 0$ for all positive integers n, we can prove that

$$D = \bigcap_{n=1}^{\infty} p_{\lambda-1}^n R = p_\lambda R$$

is a prime ideal. Let cR and dR be two right ideals in R with $cR \supset D \subset dR$ and $cRdR \subseteq D$. It follows that $cR \not\subseteq p_{\lambda-1}^n R \not\supseteq dR$ for some n. Hence

$$p_{\lambda-1}^{2n} R \subseteq cRdR \subseteq p_{\lambda-1}^{2n+1} R;$$

then $p_{\lambda-1}^{2n}(1 - p_{\lambda-1}r) = 0$ for some r in R and $p_{\lambda-1}^{2n} = 0$, which is a contradiction. If λ is a limit number, we can assume that $p_j R \neq 0$ for all $j < \lambda$ and we prove again that $D = \bigcap_{j<\lambda} p_j R = p_\lambda R$ is a prime ideal. For, let cR and dR be two right ideals of R, not contained in D but $cRdR \subseteq D$. Then for some $k < \lambda$ we have $cR \supset p_k R \subset dR$ and for $i = k + 1 < \lambda$ we obtain by (2): $p_i^2 R \subseteq p_i R \subseteq p_k R p_k R \subseteq D \subseteq p_i^2 R$. Hence $p_i(1 - p_i r) = 0$ for some r in R and $p_i = 0$ which is again a contradiction.

To prove (2) it suffices to show that $p_i p_j R = p_j R$ for $i < j = \lambda$ and $p_j \neq 0$. If λ is not a limit number, we have $p_\lambda R = \bigcap_{n=1}^{\infty} p_{\lambda-1}^n R$ and since $i < \lambda$ we obtain $\bigcap_{n=1}^{\infty} p_i^n R \supseteq p_\lambda R$. From this it follows that $p_i^n R \supset p_\lambda R \supseteq p_i p_\lambda R$ for every positive integer n and by Lemma 2 and $p_i p_\lambda R \subseteq p_i p_\lambda R$ that $p_\lambda R \subseteq p_i p_\lambda R$. The reverse relation is obvious so that finally $p_\lambda R = p_i p_\lambda R$. The same result follows if λ is a limit number, and this can be easily shown by using Lemma 2 again.

Now we determine all right ideals in R. Consider the following descending chain of right ideals:

$$R = D_0 \supset p_0 R = D_1 \supset p_0^2 R = D_2 \supset \ldots \supset p_0^n R = D_n \supset \ldots \supset \bigcap_{n=0}^{\infty} p_0^n R$$
$$= D_\omega = p_1 R \supset p_1 p_0 R = D_{\omega+1} \supset \ldots \supset D_\alpha \supset D_{\alpha+1}, \ldots,$$

where

$$D_\alpha = \begin{cases} \bigcap_{\beta<\alpha} D_\beta & \text{for a limit number } \alpha, \\ D_{\alpha-1} p_0 R & \text{otherwise.} \end{cases}$$

It is obvious that these are all the right ideals of R and that $A_\alpha = D_\alpha$ for all $\alpha < T$.

We complete the proof of the Theorem by showing that

$$A_\alpha = p_{i_1}^{m_1} p_{i_2}^{m_2} \ldots p_1^{m_{s-1}} p_0^{m_s} R \quad \text{for } \alpha = \omega^{i_1} m_1 + \omega^{i_2} m_2 + \ldots + \omega m_{s-1} + m_s$$

with integers $m_i \geqq 0$ and ordinal numbers $i_1 > i_2 > \ldots > 1 > 0$. Transfinite induction on α is used in the proof. It is enough to show that

$$(++) \qquad D = \bigcap_{n=1}^{\infty} a p_{i_k}^n R = a p_{i_k+1} R$$

for $a = p_{i_1}^{n_1} p_{i_2}^{n_2} \ldots p_{i_{k-1}}^{n_{k-1}}$ with $i_1 > i_2 > \ldots > i_{k-1} > i_k$ and $a p_{i_k}^n R \neq 0$ for all n, and to prove the similar equation $(+++)$ below.

First it is clear by our previous results that $D = A_{\omega^{i_k+1}}$ is a prime ideal if both $a = 1$ and $D = 0$. This means that R is a ring without proper zero divisors and with the order type $T = \omega^{i_k+1} + 1$. Otherwise,

$$\bigcap_{n=1}^{\infty} p_{i_k}{}^n R = p_{i_k+1} R \quad \text{and} \quad D = \bigcap_{n=1}^{\infty} a p_{i_k}{}^n R \supseteq a p_{i_k+1} R.$$

To prove the reverse relation we note that an element d in D has the form $d = a p_{i_k} r_1 = a p_{i_k}{}^2 r_2 = \ldots\ldots = a p_{i_k}{}^n r_n = \ldots$ for some elements r_n in R. If we now can prove that $ar = 0$ is possible only for elements r in $\bigcap_{n=1}^{\infty} p_{i_k}{}^n R$, then, since $a(p_{i_k} r_1 - p_{i_k}{}^n r_n) = 0$ for all n, it follows that $p_{i_k} r_1$ is contained in $p_{i_k+1} R$, so that d is an element in $a p_{i_k+1} R$. Thus, let $ar = 0$, and assume that r is not contained in $p_{i_k+1} R$. Then $rR \supseteq p_{i_k}{}^n R$ for some integer n. Hence $0 = arR \supseteq a p_{i_k}{}^n R \neq 0$, a contradiction.

Similarly, we may prove that

$$(+++) \qquad D = \bigcap_{j<i} a p_j R = a p_i R \quad \text{for a limit number } i.$$

Here we have $a = p_{i_1}{}^{n_1} p_{i_2}{}^{n_2} \ldots p_{i_k}{}^{n_k}$ with $i_1 > i_2 > \ldots > i_k \geqq i$ and $a p_j R \neq 0$ for all $j < i$. If both $a = 1$ and $D = 0$, it follows as before that D is a prime ideal and hence R is a ring with order type $\omega^i + 1$ and without proper zero divisors. Otherwise, using (2) of the Theorem, we obtain

$$a p_i R \subseteq \bigcap_{j<i} a p_j R.$$

Finally, an argument similar to the one used in the proof of $(++)$ shows that $a p_i R = \bigcap_{j<i} a p_j R$. This proves the Theorem.

The following result follows immediately from the Theorem.

COROLLARY 1. *To every element $a \neq 0$ in R there can be assigned an ordinal number $v(a) = \alpha$ with $aR = A_\alpha$ such that*

$$v(a + b) \geqq \min(v(a), v(b)) \quad \textit{and} \quad v(ab) = v(a) + v(b).$$

An element $a \neq 0$ in R generates a prime ideal $aR \neq R$ if and only if $v(a) = \omega^i$ for some power of ω and an element ϵ in R is a unit if and only if $v(\epsilon) = 0$.

The following corollary shows, in particular, that in the commutative case, or more generally if all left ideals are two-sided, only the order types $T \leqq \omega + 1$ can appear.

COROLLARY 2. *If R is a ring having property* (W) *and with maximal condition for principal left ideals, then $T \leqq \omega + 1$.*

Proof. Assume that $T > \omega + 1$. Then there exists the prime ideal $p_1 R \neq 0$, and $p_0 p_1 = p_1 \epsilon_{0,1}$ for a unit $\epsilon_{0,1}$ in R. But then

$$0 \subset Rp_1 \subset Rp_1\epsilon_{0,1}{}^{-1} \subset Rp_1\epsilon_{0,1}{}^{-2} \subset \ldots \subset Rp_1\epsilon_{0,1}{}^{-n} \subset Rp_1\epsilon_{0,1}{}^{-n-1} \subset \ldots$$

is a strictly ascending chain of principal left ideals.

COROLLARY 3. *An integral domain R has property* (W) *if and only if R is a right hereditary local right Ore domain.*

Proof. If R is a local ring, all projective right ideals are free, and since R is a right Ore domain, all the right ideals are principal. Let p_0R be the maximal ideal of R; then the transfinite powers of p_0R are all the right ideals in R and this proves property (W).

In particular, we know that the projective right global dimension of R is at most 1.

The projective left global dimension of an integral domain R having property (W) may be approximated by using the same arguments as in (**5**). Thus, let $T = \omega^{I+1} + 1$ be the order type of R, where $I = \omega_n$ is the first ordinal number of cardinality $\aleph_n$ and consider the chain

$$Rp_I \subset Rp_I\epsilon_{0,I}^{-1} \subset Rp_I\epsilon_{1,I}^{-1} \subset \ldots \subset Rp_I\epsilon_{\alpha,I}^{-1} \subset Rp_I\epsilon_{\alpha+1,I}^{-1} \subset \ldots .$$

Using a theorem of Osofsky (**6**, p. 146), it follows that projective dimension $(\bigcup_{\alpha<I} Rp_I\epsilon_{\alpha,I}^{-1}) = n + 1$, hence left global dimension $R \geqq n + 2$.

In conclusion we construct a ring R having property (W) and order type $T = \omega^2 + 1$. Let $K = k(t_1, t_2, \ldots)$ be the quotient field of the commutative polynomial ring $k[t_1, t_2, \ldots]$ over a commutative field k in infinitely many indeterminates t_n. The localization A at y of the commutative polynomial ring $K[y]$ in one indeterminate y over K is a ring having property (W) of order type $T = \omega + 1$. Now consider the twisted power series ring $R = A[[x, \sigma]]$ with elements

$$r = \sum_{i=0}^{\infty} x^i a_i, \qquad a_i \in A,$$

and the obvious addition. Let σ be the monomorphism of A defined by $q^\sigma = q$ for q in k, $y^\sigma = t_1$ and $t_i^\sigma = t_{i+1}$ for $i = 1, 2, \ldots$. If we define multiplication in R by $ax = xa^\sigma$, then R is a ring having property (W) and order type $T = \omega^2 + 1$; see (**3**, p. 598).

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