

A NOTE ON RADO'S THEOREM

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Abstract

In this note, a theorem of Rado which characterizes the convex hull of the set of all rearrangements of a given real n -tuple in terms of the Hardy–Littlewood–Pólya spectral order relation $<$ is shown to be a consequence of a result of Hardy–Littlewood–Pólya and a strong spectral inequality.

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Introduction

Using the separation theorem for convex sets of vectors, Rado (1952) characterized the convex hull of the collection of all rearrangements of a given real n -tuple in terms of the Hardy–Littlewood–Pólya spectral order relation $<$ for n -vectors. With this characterization, he gave a new proof for the classical Muirhead theorem which he then generalized. This proof of Rado's appears to have rendered him "the first mathematician to make explicit use of results on convex sets in the discussion of doubly stochastic matrices" (see Mirsky (1963)).

In this note, it is shown that the germ of half of Rado's theorem is already contained in Hardy, Littlewood and Pólya (1934), Lemma 2, p. 47, while the other half is shown to be a direct consequence of a spectral inequality (to be given in (3) below).

1. Preliminaries

In the sequel, our notation is as follows. If $x = (x_1, x_2, \dots, x_n) \in R^n$ is any n -tuple of real numbers, we denote by $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ the n -tuple in R^n whose components are those of x arranged in non-increasing order of magnitude, that

is $x_1^* \geq x_2^* \geq \dots \geq x_n^*$ and $x_i^* = x_{\pi(i)}$, $1 \leq i \leq n$, for some permutation π of the integers $1, 2, \dots, n$. If $a = (a_1, a_2, \dots, a_n) \in R^n$ and $b = (b_1, b_2, \dots, b_n) \in R^n$, then we write $a \prec b$ to mean that

$$(1) \quad \sum_{i=1}^k a_i^* \leq \sum_{i=1}^k b_i^*, \quad 1 \leq k \leq n,$$

where equality holds for $k = n$. Moreover, $a \sim b$ means $a^* = b^*$, that is, $a_i^* = b_i^*$, $1 \leq i \leq n$, or the components of a form a permutation of those of b . Thus $a \sim b$ if and only if $a \prec b$ and $b \prec a$. If $a \sim b$, then a is said to be a *rearrangement* of b and conversely.

Following Chong (1974), we call expressions of the form $a \prec b$ *strong spectral inequalities*.

For any vectors $a, b \in R^n$, the strong spectral inequality

$$(2) \quad a + b \prec a^* + b^*$$

is easily seen to hold, by virtue of the fact that there exists a permutation π of the integers $1, 2, \dots, n$ such that

$$a_{\pi(1)} + b_{\pi(1)} \geq a_{\pi(2)} + b_{\pi(2)} \geq \dots \geq a_{\pi(n)} + b_{\pi(n)}$$

and that, for $1 \leq k \leq n$,

$$\sum_{i=1}^k (a_{\pi(i)} + b_{\pi(i)}) \leq \sum_{i=1}^k (a_i^* + b_i^*),$$

where equality holds for $k = n$. Moreover, it follows from (2) by induction that

$$(3) \quad a_1 + a_2 + \dots + a_m \prec a_1^* + a_2^* + \dots + a_m^*$$

for any m vectors $a_i \in R^n$, $i = 1, 2, \dots, m$.

If $a_i \prec b$, $i = 1, 2, \dots, m$, then it follows immediately from (3) that

$$(4) \quad r_1 a_1 + r_2 a_2 + \dots + r_m a_m \prec b$$

whenever $0 \leq r_i \leq 1$, $i = 1, 2, \dots, m$, and $\sum_{i=1}^m r_i = 1$.

REMARK. The spectral inequality (3) is also a direct consequence of the induction principle for spectral inequalities established earlier by Chong (1974), Theorem 2.1, 374–375.

2. Rado's Theorem

We shall now state Rado's Theorem and show how it can be derived from a result of Hardy, Littlewood and Pólya (1934), Lemma 2, p. 47, and the strong spectral inequality (3) obtained in Section 1 above.

THEOREM [R. Rado (1952)]. *If $\mathbf{a} \in R^n$ is any n -vector, let $\mathfrak{H}(\mathbf{a})$ denote the convex hull of the set of all rearrangements of \mathbf{a} , then an n -vector $\mathbf{x} \in \mathfrak{H}(\mathbf{a})$ if and only if $\mathbf{x} \prec \mathbf{a}$.*

PROOF. First, we prove that the condition is necessary. If $\mathbf{x} \in \mathfrak{H}(\mathbf{a})$, then there are numbers r_i and vectors \mathbf{a}_i , $i = 1, 2, \dots, m$, for some $m \leq n!$ such that $0 < r_i \leq 1$, $\mathbf{a}_i \sim \mathbf{a}$, $i = 1, 2, \dots, m$, $\sum_{i=1}^m r_i = 1$ and $\mathbf{x} = \sum_{i=1}^m r_i \mathbf{a}_i$. It then follows from (4) that $\mathbf{x} \prec \mathbf{a}$.

Conversely, suppose $\mathbf{x} \prec \mathbf{a}$. Since the case that $\mathbf{x} \sim \mathbf{a}$ is trivial, we assume $\mathbf{x} \sim \mathbf{a}$. If we write each vector as a column vector, then, by Hardy, Littlewood and Pólya (1934), Lemma 2, p. 47, \mathbf{x} can be derived from \mathbf{a} by a finite number of transformations T of the form $rI + (1-r)P$, where $0 < r < 1$, I is the identity matrix and P is an $n \times n$ permutation matrix (compare the transformation T which is referred to as (2.19.3) in Hardy, Littlewood and Pólya (1934), p. 46). Since the product of permutation matrices is again a permutation matrix, the product of a finite number of matrices of the form $rI + (1-r)P$ is easily seen to be a convex combination of permutation matrices. Moreover, since $P\mathbf{a} \sim \mathbf{a}$ for any $n \times n$ permutation matrix P , we see that \mathbf{x} can be expressed as a convex combination of the rearrangements of \mathbf{a} , that is, $\mathbf{x} \in \mathfrak{H}(\mathbf{a})$.

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