EMBEDDING SEMIRINGS IN SEMIRINGS WITH MULTIPLICATIVE UNIT

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A *topological semiring* is a system $(S, +, \cdot)$ where $(S, +)$ and $(S, \cdot)$ are topological semigroups and $\cdot$ distributes across $+$ as in a ring; that is, for all x, y, z in S,

$$x \cdot (y+z) = (x \cdot y)+(x \cdot z),$$
$$(x+y) \cdot z = (x \cdot z)+(y \cdot z).$$

The operations $+$ and $\cdot$ are called *addition* and *multiplication* respectively.

If S is any topological semiring and we adjoin to S an element 0 as an isolated point and let 0 be a multiplicative zero and either an additive unit or an additive zero for $S' = S \cup \{0\}$, then it can be easily seen that S' is also a topological semiring. Thus it is always possible to embed S in a semiring S' with multiplicative zero; also S' is compact when S is.

Selden has shown in Theorem 7 of [6] (see also [7]) that each additive group in a compact semiring with multiplicative unit must be totally disconnected. This means that the semiring $(C, +, \cdot)$, where $(C, +)$ is the circle group and $x \cdot y = 0$ for all $x, y \in C$, cannot be embedded in a compact semiring with multiplicative unit. We investigate here conditions under which it is possible to embed a semiring in a semiring with multiplicative unit. In particular, we derive in Theorem 3 a necessary and sufficient condition for the embedding of a compact additively commutative semiring in a compact semiring with multiplicative unit to be possible. The special case of embedding a compact ring in a compact ring with unit is also dealt with.

It is first necessary to establish some points of notation. If x is a member of an additive semigroup and n is a positive integer, we shall use nx to mean the semigroup sum of n elements each equal to x. Note that if 1 is a multiplicative left unit of a semiring then n denotes the semiring sum of n elements each equal to 1. Hence nx also equals the product of n and x. For any x in $(S, +, \cdot)$, let

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$$0[+](x) = \{nx|\ n \text{ a positive integer}\},$$
$$\Gamma[+](x) = 0[+](x)^-,$$
$$K[+](x) = \bigcap_{n=1}^{\infty} \{mx|m \geqq n\}^-,$$

where $^-$ denotes topological closure. Then $\Gamma[+](x)$ is a commutative (additive) semigroup. If it is compact, then $K[+](x)$ is the minimal ideal of $\Gamma[+](x)$ and is also the maximal additive group in $\Gamma[+](x)$ (see [5], Theorem **3.1.1** or [2], Theorems **3.3** and **3.4**). A semiring is said to be *additively* Γ*-compact* if $\Gamma[+](x)$ is compact for each x in S.

S is said to be a *subsemiring* of a topological semiring T if and only if for each x, y in S, both $x+y$ and $x \cdot y$ are in S; S, given the relative topology, is a topological semiring. We shall say that a topological semiring S can be *embedded* in a topological semiring T if there is a subsemiring S_1 of T which is topologically isomorphic with S; note that S_1 need not be topologically closed in T.

If S is an additively Γ-compact subsemiring of a topological semiring T then, for each x in S, the closure of $0[+](x)$ in the relative topology of S is compact so that it is also compact in the topology of T. Because T is Hausdorff, it follows that the closure of $0[+](x)$ in the relative topology of S is the same as its closure in the topology of T. Thus we can use the symbol $\Gamma[+](x)$ without fear of confusion.

The first result is a topological extension of a familiar construction (see, for example, [4], page **49**).

THEOREM 1. *Any additively commutative topological semiring S can be embedded in some additively commutative topological semiring T with multiplicative unit. Further, T can be made locally compact when S is locally compact.*

PROOF. We first adjoin to S an element α as an isolated point and let α be an additive unit and a multiplicative zero for $S' = S \cup \{\alpha\}$. Let N denote the (locally compact) semiring of non-negative integers with ordinary addition and multiplication. Then we put $T = N \times S'$ and define addition $*$ and multiplication $\circ$ on T by

$$(n_1, x_1) * (n_2, x_2) = (n_1+n_2, x_1+x_2),$$
$$(n_1, x_1) \circ (n_2, x_2) = (n_1 n_2, x_1 x_2 + n_1 x_2 + n_2 x_1),$$

where $0x = \alpha$ for all x in S'. It is a simple matter to check that $(T, *, \circ)$ is a topological semiring which is locally compact when S is locally compact. Also $(1, \alpha)$ is a multiplicative unit for T and $\{0\} \times S$ is a subsemiring topologically isomorphic with S.

In what follows we are concerned with additively Γ-compact semirings.

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For each x, $\Gamma[+](x)$ is then a compact monothetic semigroup and the structure of all such semigroups, first found by Hewitt in [2], is given in Theorems 3.1.6 and 3.1.7 of [5]. In broad outline, our procedure is to construct what is roughly speaking the largest relevant compact monothetic semigroup and use it in a cartesian product with S' (as in Theorem 1) to be the space in which S is embedded when S is additively commutative.

LEMMA 1. *If S is a topological semiring then $\Gamma[+](x)$ is a subsemiring if and only if $x^2 \in \Gamma[+](x)$.*

PROOF. Suppose $x^2 \in \Gamma[+](x)$. Then if p, q are positive integers, $(pq)x^2 \in \Gamma[+](x)$; but $(pq)x^2 = (px)(qx)$ and so $(px)(qx) \in \Gamma[+](x)$. Hence $\{0[+](x)\}^2 \subset 0[+](x)$ and it follows from the continuity of multiplication that $\{\Gamma[+](x)\}^2 \subset \Gamma[+](x)$.

LEMMA 2. *If S is a topological semiring and x is a multiplicative idempotent, then $\Gamma[+](x)$ is a subsemiring for which x is a multiplicative unit.*

PROOF. It follows from Lemma 1 that $\Gamma[+](x)$ is a subsemiring. Also, for any positive integer m, $(mx) \cdot x = mx^2 = mx = x \cdot (mx)$. Thus x is a multiplicative unit for $0[+](x)$ and hence for $\Gamma[+](x)$.

LEMMA 3. *Let S be a topological semiring containing an element x for which $\Gamma[+](x)$ is compact. Then for each y in S,*

(i) $y \cdot \Gamma[+](x) = \Gamma[+](yx)$ *and* $\Gamma[+](x) \cdot y = \Gamma[+](xy)$;

(ii) $y \cdot K[+](x) = K[+](yx)$ *and* $K[+](x) \cdot y = K[+](xy)$.

PROOF. The mapping $\phi : S \to S$ given by $\phi(z) = yz$ is continuous and $\phi(0[+](x)) = 0[+](yx)$. It follows from the compactness of $0[+](x)^-$ that

$$\phi(0[+](x)^-) = 0[+](yx)^- = \Gamma[+](yx)$$

(see Corollary 2, page 101 and Prop. 9, page 61 of [1]). Hence $\Gamma[+](yx)$ is compact and so $K[+](yx)$ exists. Because $K[+](x)$ is an additive group and ϕ is an additive homomorphism onto $y \cdot K[+](x)$, we see that $y \cdot K[+](x)$ is an additive group, so that $y \cdot K[+](x) \subset K[+](yx)$, the maximal additive group in $\Gamma[+](yx)$. On the other hand, if $z \in \Gamma[+](yx)$ and $w \in y \cdot K[+](x)$, then there are z_1, w_1 in $\Gamma[+](x)$, $K[+](x)$ respectively with $z = yz_1$, $w = yw_1$, and therefore $z+w = y(z_1+w_1) \in y \cdot K[+](x)$ since $K[+](x)$ is an additive ideal of $\Gamma[+](x)$. Thus $y \cdot K[+](x)$, being an additive ideal of $\Gamma[+](yx)$, contains the minimal such ideal $K[+](yx)$, and the result follows.

We now make the construction previously referred to.

EXAMPLE. Let P be the set of prime integers and put

$$N_1 = \mathop{\times}_{\rho \in P} \Delta_\rho$$

where, for each $\rho \in P$, Δ_ρ is the (compact) ring of ρ-adic integers (see, for example, [3], § 10) and N_1 is given the product topology. If we give N_1 coordinate-wise addition and multiplication, N_1 is a compact ring. For any $\rho \in P$ we let 0_ρ be the additive unit and 1_ρ be the multiplicative unit of Δ_ρ. Then let $u \in N_1$ be such that u_ρ, the coordinate of u in Δ_ρ, is equal to 1_ρ for all $\rho \in P$. Clearly u is a multiplicative unit for N_1. It also follows that $\{nu | n \text{ a positive integer}\}$ is dense in N_1; we prove this in Lemma 4 below. Thus the additive group of N_1 is monothetic with generator u.

Let N_2 be the set of positive integers and put $N_3 = N_1 \cup N_2$. We can make N_3 a compact monothetic additive semigroup by proceeding as in Theorem 3.1.7 of [5] (see also Theorem 5.3 of [2]). We take addition in N_2 to be ordinary addition, put

$$x+m = x+mu = m+x \quad \text{if } x \in N_1 \text{ and } m \in N_2,$$

and retain the same addition in N_1. We define a topology on N_3 by letting each point in N_2 be isolated and, for each x in N_1, we take

$$\left\{ V_n^*(x) \;\middle|\; \begin{array}{l} V_n^*(x) = V(x) \cup \{m | m \geqq n \text{ and } mu \in V(x)\}, \text{ where} \\ V(x) \text{ is any neighbourhood of } x \text{ in } N_1 \text{ and } n \geqq 1 \end{array} \right\}$$

as the set of all neighbourhoods of x. It is shown in [5], Theorem 3.1.7 (see also [2], Theorem 5.3) that N_3 is a compact additive semigroup with $N_2^- = \Gamma[+](1) = N_3$ and $K[+](1) = N_1$.

Finally we make N_3 a semiring by taking ordinary multiplication on N_2, putting

$$m \cdot x = x \cdot m = mx \quad \text{if} \quad x \in N_1,\ m \in N_2,$$

and retaining the multiplication on N_1. It is not difficult to check that N_3 becomes a compact semiring with multiplicative unit 1.

LEMMA 4. *Let u and N_1 be as in the example above. Then* $\Gamma[+](u) = N_1$.

PROOF. It follows from Theorem 25.16 of [3] that there is an element v of N_1 such that

$$\{nv | n \text{ any integer}\}$$

is dense in N_1 and so, as shown on page 109 of [5] (see also § 2 of [2]), $\Gamma[+](v) = N_1$. For any $\rho \in P$, let v_ρ be the coordinate of v which is in Δ_ρ. Because $\Gamma[+](v_\rho)$ must be Δ_ρ, it follows from Lemma 3 that

$$\Delta_\rho = \Gamma[+](v_\rho) = \Gamma[+](1_\rho \cdot v_\rho) = \Gamma[+](1_\rho) \cdot v_\rho = \Delta_\rho \cdot v_\rho$$

since $\Gamma[+](1_\rho) = \Delta_\rho$ (see § 10.6 of [3]). Thus there exists an element w_ρ of Δ_ρ with $1_\rho = w_\rho v_\rho$. Let w be the element of N_1 which has its coordinate in Δ_ρ equal to w_ρ for all $\rho \in P$; then $wv = u$. Hence, since u is a multiplicative unit for N_1,

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$$N_1 = uN_1 = (wv)N_1 = w(vN_1) \subset wN_1 \subset N_1$$

and we see that $wN_1 = N_1$. Thus, by Lemma 3,

$$\Gamma[+](u) = \Gamma[+](wv) = w\Gamma[+](v) = wN_1 = N_1.$$

Our main effort is devoted to proving the following theorem.

THEOREM 2. *Let S be an additively commutative and additively Γ-compact semiring. The following are equivalent.*

(i) *S can be embedded in an additively commutative and additively Γ-compact semiring with multiplicative unit.*

(ii) *S can be embedded in an additively Γ-compact semiring with multiplicative left unit.*

(iii) *There is a continuous extension* $\Psi : N_3 \times S \to S$ *of the mapping* $\psi : N_2 \times S \to S$ *defined by* $\psi(n, x) = nx$ *for* $n \in N_2$, $x \in S$.

Clearly (i) implies (ii). We shall prove the theorem by showing that (ii)⇒(iii) and (iii)⇒(i). That (ii)⇒(iii) follows from the following more general result.

LEMMA 5. *Let S be any additively Γ-compact subsemiring of a topological semiring T with multiplicative left unit* α *such that* $\Gamma[+](\alpha)$ *is compact. Then there is a continuous additive extension* $\Psi : N_3 \times S \to S$ *of the mapping* $\psi : N_2 \times S \to S$ *defined by* $\psi(n, x) = nx$. (Note that Ψ is uniquely determined by ψ because N_2 is dense in N_3.)

PROOF. As $\Gamma[+](\alpha)$ is a compact subsemiring in which α is a multiplicative unit (Lemma 2), it is a consequence of [6], Theorem 7 that $K[+](\alpha)$ is totally disconnected, and hence 0-dimensional (Theorem 3.5 of [3]). Because $K[+](\alpha)$ is a monothetic additive group with generator $\alpha' = \alpha + e$, where e is the unit of $K[+](\alpha)$ ([5], Theorem 3.1.2 or [2], Theorems 3.2 and 3.4), it follows from Theorem 25.16 of [3] that $(K[+](\alpha), +)$ is topologically isomorphic with a cartesian product $\times_{\rho \in P} A_\rho$ where for each ρ in P, A_ρ is either the trivial group with one member or the group $Z(\rho^{r_\rho})$ of residues modulo ρ^{r_ρ} for some integer $r_\rho \geqq 1$ or the group Δ_ρ. (If $K[+](\alpha)$ is finite then it is cyclic and the result follows from Theorem 17, Chapter III of [9].) In what follows we shall assume that $K[+](\alpha)$ is identical with $\times_{\rho \in P} A_\rho$. For each $\rho \in P$ we introduce a multiplication $\circ$ on A_ρ as the natural ring multiplication and denote by 1_ρ its multiplicative unit. We give $\times_{\rho \in P} A_\rho$ the coordinate-wise multiplication, for which the element β, whose coordinate in A_ρ is 1_ρ for all ρ, is a multiplicative unit. It is clear that for each $\rho \in P$ there is a continuous additive homomorphism ϕ'_ρ from Δ_ρ onto A_ρ with $\phi'_\rho(1_\rho) = 1_\rho$. (If A_ρ is isomorphic with $Z(\rho^{r_\rho})$, put

$$\phi'_\rho(x) = x_0 + x_1\rho + \cdots + x_{r_\rho - 1}\rho^{r_\rho - 1}$$

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for each $x = (x_0, x_1, x_2, \cdots)$ in Δ_ρ.) If for each x in N_1 we let x_ρ be the coordinate of x in Δ_ρ and we define a function $\phi' : N_1 \to K[+](\alpha)$ by putting the coordinate of $\phi'(x)$ in A_ρ equal to $\phi'_\rho(x_\rho)$ for all $\rho \in P$ and all x in N_1, then ϕ' is clearly a continuous additive homomorphism with $\phi'(u) = \beta$ and $\phi'(N_1) = K[+](\alpha)$. The mapping $f : K[+](\alpha) \to K[+](\alpha)$ given by $f(x) = \alpha' \circ x$ is a continuous additive homomorphism and $f(\beta) = \alpha' \circ \beta = \alpha'$. Also $f(K[+](\alpha))$ is closed and contains $0[+](\alpha')$ and hence its closure $\Gamma[+](\alpha') = K[+](\alpha)$; hence f maps $K[+](\alpha)$ onto $K[+](\alpha)$. We now see that the mapping $\phi : N_1 \to K[+](\alpha)$ given by $\phi(x) = f(\phi'(x))$ is a continuous additive homomorphism of N_1 onto $K[+](\alpha)$ for which

$$\phi(u) = f(\phi'(u)) = f(\beta) = \alpha'.$$

We extend ϕ to become $\Phi : N_3 \to \Gamma[+](\alpha)$ by putting $\Phi(n) = n\alpha$ if $n \in N_2$. It is not difficult to show that Φ is a continuous additive homomorphism of N_3 onto $\Gamma[+](\alpha)$. (However the proof must be split into two cases according as $\Gamma[+](\alpha) \setminus K[+](\alpha)$ is infinite or finite, and use must be made of the characterization of $\Gamma[+](\alpha)$ in either [5], Theorems 3.1.7 and 3.1.6 respectively or in [2], Theorem 5.6.)

Finally, we put $\Psi(y, x) = \Phi(y) \cdot x$ (where $\cdot$ is multiplication in T) for each y in N_3 and x in S. For each x in S,

$$\Psi(N_3 \times \{x\}) \subset \Phi(N_3) \cdot x = \Gamma[+](\alpha) \cdot x = \Gamma[+](\alpha x) = \Gamma[+](x) \subset S,$$

and so Ψ maps $N_3 \times S$ into S. Clearly Ψ is continuous and, for all n in N_2,

$$\Psi(n, x) = \Phi(n) \cdot x = n\Phi(1) \cdot x = (n\alpha) \cdot x = n(\alpha \cdot x) = nx = \psi(n, x).$$

We now show that (iii) $\Rightarrow$ (i).

Lemma 6. *Let S be an additively commutative and additively Γ-compact semiring for which there is a continous extension $\Psi : N_3 \times S \to S$ of $\psi : N_2 \times S \to S$ defined by $\psi(n, x) = nx$. Then S can be embedded in an additively commutative and additively Γ-compact semiring with multiplicative unit.*

Proof. We first adjoin an element γ as an isolated point to S so that γ is an additive unit and a multiplicative zero for $S' = S \cup \{\gamma\}$; then S' is an additively Γ-compact semiring. We also adjoin an element 0 as an isolated point to N_3 so that 0 is an additive unit and a multiplicative zero for $N_4 = N_3 \cup \{0\}$; then N_4 is a compact semiring. We extend Ψ to $N_4 \times S'$ by putting $\Psi(0, x) = \Psi(y, \gamma) = \gamma$ for all x in S', y in N_4; then Ψ is continuous on $N_4 \times S'$. Let $T = N_4 \times S'$ and define addition $*$ and multiplication $\circ$ on T by

$$(y_1, x_1) * (y_2, x_2) = (y_1 + y_2, x_1 + x_2),$$
$$(y_1, x_1) \circ (y_2, x_2) = (y_1 y_2, x_1 x_2 + \Psi(y_1, x_2) + \Psi(y_2, x_1)).$$

Then clearly $*$, $\circ$ are continuous and $*$ is associative and commutative. To complete the proof that T is a semiring we first note that, for all x_1, x_2 in S' and y_1, y_2 in N_4,

(a) $[\Psi(y_1, x_1)] \cdot x_2 = \Psi(y_1, x_1 x_2) = x_1 \cdot [\Psi(y_1, x_2)]$;

(b) $\Psi[y_1, \Psi(y_2, x_1)] = \Psi(y_1 y_2, x_1)$;

(c) $\Psi(y_1, x_1+x_2) = \Psi(y_1, x_1)+\Psi(y_1, x_2)$;

(d) $\Psi(y_1+y_2, x_1) = \Psi(y_1, x_1)+\Psi(y_2, x_1)$.

(These properties are clear for all x_1, x_2 in S' and y_1, y_2 in $N_2 \cup \{0\}$ because Ψ is an extension of ψ. The results follow from the continuity of Ψ and the fact that N_2 is dense in N_3.) It is a simple matter to use (a)—(d) and the commutativity of $+$ to check the distributive laws and the associativity of $\circ$.

That T is additively Γ-compact follows because

$$\begin{aligned}\Gamma[*](y, x) &= \{n(y, x) | n \geqq 1\}^- \\ &= \{(ny, nx) | n \geqq 1\}^- \subset \Gamma[+](y) \times \Gamma[+](x).\end{aligned}$$

Also, $(1, \gamma)$ is a multiplicative unit for T since

$$\begin{aligned}(y, x) \circ (1, \gamma) &= (y \cdot 1, x\gamma + \Psi(y, \gamma) + \Psi(1, x)) \\ &= (y, \gamma+\gamma+x) = (y, x) = (1, \gamma) \circ (y, x).\end{aligned}$$

Finally we note that $\{0\} \times S$ is a subsemiring which is topologically isomorphic with S since

$$\begin{aligned}(0, x_1) * (0, x_2) &= (0+0, x_1+x_2) = (0, x_1+x_2), \\ (0, x_1) \circ (0, x_2) &= (0 \cdot 0, x_1 x_2 + \Psi(0, x_2) + \Psi(0, x_1)) \\ &= (0, x_1 x_2 + \gamma + \gamma) = (0, x_1 x_2).\end{aligned}$$

When S is compact we have the following result.

THEOREM 3. *Let S be a compact additively commutative semiring. Then the following are equivalent.*

(i) *S can be embedded in a compact additively commutative semiring with multiplicative unit.*

(ii) *S can be embedded in an additively Γ-compact semiring with multiplicative left unit.*

(iii) *There is a continuous extension $\Psi : N_3 \times S \to S$ of the mapping $\psi : N_2 \times S \to S$ defined by $\psi(n, x) = nx$.*

(iv) *The mapping $\psi : N_2 \times S \to S$ is uniformly continuous.*

PROOF. As S is compact, the semiring T constructed in Lemma 6 is compact; hence the equivalence of (i), (ii) and (iii). Because $N_3 \times S$ is a compact Hausdorff space, it can be regarded as a uniform space (see Theorem

1, page 225 of [1]). Then if $N_2 \times S$ is given its relative uniformity as a subset of $N_3 \times S$, the equivalence of (iii) and (iv) follows from Corollary 2 to Theorem 2, page 228 of [1].

Our analysis includes the embedding of rings in rings with unit as a special case.

If S is any topological ring and M is the ring of integers then S can be embedded in the product ring $M \times S$, which is locally compact when S is; the construction is given in [4], page 49, and is similar to that in Theorem 1.

In an additively Γ-compact ring, $\Gamma[+](x)$ is a group and so is identical with $K[+](x)$. (This follows immediately from the Corollary to Theorem 1.1.10 of [5]. Alternatively, $\Gamma[+](x)$ contains an additive idempotent (the additive unit of $K[+](x)$); this idempotent must be the additive unit of the ring and so it follows from [8], Theorem 3.2 that $\Gamma[+](x)$ is a group.) In this case we have the following analogue of Lemma 5.

LEMMA 7. *Let S be an additively Γ-compact subring of a topological semi-ring T with multiplicative left unit α such that $\Gamma[+](\alpha)$ is compact. Then there is a continous extension $\bar{\chi}: N_1 \times S \to S$ of the mapping $\chi: \{nu | n \geqq 1\} \times S \to S$ defined by $\chi(nu, x) = nx$.*

PROOF. Let $\bar{\chi}$ be the restriction to $N_1 \times S$ of the function $\Psi: N_3 \times S \to S$ defined in Lemma 5. Then for each positive integer n and each x in S,

$$\bar{\chi}(nu, x) = \Phi(nu) \cdot x = n\Phi(u) \cdot x = (n\alpha') \cdot x = n(\alpha' x).$$

But

$$\alpha' x = (\alpha + e)x = \alpha x + ex = x + ex,$$

where e is the additive identity of $K[+](\alpha)$. Now

$$ex + ex = (e+e)x = ex$$

and

$$ex \in K[+](\alpha) \cdot x = K[+](\alpha x) = K[+](x)$$

so that ex is the additive identity of $K[+](x)$. However $K[+](x) = \Gamma[+](x)$ because S is a ring and so $x + ex = x$. Thus $\bar{\chi}(nu, x) = nx$ which means that $\bar{\chi}$ is an extension of χ.

Conversely, if S is an additively Γ-compact ring for which $\bar{\chi}$ of Lemma 7 exists, then we can construct an additively Γ-compact ring on $N_1 \times S$ (using $\bar{\chi}$ in place of the Ψ of Lemma 6), and S is topologically isomorphic with $\{\varepsilon\} \times S$ where ε is the additive unit of N_1. Also, if γ is the additive unit of S, (u, γ) is a multiplicative unit for the ring $N_1 \times S$.

Hence we have the following results.

THEOREM 4. *Let S be an additively Γ-compact ring. Then the following are equivalent.*

(i) *S can be embedded in an additively Γ-compact ring with unit.*

(ii) *S can be embedded in an additively Γ-compact semiring with left unit.*

(iii) *There is a continuous extension* $\bar{\chi} : N_1 \times S \to S$ *of the mapping* $\chi : \{nu|n \geqq 1\} \times S \to S$ *defined by* $\chi(nu, x) = nx$.

THEOREM 5. *Let S be a compact ring. Then the following are equivalent.*

(i) *S can be embedded in a compact ring with unit.*

(ii) *S can be embedded in an additively Γ-compact semiring with left unit.*

(iii) *There is a continuous extension* $\bar{\chi} : N_1 \times S \to S$ *of the mapping* $\chi : \{nu|n \geqq 1\} \times S \to S$ *defined by* $\chi(nu, x) = nx$.

(iv) *The mapping* $\chi : \{nu|n \geqq 1\} \times S \to S$ *is uniformly continuous.*

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