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AUTOMORPHISMS OF QUANTUM MATRICES

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Abstract. We study the automorphism group of the algebra $\mathcal{O}_q(M_n)$ of $n \times n$ generic quantum matrices. We provide evidence for our conjecture that this group is generated by the transposition and the subgroup of those automorphisms acting on the canonical generators of $\mathcal{O}_q(M_n)$ by multiplication by scalars. Moreover, we prove this conjecture in the case when $n = 3$.

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1. Introduction. Let $\mathbb{K}$ be a field and q be an element in $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. We assume that q is not a root of unity. The quantisation of the ring of regular functions on $m \times n$ matrices with entries in $\mathbb{K}$ is denoted by $\mathcal{O}_q(M_{m,n})$; this is the $\mathbb{K}$-algebra generated by the $m \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i \leq m$ and $1 \leq \alpha \leq n$, subject to the following relations:

$$\begin{array}{ll} Y_{i,\beta} Y_{i,\alpha} = q^{-1} Y_{i,\alpha} Y_{i,\beta}, & (\alpha < \beta); \\ Y_{j,\alpha} Y_{i,\alpha} = q^{-1} Y_{i,\alpha} Y_{j,\alpha}, & (i < j); \\ Y_{j,\beta} Y_{i,\alpha} = Y_{i,\alpha} Y_{j,\beta}, & (i < j, \alpha > \beta); \\ Y_{j,\beta} Y_{i,\alpha} = Y_{i,\alpha} Y_{j,\beta} - (q - q^{-1}) Y_{i,\beta} Y_{j,\alpha}, & (i < j, \alpha < \beta). \end{array}$$

It is well known that $\mathcal{O}_q(M_{m,n})$ is a Noetherian domain that can be presented as an iterated Ore extension over the base field $\mathbb{K}$ with the indeterminates $Y_{i,\alpha}$ adjoined in lexicographic order. Moreover, as all the defining relations of the algebra are quadratic, $\mathcal{O}_q(M_{m,n})$ is a graded algebra with all the indeterminates $Y_{i,\alpha}$ in degree 1.

This paper is concerned with the symmetries of quantum matrices. More precisely, we are studying the automorphism group of this family of algebras. As usual in the quantum setting, it is to be expected that the automorphism group of $\mathcal{O}_q(M_{m,n})$ is quite small (see for instance [**3**] and references therein).

This paper is dedicated to Kenny Brown and Toby Stafford for their 60th birthdays

In the case of $\mathcal{O}_q(M_{m,n})$, there are two classes of automorphisms that are well known:

1. The set $\mathcal{H}$ of automorphisms acting on the indeterminates $Y_{i,\alpha}$ by multiplication by nonzero scalars; this subgroup of $\mathrm{Aut}(\mathcal{O}_q(M_{m,n}))$ is isomorphic to the torus $(\mathbb{K}^*)^{m+n-1}$ [**3**, Corollary 4.11 and its proof];
2. In the square case, where $m = n$, the transposition τ sending $Y_{i,\alpha}$ to $Y_{\alpha,i}$ is an automorphism that generates a subgroup of order 2 of $\mathrm{Aut}(\mathcal{O}_q(M_n))$.

In the case where $m \neq n$, we proved in [**3**] that $\mathrm{Aut}(\mathcal{O}_q(M_{m,n})) = \mathcal{H}$. Unfortunately, the methods used in that paper are not sufficient to resolve the square case. However, it was proved by Alev and Chamarie [**1**] that $\mathrm{Aut}(\mathcal{O}_q(M_2)) = \mathcal{H} \rtimes \langle\tau\rangle$. In view of these results, it is natural to conjecture the following result.

CONJECTURE 1.1. $\mathrm{Aut}(\mathcal{O}_q(M_n)) = \mathcal{H} \rtimes \langle\tau\rangle$.

The main aim of this paper is to provide evidence for this conjecture, and also to prove it in the case when $n = 3$.

Set $R := \mathcal{O}_q(M_n)$, $G := \mathcal{H} \rtimes \langle\tau\rangle$, and let $\sigma \in \mathrm{Aut}(R)$. In Section 2, we prove that there exists $g \in G$ such that:

$$g \circ \sigma(Y_{i,\alpha}) - Y_{i,\alpha} \text{ is a sum of homogeneous terms of degree } \geq 2. \tag{1}$$

Of course, we conjecture that $g \circ \sigma = \mathrm{id}$. The above result (1) already has interesting consequences. Indeed, it follows from a result of Alev and Chamarie [**1**, Lemme 1.4.2] that such a $g \circ \sigma$ belongs to the subalgebra of $\mathrm{End}_{\mathbb{K}}(R)$ generated by the derivations of R. As the derivations of R were computed in [**4**], we can for instance prove that every normal element of R is fixed by $g \circ \sigma$ (an element u is *normal* in R if $uR = Ru$).

Before going any further, let us mention that the normal elements of R have been described in [**3**]. They are closely related to distinguished elements of R called *quantum minors*. Recall that if $I := \{i_1 < \cdots < i_t\}$, $\Lambda = \{\alpha_1 < \cdots < \alpha_t\} \subseteq \{1, \ldots, n\}$ with $|I| = |\Lambda| = t \neq 0$, then the quantum minor $[I|\Lambda] = [i_1, \ldots, i_t|\alpha_1, \ldots, \alpha_t]$ is defined by

$$[I|\Lambda] = [i_1, \ldots, i_t|\alpha_1, \ldots, \alpha_t] := \sum_{w \in S_t} (-q)^{l(w)} Y_{i_1,\alpha_{w(1)}} Y_{i_2,\alpha_{w(2)}} \cdots Y_{i_t,\alpha_{w(t)}},$$

where l is the usual length function on permutations.

It is well known that the quantum minors b_i with $i \in \{1, \ldots, 2n-1\}$ defined by

$$b_i := \begin{cases} [1, \ldots, i|n-i+1, \ldots, n] & \text{if } 1 \leq i \leq n \\ [i-n+1, \ldots, n|1, \ldots, 2n-i] & \text{otherwise,} \end{cases}$$

are normal in R, so that the main result of Section 2 shows that

$$g \circ \sigma(b_i) = b_i, \text{ for all } i \in \{1, \ldots, 2n-1\}.$$

Note that $\Delta := b_n$ is the so-called *quantum determinant* of R. As we assume that q is not a root of unity, the centre of R is precisely the polynomial algebra in the quantum determinant Δ, and so the previous result shows in particular that every element in the centre of R is left invariant by $g \circ \sigma$.

In Section 2, we use (1) as well as graded arguments in order to prove that when $n = 3$ we indeed have $g \circ \sigma = \mathrm{id}$, so that Conjecture 1.1 is true in this case.

Throughout this paper, we set $[\![a, b]\!] := \{i \in \mathbb{N} \mid a \leq i \leq b\}$ and we assume $n \geq 3$.

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2. The automorphism group of $\mathcal{O}_q(M_n)$: reduction step. In this section, we investigate the group of automorphisms of $R = \mathcal{O}_q(M_n)$. We will be using graded arguments, as well as the induced actions of $\mathrm{Aut}(R)$ on the set of height-one prime ideals, on the centre and on the set of normal elements of R.

In the following, we will use several times the following well-known result concerning normal elements of $R = \mathcal{O}_q(M_n)$.

LEMMA 2.1. *Let u and v be two nonzero normal elements of R such that $\langle u\rangle = \langle v\rangle$. Then there exist $\lambda, \mu \in \mathbb{K}^*$ such that $u = \lambda v$ and $v = \mu u$.*

2.1. Torus automorphisms of $\mathcal{O}_q(M_n)$. Recall from Section 1 that $\mathcal{H}$ denotes the subgroup of those automorphisms of R acting on the indeterminates $Y_{i,\alpha}$ by multiplication by nonzero scalars. The proof of [**3**, Corollary 4.11] shows that $\mathcal{H}$ is isomorphic to the torus $(\mathbb{K}^*)^{2n-1}$. More precisely, for any $h := (a_1, \dots, a_n, b_1, \dots, b_{n-1}) \in (\mathbb{K}^*)^{2n-1}$, define an automorphism σ_h in $\mathcal{H}$ as follows:

$$\sigma_h(Y_{i,\alpha}) = \begin{cases} a_i b_\alpha Y_{i,\alpha} & \text{if } \alpha < n \\ a_i Y_{i,\alpha} & \text{if } \alpha = n. \end{cases}$$

The proof of [**3**, Corollary 4.11] shows that the map $h \mapsto \sigma_h$ from $(\mathbb{K}^*)^{2n-1}$ to $\mathcal{H}$ is an isomorphism. The elements of $\mathcal{H}$, that is the automorphisms σ_h with $h \in (\mathbb{K}^*)^{2n-1}$, are called the *torus automorphisms* to R.

2.2. Height-one prime ideals of $\mathcal{O}_q(M_n)$. In [**3**, Propositions 3.5 and 3.6], we have described the height-one primes of R. We now recall the results that we have obtained.

PROPOSITION 2.2. *For any height-one prime ideal P of $\mathcal{O}_q(M_n)$, there exists an irreducible polynomial $V = \sum_{i_1=0}^{r_1} \cdots \sum_{i_n=0}^{r_n} a_{i_1,\dots,i_n} X_1^{i_1}, \dots, X_n^{i_n} \in \mathbb{K}[X_1, \dots, X_n]$ (where $r_i = \deg_{X_i} V$ for all $i \in \{1, \dots, n\}$) such that $P = \langle u\rangle$, where*

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_n=0}^{r_n} a_{i_1,\dots,i_n} \prod_{j=1}^{n} b_j^{i_j} b_{n+j}^{r_j - i_j}.$$

(By convention, we set $b_{2n} := 1$.)
Moreover, u is normal in R.

2.3. q-commutation, gradings and automorphisms. Recall that the relations that define $R = \mathcal{O}_q(M_n)$ are all quadratic, so that $R = \oplus_{i\in\mathbb{N}} R_i$ is a $\mathbb{N}$-graded algebra, the canonical generators $Y_{i,\alpha}$ of R having degree 1. Note, for later use, that a $t \times t$ quantum minor of R is a homogeneous element of degree t with respect to this grading of R. In the following, R will always be endowed with this grading.

In [**3**, Corollary 4.3], we have shown the following result.

PROPOSITION 2.3. *Let σ be an automorphism of $R = \mathcal{O}_q(M_n)$ and x be an homogeneous element of degree d of R. Then, $\sigma(x) = y_d + y_{>d}$, where $y_d \in R_d \setminus \{0\}$ and $y_{>d} \in R_{>d}$.*

Note that the torus automorphisms of R preserve degrees. We finish this section by recording the following result for later use.

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LEMMA 2.4. *Let* $\sigma \in \mathrm{Aut}(R)$ *such that there exist nonzero scalars* $\lambda_{i,\alpha}$ *with*

$$\sigma(Y_{i,\alpha}) - \lambda_{i,\alpha} Y_{i,\alpha} \in R_{\geq 2} \text{ for all } (i,\alpha).$$

Then there exists a torus automorphism $\sigma_h \in \mathcal{H}$ *such that*

$$\sigma_h \circ \sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2} \text{ for all } (i,\alpha).$$

Proof. Assume $i < j$ and $\alpha < \beta$. Applying σ to the relation $Y_{j,\beta} Y_{i,\alpha} = Y_{i,\alpha} Y_{j,\beta} - (q - q^{-1}) Y_{i,\beta} Y_{j,\alpha}$, and then identifying the degree 2 components, yields

$$\lambda_{i,\alpha}\lambda_{j,\beta} = \lambda_{i,\beta}\lambda_{j,\alpha}$$

for all $i < j$ and $\alpha < \beta$. Hence, the matrix $(\lambda_{i,\alpha})$ has rank one, so that there exist $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}, b_n = 1 \in \mathbb{K}^*$ such that

$$\lambda_{i,\alpha} = a_i b_\alpha$$

for all (i,α). Set $h = (a_1^{-1}, \ldots, a_n^{-1}, b_1^{-1}, \ldots, b_{n-1}^{-1}) \in (\mathbb{K}^*)^{2n-1}$. Then one easily checks that the automorphism $\sigma_h \in \mathcal{H}$ has the property that $\sigma_h \circ \sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}$ for all (i,α). □

2.4. Automorphism group of $\mathcal{O}_q(M_n)$: action on the centre. Recall that the centre of $R = \mathcal{O}_q(M_n)$ is the polynomial ring $\mathbb{K}[\Delta]$, where Δ denotes the quantum determinant of R. We now apply the results of the previous section to $R = \mathcal{O}_q(M_n)$ to prove that the quantum determinant Δ of R is an eigenvector of every automorphism of R.

PROPOSITION 2.5. *Let* σ *be an automorphism of* R*. Then there exists* $\mu \in \mathbb{K}^*$ *such that* $\sigma(\Delta) = \mu\Delta$.

Proof. Since σ is an automorphism of R, it induces an automorphism of the centre $\mathbb{K}[\Delta]$ of R. Hence, there exist $\mu \in \mathbb{K}^*$ and $\lambda \in \mathbb{K}$ such that $\sigma(\Delta) = \mu\Delta + \lambda$. Moreover, Δ is an homogeneous element of degree n of $R = \mathcal{O}_q(M_n)$. Hence, Proposition 2.3 shows that we must have $\sigma(\Delta) \in R_{\geq n}$. Naturally, this forces λ to be zero. □

2.5. Automorphism group of $\mathcal{O}_q(M_n)$: action on the normal element $b_1 = Y_{1,n}$.

LEMMA 2.6. *Let* $\sigma \in \mathrm{Aut}(R)$*. Then there exist* $\epsilon \in \{0, 1\}$*,* $P, Q, P', Q' \in \mathbb{K}[X]$ *such that*

$$\tau^\epsilon \circ \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1} \text{ and } \sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) = P'(\Delta)b_1 + Q'(\Delta)b_{n+1}.$$

Proof. As $\langle b_1 \rangle = \langle Y_{1,n} \rangle$ is a height-one prime ideal of R, the ideal $\langle \sigma(b_1) \rangle$ must also be a height-one prime of R. It follows from Proposition 2.2 that $\langle \sigma(Y_{1,n}) \rangle = \langle u \rangle$, where

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_n=0}^{r_n} a_{i_1,\ldots,i_n} \prod_{j=1}^{n} b_j^{i_j} b_{n+j}^{r_j - i_j}$$

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is normal in R. Hence, we deduce from Lemma 2.1 that

$$\sigma(Y_{1,n}) = \lambda u = \sum_{i_1=0}^{r_1} \cdots \sum_{i_n=0}^{r_n} a'_{i_1,\dots,i_n} \prod_{j=1}^{n} b_j^{i_j} b_{n+j}^{r_j-i_j},$$

where $\lambda \in \mathbb{K}^*$ and $a'_{i_1,\dots,i_n} := \lambda a_{i_1,\dots,i_n}$.

On the other hand, it follows from Proposition 2.3 that $\sigma(Y_{1,n}) = u_1 + u_{\geq 2}$, with $u_1 \in R_1 \setminus \{0\}$ and $u_{\geq 2} \in R_{\geq 2}$. Since b_i is homogeneous of degree i if $i \leq n$, and $2n - i$ if $i \geq n$, comparing the two expressions of $\sigma(Y_{1,n})$ that we have obtained leads to

$$\text{either } \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1} \text{ or } \sigma(Y_{1,n}) = P(\Delta)b_{n-1} + Q(\Delta)b_{2n-1}.$$

Now, the existence of ϵ such that $\tau^\epsilon \circ \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1}$ easily follows from the fact that $\tau(\Delta) = \Delta$, and $\tau(b_i) = b_{2n-i}$ for all i.

Note that the previous reasoning also applies to $\sigma^{-1} \circ \tau^\epsilon$, so that $\sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) = P'(\Delta)b_1 + Q'(\Delta)b_{n+1}$ or $\sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) = P'(\Delta)b_{n-1} + Q'(\Delta)b_{2n-1}$. Recall, from Proposition 2.5, that there exists $\mu \in \mathbb{K}^*$ such that $\sigma^{-1} \circ \tau^\epsilon(\Delta) = \mu\Delta$, so that applying $\sigma^{-1} \circ \tau^\epsilon$ to $\tau^\epsilon \circ \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1}$ leads to

$$Y_{1,n} = P(\mu\Delta)\sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) + Q(\mu\Delta)\sigma^{-1} \circ \tau^\epsilon(b_{n+1}).$$

Comparing the degree 1 part of each side using Proposition 2.3, this easily implies that the case $\sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) = P'(\Delta)b_{n-1} + Q'(\Delta)b_{2n-1}$ is impossible, so that $\sigma^{-1} \circ \tau^\epsilon(Y_{1,n}) = P'(\Delta)b_1 + Q'(\Delta)b_{n+1}$, as desired. □

2.6. Automorphism group of $\mathcal{O}_q(M_n)$: reduction step. In view of Lemma 2.6, it is natural to introduce

$$G' := \{\sigma \in \mathrm{Aut}(R) \mid \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1}\}.$$

Note that the proof of the previous lemma shows that G' is invariant under taking inverses.

LEMMA 2.7. *Set $J_r := Y_{1,1}R + Y_{1,2}R + \cdots + Y_{1,n-1}R$ and $J_c := Y_{2,n}R + Y_{3,n}R + \cdots + Y_{n,n}R$. If $\sigma \in G'$, then $\sigma(J_r) = J_r$ and $\sigma(J_c) = J_c$.*

Proof. The proof is given for the case $J := J_r$; the proof for J_c is similar.

Let $\beta \in [\![1, n-1]\!]$ and write $\sigma(Y_{1,\beta})$ in the PBW basis of R:

$$\sigma(Y_{1,\beta}) = \sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}},$$

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where Γ is a finite subset of $\mathbb{N}^{n^2}$ and each $c_{\underline{\gamma}} \neq 0$. Recall that $Y_{1,n}Y_{1,\beta} = q^{-1}Y_{1,\beta}Y_{1,n}$. Hence, applying σ to this equality leads to

$$(P(\Delta)Y_{1,n} + Q(\Delta)[2,\dots,n \mid 1,\dots,n-1])\left(\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right)$$
$$= q^{-1}\left(\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right)(P(\Delta)Y_{1,n} + Q(\Delta)[2,\dots,n \mid 1,\dots,n-1]).$$

Now, since Δ is central in R, and $[2,\dots,n \mid 1,\dots,n-1]Y_{1,n}^{-1} = b_{n+1}b_1^{-1}$ is central in the field of fractions of R, see [**3**, Theorem 3.4], we obtain

$$Y_{1,n}\left(\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right) = q^{-1}\left(\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right) Y_{1,n};$$

that is,

$$\left(\sum_{\underline{\gamma}\in\Gamma} q^{-\gamma_{1,1}-\dots-\gamma_{1,n-1}+\gamma_{2,n}+\dots+\gamma_{n,n}} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right) Y_{1,n}$$
$$= q^{-1}\left(\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}\right) Y_{1,n}.$$

As R is a domain, this implies that

$$\sum_{\underline{\gamma}\in\Gamma} q^{-\gamma_{1,1}-\dots-\gamma_{1,n-1}+\gamma_{2,n}+\dots+\gamma_{n,n}} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}} = q^{-1}\sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{n,n}^{\gamma_{n,n}}.$$

Identifying these two expressions in the PBW basis, and then using the fact that q is not a root of unity leads to

$$-\gamma_{1,1} - \dots - \gamma_{1,n-1} + \gamma_{2,n} + \dots + \gamma_{n,n} = -1$$

for all $\underline{\gamma} \in \Gamma$. In particular, for all $\underline{\gamma} \in \Gamma$, there exists $\beta_0 \in \{1,\dots,n-1\}$ such that $\gamma_{1,\beta_0} \geq 1$. Hence, $\sigma(Y_{1,\beta})$ belongs to J, and so $\sigma(J) \subseteq J$.

One can also apply this argument to σ^{-1}, so that we also have $\sigma^{-1}(J) \subseteq J$. From these two inclusions, we conclude that $\sigma(J) = J$. □

Corollary 2.8. *Set* $K_r := \langle Y_{1,1}, Y_{1,2}, \dots, Y_{1,n}\rangle = Y_{1,1}R + Y_{1,2}R + \dots + Y_{1,n}R$ *and* $K_c := \langle Y_{1,n}, Y_{2,n}, \dots, Y_{n,n}\rangle = Y_{1,n}R + Y_{2,n}R + \dots + Y_{n,n}R$. *If* $\sigma \in G'$, *then* $\sigma(K_r) = K_r$ *and* $\sigma(K_c) = K_c$.

Proof. Again, we only consider the case of $K = K_r$.

As $J = J_r \subset K$, Lemma 2.7 shows that $J \subset \sigma(K)$.

On the other hand, K is a height n prime ideal of R, so that $\sigma(K)$ is also a height n prime ideal. Moreover, since $J \subset \sigma(K)$, $Y_{1,1}, Y_{1,2}, \dots, Y_{1,n-1}$ belong to $\sigma(K)$. Now, $(q - q^{-1})Y_{1,n}Y_{i,1} = Y_{1,1}Y_{i,n} - Y_{i,n}Y_{1,1} \in \sigma(K)$ for all $i \in [\![2,n]\!]$. As $\sigma(K)$ is (completely) prime, this leads to: either $Y_{1,n} \in \sigma(K)$ or $Y_{i,1} \in \sigma(K)$, for all $i \in [\![2,n]\!]$.

We claim that the second possibility cannot happen. If it did, then $\sigma(K)$ would strictly contain the ideal generated by $Y_{i,1}$, for $i \in [\![1, n]\!]$. However, this ideal is prime and has height n, the same height as $\sigma(K)$. This is impossible.

Hence, $Y_{1,n} \in \sigma(K)$. As we already know that $Y_{1,1}$, $Y_{1,2}$, ..., $Y_{1,n-1}$ belong to $\sigma(K)$, we obtain that $K \subseteq \sigma(K)$. Now these two ideals are prime and each has height n, so that they are equal; that is, $\sigma(K) = K$. □

PROPOSITION 2.9. *Let G be the subgroup of* Aut(R) *generated by τ and the torus automorphisms. Let $\sigma \in$* Aut(R)*. Then there exists $g \in G$ such that, for all $(i, \alpha) \in [\![1, n]\!]^2$, we have*

$$g \circ \sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}.$$

Proof. In view of Lemma 2.4, it is enough to prove that there exist $g \in G$ and nonzero scalars $\lambda_{i,\alpha}$ with

$$g \circ \sigma(Y_{i,\alpha}) - \lambda_{i,\alpha} Y_{i,\alpha} \in R_{\geq 2} \text{ for all } (i, \alpha).$$

First, it follows from Lemma 2.6 that there exist $g' \in G$, and $P, Q \in \mathbb{K}[X]$ such that

$$g' \circ \sigma(Y_{1,n}) = P(\Delta)b_1 + Q(\Delta)b_{n+1} = P(\Delta)Y_{1,n} + Q(\Delta)[2, \ldots, n \mid 1, \ldots, n-1].$$

Hence, it is enough to prove Proposition 2.9 when σ is an automorphism of R such that

$$\sigma(Y_{1,n}) = P(\Delta)Y_{1,n} + Q(\Delta)[2, \ldots, n \mid 1, \ldots, n-1];$$

that is, when $\sigma \in G'$.

So, let $\sigma \in G'$. It follows from Corollary 2.8 that $\sigma(K_r) = K_r$. Hence, σ induces an automorphism of R/K_r an algebra that is isomorphic to $\mathcal{O}_q(M_{n-1,n})$ via an isomorphism that sends $Y_{i,\alpha} + K_r$ to $y_{i-1,\alpha}$, where $y_{i,\alpha}$ denote the canonical generators of $\mathcal{O}_q(M_{n-1,n})$. Hence, it follows from [**3**] that there exist $\lambda_{i,\alpha} \in \mathbb{K}^*$ such that

$$\sigma(Y_{i,\alpha}) - \lambda_{i,\alpha} Y_{i,\alpha} \in K_r,$$

for all $(i, \alpha) \in [\![2, n]\!] \times [\![1, n]\!]$.

Let $(i, \alpha) \in [\![2, n]\!] \times [\![1, n]\!]$. Then there exist $\mu_1, \ldots, \mu_n \in \mathbb{K}$ and $u_{\geq 2} \in R_{\geq 2}$ such that

$$\sigma(Y_{i,\alpha}) = \lambda_{i,\alpha} Y_{i,\alpha} + \mu_1 Y_{1,1} + \cdots + \mu_n Y_{1,n} + u_{\geq 2}. \tag{2}$$

Similarly, using the fact that $\sigma(K_c) = K_c$, we obtain that for all $(i, \alpha) \in [\![1, n]\!] \times [\![1, n-1]\!]$, there exist $\lambda'_{i,\alpha} \in \mathbb{K}^*$, $\mu'_1, \ldots, \mu'_n \in \mathbb{K}$ and $u'_{\geq 2} \in R_{\geq 2}$ such that

$$\sigma(Y_{i,\alpha}) = \lambda'_{i,\alpha} Y_{i,\alpha} + \mu'_1 Y_{1,n} + \cdots + \mu'_n Y_{n,n} + u'_{\geq 2}. \tag{3}$$

Comparing equations (2) and (3), we obtain that for all $(i, \alpha) \in [\![2, n]\!] \times [\![1, n-1]\!]$, there exist $\lambda_{i,\alpha} \in \mathbb{K}^*$, $\mu_{i,\alpha} \in \mathbb{K}$ and $v_{\geq 2} \in R_{\geq 2}$ such that

$$\sigma(Y_{i,\alpha}) = \lambda_{i,\alpha} Y_{i,\alpha} + \mu_{i,\alpha} Y_{1,n} + v_{\geq 2}. \tag{4}$$

Now, assume that $(i, \alpha) \in [\![2, n]\!] \times [\![1, n-2]\!]$. Applying σ to $Y_{i,\alpha} Y_{i,\alpha+1} = q Y_{i,\alpha+1} Y_{i,\alpha}$, and identifying the degree 2 terms, leads to

$$\begin{aligned}&(\lambda_{i,\alpha} Y_{i,\alpha} + \mu_{i,\alpha} Y_{1,n})(\lambda_{i,\alpha+1} Y_{i,\alpha+1} + \mu_{i,\alpha+1} Y_{1,n})\\ &\quad = q(\lambda_{i,\alpha+1} Y_{i,\alpha+1} + \mu_{i,\alpha+1} Y_{1,n})(\lambda_{i,\alpha} Y_{i,\alpha} + \mu_{i,\alpha} Y_{1,n})\end{aligned}$$

thanks to (4). Using the commutation relations in R, we get

$$(1-q)\lambda_{i,\alpha}\mu_{i,\alpha+1} Y_{i,\alpha} Y_{1,n} + (1-q)\lambda_{i,\alpha+1}\mu_{i,\alpha} Y_{i,\alpha+1} Y_{1,n} + (1-q)\mu_{i,\alpha}\mu_{i,\alpha+1} Y_{1,n}^2 = 0.$$

As $q - 1 \neq 0$ and $\lambda_{i,\alpha}\lambda_{i,\alpha+1} \neq 0$, this forces $\mu_{i,\alpha} = 0$ and $\mu_{i,\alpha+1} = 0$. Hence, we have just proved that for all $(i, \alpha) \in [\![2, n]\!] \times [\![1, n-1]\!]$, there exist $\lambda_{i,\alpha} \in \mathbb{K}^*$, and $v_{\geq 2} \in R_{\geq 2}$ such that

$$\sigma(Y_{i,\alpha}) = \lambda_{i,\alpha} Y_{i,\alpha} + v_{\geq 2},$$

as required.

Now let $i \in [\![2, n]\!]$. As $Y_{i,n} Y_{1,n} = q^{-1} Y_{1,n} Y_{i,n}$, we must have

$$\sigma(Y_{i,n})\sigma(Y_{1,n}) = q^{-1}\sigma(Y_{1,n})\sigma(Y_{i,n});$$

that is,

$$\begin{aligned}&(\lambda_{i,n} Y_{i,n} + \mu_1 Y_{1,1} + \cdots + \mu_n Y_{1,n} + u_{\geq 2})(P(\Delta)b_1 + Q(\Delta)b_{n+1}) =\\ &\qquad q^{-1}(P(\Delta)b_1 + Q(\Delta)b_{n+1})(\lambda_{i,n} Y_{i,n} + \mu_1 Y_{1,1} + \cdots + \mu_n Y_{1,n} + u_{\geq 2}).\end{aligned}$$

As Δ and $b_{n+1}b_1^{-1}$ are central in the field of fractions of R, we obtain

$$\begin{aligned}&(\lambda_{i,n} Y_{i,n} + \mu_1 Y_{1,1} + \cdots + \mu_n Y_{1,n} + u_{\geq 2})b_1\\ &\quad = q^{-1} b_1(\lambda_{i,n} Y_{i,n} + \mu_1 Y_{1,1} + \cdots + \mu_n Y_{1,n} + u_{\geq 2}).\end{aligned}$$

One can easily check that this forces $\mu_1 = \cdots = \mu_n = 0$.

Hence, for all $i \in [\![2, n]\!]$, there exist $\lambda_{i,n} \in \mathbb{K}^*$ such that

$$\sigma(Y_{i,n}) - \lambda_{i,n} Y_{i,n} \in R_{\geq 2}.$$

Similarly, for all $\alpha \in [\![1, n-1]\!]$, there exist $\lambda_{1,\alpha} \in \mathbb{K}^*$ such that

$$\sigma(Y_{1,\alpha}) - \lambda_{1,\alpha} Y_{1,\alpha} \in R_{\geq 2}.$$

To conclude, it just remains to prove that there exists $\lambda_{1,n} \in \mathbb{K}^*$ such that $\sigma(Y_{1,n}) - \lambda_{1,n} Y_{1,n} \in R_{\geq 2}$. This follows easily from Lemma 2.3 and the fact that $\sigma \in G'$. $\square$

2.7. Summary. Recall that we conjecture that Aut(R) is the semidirect product of $\mathcal{H}$ and the subgroup of the order of two generated by the transposition τ. We set $G = \mathcal{H} \rtimes \langle \tau \rangle$. The previous result shows that for all $\sigma \in \mathrm{Aut}(R)$, there exists $g \in G$ such that

$$g \circ \sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}$$

for all $(i, \alpha) \in [\![1, n]\!]^2$.

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So to prove Conjecture 1.1 it is enough to prove that the only automorphism σ of R such that

$$\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}, \tag{5}$$

for all $(i, \alpha) \in [\![1, n]\!]^2$, is the identity automorphism.

Automorphisms satisfying the above property (5) are closely related to derivations of R. Indeed, let $D(R)$ denote the subalgebra of $\mathrm{End}_{\mathbb{K}}(R)$ generated by the $\mathbb{K}$-linear derivations of R. Alev and Chamarie proved [**1**, Lemme 1.4.1] that there exists a family $(d_l)_{l>0}$ of elements of $D(R)$ such that for any element $x \in R_i$ we have

$$\sigma(x) = x + \sum_{l>0} d_l(x) \tag{6}$$

with $d_l(x)$ homogeneous of degree $l + i$. In [**4**], we computed the derivations of the algebra R. Interestingly, it easily follows from [**4**, Theorem 2.9] that $d(b_i) \in \langle b_i \rangle$, for each derivation d of R. Hence, the same is true for any element of $D(R)$, and so we deduce the following result from the above discussion.

Proposition 2.10. *Let $\sigma \in \mathrm{Aut}(R)$ such that $\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}$, for all $(i, \alpha) \in [\![1, n]\!]^2$. Then $\sigma(b_i) = b_i$ for all $i \in \{1, \ldots, 2n-1\}$.*

Proof. The above discussion shows that $d_l(b_i) \in \langle b_i \rangle$ for all $l > 0$. Hence, we deduce from (6) that $\sigma(b_i) \in \langle b_i \rangle$. Consequently, $\sigma(b_i) = \lambda_i b_i$ with $\lambda_i \in \mathbb{K}^*$, by Lemma 2.1. On the other hand,

$$\sigma(b_i) = b_i + \sum_{l>0} d_l(b_i),$$

with $d_l(b_i)$ homogeneous of degree $l + \deg(b_i)$. Comparing the components with degree equal to the degree of b_i, we obtain $\lambda_i = 1$, so that $\sigma(b_i) = b_i$, as desired. □

3. Automorphisms of 3×3 quantum matrices. In this section, R denotes the algebra of 3×3 quantum matrices. We prove our conjecture in the case when $n = 3$. As explained in the previous section, all we need to do is to prove that the only automorphism $\sigma \in \mathrm{Aut}(R)$ such that

$$\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2},$$

for all $(i, \alpha) \in [\![1, 3]\!]^2$, is the identity automorphism. Observe that for such an automorphism, $\sigma(Y_{i,\alpha}) = Y_{i,\alpha}$ if and only if $\deg(\sigma(Y_{i,\alpha})) = 1$.

Lemma 3.1. *Let $[I|\Lambda]$ be a $t \times t$ quantum minor and suppose that σ is an automorphism such that $\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}$, for all $(i, \alpha) \in [\![1, 3]\!]^2$. Then $\sigma([I|\Lambda]) - [I|\Lambda] \in R_{\geq t+1}$. As a consequence, $\sigma([I|\Lambda]) = [I|\Lambda]$ if and only if $\deg(\sigma([I|\Lambda])) = t$.*

Proof. Easy, by induction, with $t = 1$ being given by the observation immediately preceding the statement of this lemma. □

Let $\sigma \in \mathrm{Aut}(R)$ be such that

$$\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2},$$

for all $(i, \alpha) \in [\![1, 3]\!]^2$.

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Set $d_{i,\alpha} := \deg(\sigma(Y_{i,\alpha}))$, for all $(i,\alpha) \in [\![1,3]\!]^2$. Our aim is to prove that $d_{i,\alpha} = 1$ for all (i,α); so that σ is then the identity automorphism. We note first that $d_{1,3} = d_{3,1} = 1$ by Proposition 2.10.

In the following lemma, we will use several times the anti-endomorphism $\Gamma : \mathcal{O}_q(M_n) \to \mathcal{O}_q(M_n)$ defined on generators by $\Gamma(Y_{i,\alpha}) = (-q)^{i-\alpha}[\widetilde{\alpha}|\widetilde{\imath}]$, see [**5**, Corollary 5.2.2]. Here, if $I \subseteq \{1,\dots,n\}$, then $\widetilde{I} := \{1,\dots,n\} \setminus I$, and $\widetilde{\imath} := \widetilde{\{i\}}$ for any $i \in \{1,\dots,n\}$. The effect of Γ on 2×2 quantum minors is given by $\Gamma([I|\Lambda]) = (-q)^{I-\Lambda}[\widetilde{\Lambda}|\widetilde{I}]\Delta$, see [**2**, Lemma 4.1], where the superscript $I-\Lambda$ denotes the difference between the sum of the entries of I and the sum of the entries of Λ.

LEMMA 3.2. *Let* $\sigma \in \mathrm{Aut}(R)$ *be such that*

$$\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2},$$

for all $(i,\alpha) \in [\![1,3]\!]^2$. *Then* $d_{1,1} = d_{3,3} = 1$.

Proof. Assume to the contrary that $d_{1,1} + d_{3,3} > 2$.

Recall from Proposition 2.10 that $b_2 = \sigma(b_2) = \sigma(Y_{1,2})\sigma(Y_{2,3}) - q\sigma(Y_{1,3})\sigma(Y_{2,2})$, so that

$$b_2 = \sigma(Y_{1,2})\sigma(Y_{2,3}) - qY_{1,3}\sigma(Y_{2,2}).$$

Hence, comparing the degrees on both sides, we obtain

$$d_{1,2} + d_{2,3} = 1 + d_{2,2}.$$

Similarly, by using b_4, we obtain

$$d_{2,1} + d_{3,2} = 1 + d_{2,2}.$$

Suppose that $d_{1,1} + d_{2,2} \leq d_{1,2} + d_{2,1}$ and that $d_{2,2} + d_{3,3} \leq d_{2,3} + d_{3,2}$. Then $d_{1,1} + 2d_{2,2} + d_{3,3} \leq d_{1,2} + d_{2,1} + d_{2,3} + d_{3,2} = 2 + 2d_{2,2}$, by using the above two equations. It follows that $d_{1,1} = d_{3,3} = 1$, a contradiction to the initial assumption.

So either $d_{1,1} + d_{2,2} > d_{1,2} + d_{2,1}$ or $d_{2,2} + d_{3,3} > d_{2,3} + d_{3,2}$. By symmetry, we can assume that $d_{1,1} + d_{2,2} > d_{1,2} + d_{2,1}$. In this case, we easily get that $\deg(\sigma([1,2|1,2])) = d_{1,1} + d_{2,2}$.

Applying Γ to $[1,3|1,3] = Y_{1,1}Y_{3,3} - qY_{1,3}Y_{3,1}$ gives the relation $Y_{2,2}[1,2,3|1,2,3] = [1,2|1,2][2,3|2,3] - q[2,3|1,2][1,2|2,3]$. Thus

$$\sigma(Y_{2,2})\Delta = \sigma([1,2|1,2])\sigma([2,3|2,3]) - q[2,3|1,2][1,2|2,3].$$

Comparing degrees, we obtain

$$d_{2,2} + 3 = d_{1,1} + d_{2,2} + e,$$

where $e := \deg(\sigma([2,3|2,3])) \geq 2$. This forces $d_{1,1} = 1$ and $e = 2$, so that $\sigma(Y_{1,1}) = Y_{1,1}$ and $\sigma([2,3|2,3]) = [2,3|2,3]$.

Applying σ to the quantum Laplace expansion $\Delta = Y_{1,1}[2,3|2,3] - qY_{1,2}[2,3|1,3] + q^2Y_{1,3}[2,3|1,2]$, we obtain

$$\Delta = Y_{1,1}[2,3|2,3] - q\sigma(Y_{1,2})\sigma([2,3|1,3]) + q^2Y_{1,3}[2,3|1,2].$$

Hence, $\sigma(Y_{1,2})\sigma([2,3|1,3]) = Y_{1,2}[2,3|1,3]$. Thus, $\sigma(Y_{1,2}) = Y_{1,2}$ and $\sigma([2,3|1,3]) = [2,3|1,3]$. Similarly, we obtain $\sigma(Y_{2,1}) = Y_{2,1}$ and $\sigma([1,3|2,3]) = [1,3|2,3]$.

So σ acts as identity on the following elements of R: $Y_{3,1}$, $Y_{2,1}$, $Y_{1,1}$, $Y_{1,2}$, $Y_{1,3}$, $[1,2|2,3]$, $[1,3|2,3]$, $[2,3|2,3]$, $[2,3|1,3]$ and $[2,3|1,2]$.

Applying Γ to $[1,3|1,2] = Y_{1,1}Y_{3,2} - qY_{1,2}Y_{3,1}$ produces

$$\begin{aligned} Y_{3,2}\Delta &= [1,3|1,2][2,3|2,3] - q[2,3|1,2][1,3|2,3] \\ &= \{Y_{1,1}Y_{3,2} - qY_{1,2}Y_{3,1}\}[2,3|2,3] - q[2,3|1,2][1,3|2,3] \end{aligned}$$

which can be re-arranged to give

$$\left\{\Delta - Y_{1,1}[2,3|2,3]\right\} Y_{3,2} = -q\left\{Y_{1,2}Y_{3,1}[2,3|2,3] + [2,3|1,2][1,3|2,3]\right\}.$$

In this equation, all terms except $Y_{3,2}$ are already known to be fixed by σ; so $\sigma(Y_{3,2}) = Y_{3,2}$ also.

Finally, all terms in $[2,3|1,2] = Y_{2,1}Y_{3,2} - qY_{2,2}Y_{3,1}$ except $Y_{2,2}$ are now known to be fixed by σ; so $\sigma(Y_{2,2}) = Y_{2,2}$ and $d_{2,2} = 1$. As we have already shown that $d_{1,1} = 1$, we obtain $d_{1,1} + d_{2,2} = 2 = d_{1,2} + d_{2,1}$, a contradiction! □

PROPOSITION 3.3. *Let $\sigma \in \mathrm{Aut}(R)$ be such that $\sigma(Y_{i,\alpha}) - Y_{i,\alpha} \in R_{\geq 2}$, for all $(i,\alpha) \in [\![1,3]\!]^2$. Then $\sigma(Y_{i,\alpha}) = Y_{i,\alpha}$ for all $i,\alpha \in \{1,2,3\}$.*

Proof. It is enough to prove that $d_{i,\alpha} = 1$ for all $i,\alpha \in \{1,2,3\}$.

We already know from Proposition 2.10 and Lemma 3.2 that σ leaves invariant the following quantum minors:

$$Y_{3,1},\ Y_{1,1},\ Y_{1,3},\ Y_{3,3},\ [1,2|2,3],\ [1,3|1,3],\ [2,3|1,2],\ [1,2,3|1,2,3].$$

One can easily check that

$$[1,2|1,3][1,3|2,3] = Y_{1,3}[1,2,3|1,2,3] + q[1,3|1,3][1,2|2,3],$$

by applying Γ to the formula for $[1,2|2,3]$ and re-arranging. As all the minors on the right-hand side are left invariant by σ, this implies

$$\sigma([1,2|1,3][1,3|2,3]) = [1,2|1,3][1,3|2,3].$$

As usual, it follows that $\sigma([1,2|1,3]) = [1,2|1,3]$ and $\sigma([1,3|2,3]) = [1,3|2,3]$.

Similarly, one obtains $\sigma([1,3|1,2]) = [1,3|1,2]$ and $\sigma([2,3|1,3]) = [2,3|1,3]$.

By a quantum Laplace expansion, we have:

$$[1,3|1,3]Y_{2,1} = q[2,3|1,3]Y_{1,1} + q^{-1}[1,2|1,3]Y_{3,1}.$$

As all of the minors on the right-hand side are left invariant by σ, this implies

$$\sigma(Y_{2,1}[1,3|1,3]) = Y_{2,1}[1,3|1,3].$$

As usual, this implies that $\sigma(Y_{2,1}) = Y_{2,1}$ (and $\sigma([1,3|1,3]) = [1,3|1,3]$).

Similarly, one can prove that $\sigma(Y_{1,2}) = Y_{1,2}$, $\sigma(Y_{3,2}) = Y_{3,2}$ and $\sigma(Y_{2,3}) = Y_{2,3}$.

It just remains to prove that $\sigma(Y_{2,2}) = Y_{2,2}$. This easily follows from the facts that $[1,2|2,3] = Y_{1,2}Y_{2,3} - qY_{2,2}Y_{1,3}$ and that σ leaves invariant all these quantum minors except maybe $Y_{2,2}$. □

From this proposition and Proposition 2.9, we deduce our main theorem:

THEOREM 3.4. *The automorphism group of the algebra of* 3×3 *quantum matrices is the semidirect product of the torus automorphisms and the cyclic group of order* 2 *given by the transpose automorphism.*

After this paper was completed, Conjecture 1.1 was proved in [**6**].

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