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ON CONVEX FUNCTIONS HAVING POINTS OF GATEAUX DIFFERENTIABILITY WHICH ARE NOT POINTS OF FRÉCHET DIFFERENTIABILITY

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ABSTRACT We study the relationships between Gateaux, Fréchet and weak Hadamard differentiability of convex functions and of equivalent norms As a consequence we provide related characterizations of infinite dimensional Banach spaces and of Banach spaces containing l_1 Explicit examples are given Some renormings of WCG Asplund spaces are made in this vein

0. **Introduction.** Let us consider the supremum norm $\|\cdot\|$ on c_0. We may observe that if $\|\cdot\|$ is Gateaux differentiable at some point, then it is Fréchet differentiable there. Indeed, this is so because the dual norm is weak* Kadec, which means that the weak* and the norm topologies coincide on the dual unit sphere. Similarly, once Talagrand [T] constructs a Gateaux smooth norm on $C([0,\Omega])$, then this norm is automatically Fréchet smooth. But in this case it is not caused by the weak* Kadecness of the corresponding dual norm. In fact, as in the proof of [T, Théorème 3] it can be shown that $C([0,\Omega])^*$ admits no equivalent dual weak* Kadec norm. An example of Haydon [H] shows that there exists an Asplund space admitting not only no Fréchet but even no Gateaux smooth norm. All these situations show that the phenomenon of Fréchet differentiability is in some cases close to that of Gateaux differentiability. On the other hand Phelps [Ph, p. 80] constructed a norm on l_1 which is everywhere Gateaux (except at the origin) but is nowhere Fréchet differentiable, see also [DGZ, Proposition III.4.5]. (For a classification of separable spaces that admit such norms see [DGZ, Theorem III.1.9].) Less drastically, if a Banach space is weak Asplund, or even a Gateaux differentiability space [Ph, p. 90] and we know that it is not Asplund, then there exists a convex continuous function, even an equivalent norm, having points of Gateaux but not Fréchet smoothness. A real valued function f on a Banach space will be called a PGNF-*function* if there exists a point at which f is Gateaux but not Fréchet differentiable. This point will be called a *special point* for f.

Our note is devoted mainly to the question: *Does there exist on every Banach space a convex continuous* PGNF-*function?* In Section 1 we show that the occurrence of a

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convex PGNF-function is equivalent, among other facts, to the conclusion of Josefson-Nissenzweig theorem, that is, that there exists a sequence $\{\xi_n\}$ in the dual such that $\xi_n \longrightarrow 0$ weakly* but $\inf_n \|\xi_n\| > 0$ [D2, Chapter XII]. Thus our question has a positive answer in every infinite dimensional Banach space. In what follows such a sequence $\{\xi_n\}$ will be called a JN-*sequence*. As a byproduct we also obtain a characterization of Banach spaces containing l_1 via a variant of the PGNF-function for an intermediate notion of weak Hadamard differentiability. In Section 2 we try to construct a PGNF-norm otherwise as smooth as the space in question is. In particular, we are able to do so in WCG Asplund spaces.

For notation and concepts not introduced in the text we refer to the books [D1], [Ph2], [DGZ].

1. **Statements equivalent to the existence of a convex PGNF-function.** Let X be a Banach space and let $\mathcal{B}$ be a bornology on X, that is $\mathcal{B}$ is a family of bounded sets of X such that $\cup\{B : B \in \mathcal{B}\} = X$. We say that a sequence $\{\xi_n\}$ in X^* *$\mathcal{B}$-converges* to $\xi \in X^*$ if

$$\sup |\langle \xi_n - \xi, B\rangle| \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad \text{for each } B \in \mathcal{B}.$$

A function $f \colon X \longrightarrow \mathbb{R}$ is called *$\mathcal{B}$-differentiable* at $x \in X$ if there is $\xi \in X^*$ such that for each $B \in \mathcal{B}$

$$\frac{1}{t}[f(x+th) - f(x) - \langle \xi, th\rangle] \longrightarrow 0 \quad \text{as } t \downarrow 0$$

uniformly for $h \in B$. Thus, if $\mathcal{B} = \big\{\{x\} : x \in X\big\}$, $\mathcal{B}$-convergence becomes weak* convergence and $\mathcal{B}$-differentiability is nothing else than Gateaux differentiability. Similarly for $\mathcal{B}$ consisting of all bounded sets we get norm convergence and Fréchet differentiability.

PROPOSITION 1. *Let $(Y, \|\cdot\|)$ be a Banach space and consider on $Y \times \mathbb{R}$ a bornology $\mathcal{B}$. Let $\{\zeta_n\}$ be a bounded sequence in Y^* and let $\{\gamma_n\} \subset [\frac{1}{2}, 1)$ be such that* $\lim_n \gamma_n = 1$. *Then $\{\zeta_n, -\gamma_n)\}$ $\mathcal{B}$-converges to $(0, -1)$ if and only if the function $f \colon Y \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by*

$$f(y, t) = \sup |\langle \zeta_n, y\rangle - \gamma_n t|, \quad (y, t) \in Y \times \mathbb{R},$$

is $\mathcal{B}$-differentiable at $(0, -1)$ with derivative $f'\big((0, -1)\big)(y, t) = -t$, $(y, t) \in Y \times \mathbb{R}$.

PROOF. Assume $\{(\zeta_n, -\gamma_n)\}$ $\mathcal{B}$-converges to $(0, -1)$. Fix any $B \in \mathcal{B}$ and any $\epsilon > 0$. As B and $\{\zeta_n\}$, $\{\gamma_n\}$ are bounded, the definition of f ensures the existence of $\delta > 0$ such that

$$f\big((0, -1) + \tau(y, t)\big) = \sup_n [\langle \zeta_n, \tau y\rangle + \gamma_n - \gamma_n \tau t]$$

for all $(y, t) \in B$ and all $\tau \in (-\delta, \delta)$. Now we find m so large that

$$\sup |\langle (\zeta_n, -\gamma_n) - (0, -1), B\rangle| < \epsilon \quad \text{whenever } n > m.$$

Then for every $\tau \in (0,\delta)$ satisfying moreover

$$0 < \tau < \inf\left\{ \frac{1-\gamma_n}{|\langle \zeta_n, y\rangle| + |t|(1-\gamma_n)} : (y,t) \in B, n = 1,2,\ldots,m \right\}$$

and every $(y,t) \in B$ we have

$$\frac{1}{\tau}\Big[f\big((0,-1) \pm \tau(y,t)\big) - f(0,-1) \pm \tau t\Big] = \sup_n \Big[\langle \zeta_n, \pm y\rangle \pm t(1-\gamma_n) - \frac{1}{\tau}(1-\gamma_n\Big]$$
$$= \max\big\{\max_{n\le m}[\cdots], \max_{n>m}[\cdots]\big\} < \max\{0,\epsilon\} = \epsilon.$$

And since $\epsilon > 0$ was arbitrary and f is convex, we conclude that f is $\mathcal{B}$-differentiable at $(0,-1)$ with derivative $f'(0,-1) = (0,-1)$.

Conversely, assume that f is $\mathcal{B}$-differentiable at $(0,-1)$ and $f'(0,-1) = (0,-1)$. We are to show that $(\zeta_n, -\gamma_n)$ $\mathcal{B}$-converges to $(0,-1)$. Fix any $\epsilon > 0$. Then there is $\delta > 0$ such that

$$\frac{1}{\tau}[f(0,-1) \pm \tau(y,t)) - f(0,-1) \pm \tau t] < \epsilon$$

whenever $0 < \tau < \delta$ and $(y,t) \in B$. From the previous paragraph we know there is $\tau \in (0,\delta)$ such that

$$f\big((0,-1) \pm \tau(y,t)\big) = \sup_n [\langle \zeta_n, \pm\tau y\rangle + \gamma_n \mp \gamma_n \tau t]$$

for all $(y,t) \in B$. Thus for all n and all $(y,t) \in B$ we have

$$\pm\langle \zeta_n, y\rangle + \frac{\gamma_n}{\tau} \mp \gamma_n t - \frac{1}{\tau} \pm t < \epsilon$$

or, for all n,

$$\sup |\langle (\zeta_n, -\gamma_n) - (0,-1), B\rangle| - \frac{1-\gamma_n}{\tau} \le \epsilon.$$

Hence

$$\limsup_n \sup |\langle (\zeta_n, -\gamma_n) - (0,-1), B\rangle| \le \epsilon.$$

But $\epsilon > 0$ was arbitrary; so $(\zeta_n, -\gamma_n)$ $\mathcal{B}$-converges to $(0,-1)$. ∎

THEOREM 1. *For a Banach space $(X, \|\cdot\|)$ the following statements are equivalent:*

(i) There exists on X a convex continuous PGNF-*function.*

(ii) There exists on X an equivalent PGNF-*norm.*

(iii) There exists on X^* an equivalent dual norm which is not weak* Kadec.

(iv) There exists a JN-*sequence in X^*.*

(v) There exists a linear continuous noncompact operator $T: X \longrightarrow c_0$.

(vi) X is infinite dimensional.

PROOF. (i) $\Rightarrow$ (iv). Let $f: X \longrightarrow \mathbb{R}$ be a convex continuous PGNF-function with a special point $x_0 \in X$. Then by Šmulyan's test [DGZ, Chapter 1, Corollary 1.5], there are $x_n \in X$ converging in norm to x_0 and ξ_n in the subdifferential $\partial f(x_n)$ of f at x_n such that

$\xi_n \longrightarrow f'(x_0)$ weakly* but $\inf_n \|\xi_n - f'(x_0)\| > 0$ Thus $\{\xi_n - f'(x_0)\}$ is a JN-sequence (ii) $\Rightarrow$ (iii) can be proved in the same way (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) are trivial, while (v) $\Rightarrow$ (vi) is obvious For the proof of (iv) $\Rightarrow$ (v) we can follow [D2, Chapter XII, Exercise 1] (vi) $\Rightarrow$ (iv) is the deeper Josefson-Nissenzweig theorem [D2, Chapter XII]

It remains to prove (iv) $\Rightarrow$ (ii) Let $\{\xi_n\}$ be a JN-sequence in X^* Write X as $Y \times \mathbb{R}$ where $Y \times \{0\}$ is a closed hyperplane in X and put $\zeta_n(y) = \langle \xi_n, (y, 0)\rangle$, $y \in Y$ Then $\{\zeta_n\}$ is a JN-sequence in Y^* Let f $Y \times \mathbb{R} \longrightarrow [0, +\infty)$ be the function constructed in Proposition 1 for our $\{\zeta_n\}$ and some $\{\gamma_n\} \subset [\frac{1}{2}, 1)$, $\gamma_n \uparrow 1$ Applying Proposition 1 twice for different $\mathcal{B}$ we can conclude that f is Gateaux but not Fréchet differentiable at $(0, -1)$ Put now

$$|||(y,t)||| = \max\Big[f(y,t), \frac{1}{2}(\|y\| + |t|)\Big], \quad (y,t) \in Y \times \mathbb{R}$$

Then $|||\ \ |||$ is an equivalent norm on $Y \times \mathbb{R}$ and

$$|||(0,-1)||| = f(0,-1) = 1 > \frac{1}{2}(\|0\| + 1)$$

Thus $|||\ \ |||$ has the same differentiability property at $(0, -1)$ as f, so $|||\ \ |||$ is a PGNF-norm ∎

We recall that a function f $X \longrightarrow \mathbb{R}$ is said to be *weak Hadamard differentiable* at $x \in X$ if there is $\xi \in X^*$ such that

$$\frac{1}{t}[f(x+th) - f(x) - \langle \xi, th\rangle] \longrightarrow 0 \quad \text{as } t \downarrow 0$$

uniformly for $h \in K$, where K is any weakly compact set in X (This corresponds to using the bornology of all weakly compact sets) In a reflexive space this is clearly the same as Fréchet differentiability for any function, while in l_1 it coincides with Gateaux differentiability for any Lipschitz function For convex functions the situation is much more interesting Indeed, as a parallel to Theorem 1 we may formulate

THEOREM 2 *For a Banach space $(X, \|\ \ \|)$ the following assertions are equivalent*

(i) There exist a convex continuous function (an equivalent norm) on X and $x_0 \in X$ at which it is weak Hadamard but not Frechet differentiable

(ii) There exist an equivalent dual norm $|||\ \ |||$ on X^* and $\xi_0, \xi_1, \xi_2,$ in X^* such that $|||\xi_n||| = |||\xi_0||| = 1$, $\xi_n \longrightarrow \xi_0$ in the Mackey topology but $\inf_n |||\xi_n - \xi_0||| > 0$

(iii) X is not sequentially reflexive, that is, there is a sequence $\{\xi_n\}$ in X^* such that $\xi_n \longrightarrow 0$ in the Mackey topology on X^* but $\inf_n \|\xi_n\| > 0$

(iv) There exists a linear completely continuous, that is, sequentially weak to norm continuous, noncompact operator T $X \longrightarrow c_0$

(v) X contains an isomorphic copy of l_1

PROOF (iii) $\Leftrightarrow$ (v) is due to Ørno [Ø] and can be found in the Appendix, see also [B], [BF] For the remaining implications one follows the proof of Theorem 1 ∎

It should be noted that a JN-sequence can always be chosen to be basic [KP], [M] and thus the operator T in both the above theorems can moreover be constructed with dense range

Let us consider further the explicit construction of a convex continuous PGNF-function. Theorem 1 ensures this is possible in every infinite dimensional Banach space and Proposition 1 shows how to construct such a function once a JN-sequence is at hand. But the construction of a JN-sequence is not trivial for a general Banach space, (see [D2, Chapter XII]). Let us record some situations when a JN-sequence can be found easily.

Let X be an infinite dimensional separable Banach space. Let $\{\xi_n\}$ be an infinite sequence of distinct points in the unit ball B_{X^*} of X^* with distance between distinct members bounded away from zero. Because B_{X^*} with the weak* topology is a metrizable compact, there is a subsequence $\{\xi_{n_i}\}$ converging to some ξ weakly*. And clearly $\{\xi_{n_i}\}$ do not converge (to ξ) in norm. Hence $\{\xi_{n_i} - \xi\}$ is a JN-sequence. Now we can easily use this sequence for the construction of a JN-sequence for every Banach space having a separable infinite dimensional quotient space. Indeed, if Q is a linear continuous operator from X onto a separable infinite dimensional space Y and if $\{\zeta_n\}$ is a JN-sequence in Y^*, then $\{Q^*\zeta_n\}$ is a JN-sequence in X^*. Thus, in particular, we can find a JN-sequence for WCG spaces, or more generally for WCD spaces, for the duals of Asplund spaces [DGZ, Chapter VI, §§2, 3] for $L_\infty([0,1])$ and for ℓ_∞, [LT, p. 111].

The next proposition presents a different way of constructing a PGNF-function in a large class of Banach spaces. Given a nonempty set K in X let d_K be the corresponding distance function, that is,

$$d_K(x) = \inf\{\|x-k\| : k \in K\}, \quad x \in X.$$

Clearly d_K is a Lipschitz function with $d_K(x) \geq 0 = d_K(k)$ for all $x \in X$ and all $k \in K$.

PROPOSITION 2. *Let $K \subset X$ be a convex closed set with $0 \in K$. Then the following statements are equivalent:*

(i) d_K is Gateaux differentiable at 0.

(ii) 0 is not a support point of K, that is, $\sup\langle \xi, K\rangle > 0$ *for every* $0 \neq \xi \in X^*$.

(iii) $\cup nK$ is dense in X.

Further d_K is Fréchet differentiable at 0 *if and only if* $0 \in \operatorname{int} K$.

PROOF. (i) $\Rightarrow$ (ii). Take any $0 \neq \xi \in X^*$. Find $h \in X$ so that $\langle \xi, h\rangle > 0$. Then

$$\begin{aligned} 0 &= \lim_{t\downarrow 0} \frac{1}{t} d_K(th) = \lim_{t\downarrow 0} \frac{1}{t} \inf_{k\in K} \|th-k\| \\ &\geq \limsup_{t\downarrow 0} \frac{1}{t} \inf_{k\in K} \langle \xi, th-k\rangle \\ &= \langle \xi, h\rangle - \lim_{t\downarrow 0} \frac{1}{t} \sup\langle \xi, K\rangle > (-\infty)\sup\langle \xi, K\rangle \end{aligned}$$

and therefore $\sup\langle \xi, K\rangle > 0$.

(ii) $\Rightarrow$ (iii). Assume (iii) is false. Then there is $x \in X$ not lying in $\overline{\cup nK}$. Hence there exists $0 \neq \xi \in X^*$ such that $\langle \xi, x\rangle > \sup\{\langle \xi, nk\rangle : k \in K, n = 1, 2, \ldots\}$. Then necessarily $\sup\langle \xi, K\rangle \leq 0$ so that (ii) is violated.

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(iii) $\Rightarrow$ (i). Take any $\xi \in \partial d_K(0)$. Then for each $k \in K$ we have $\langle \xi, k\rangle \leq d_K(k) - d_K(0) = 0$ and so, by (iii), $\langle \xi, h\rangle \leq 0$ for all $h \in X$. Thus $\xi = 0$ and d_K must be Gateaux differentiable at 0.

Assume now $0 \in \operatorname{int} K$. Then $d_K(x) = 0$ whenever $\|x\|$ is sufficiently small, whence d_K is Fréchet differentiable at 0. Finally, let $0 \notin \operatorname{int} K$. According to the Bishop-Phelps theorem [Ph, pp. 50, 51], there are support points $k_n \in K$ converging to 0. Take $\xi_n \in X^*, \|\xi_n\| = 1$, such that $\langle \xi_n, k_n\rangle = \sup\langle \xi_n, K\rangle$. Then we can easily check that $\xi_n \in \partial d_K(k_n)$. And since $\partial d_K(0)$ contains 0, Šmulyan's test says that d_K is not Fréchet differentiable at 0. ∎

Having this proposition we can construct a convex continuous PGNF-function whenever we have at hand a convex closed set $K \subset X$ such that $0 \in K \setminus \operatorname{int} K$ and $\cup nK$ is dense in X. Such a set can easily be constructed if X has a separable infinite dimensional quotient, X/Z, say. Indeed, let Q be the associated quotient operator. Choose a sequence $\{x_n\}$ in the unit ball of X such that $\{Qx_n\}$ is dense in the unit ball of X/Z and define $H = \overline{\text{co}}\{\pm\frac{1}{n}x_n : n = 1, 2, \ldots\}, K = H + Z$. Then $\cup nK$ is dense in X. Further, H is compact and hence so is $Q(K) = Q(H)$. It follows $\operatorname{int} Q(K) = \emptyset$, and finally $\operatorname{int} K = \emptyset$ because Q is open.

A similar construction gives a convex closed $K \subset X$ with $0 \in K \setminus \operatorname{int} K$ and $\overline{\cup nK} = X$ if X can be written as $\overline{Y + Z}$, where $Y \cap Z = \{0\}$ and Y is infinite dimensional and separable. This is the case, in particular, if $X = \ell_\infty$ [R] or if X has a Markuševič basis, see, *e.g.* [V] or references therein.

Unfortunately we must leave open the question of *whether every Banach space admits a convex closed set K such that* $0 \in K \setminus \operatorname{int} K$ *and* $\cup nK$ *is dense*. Possibilities of constructing such K from a JN-sequence are discussed in [BFa].

2. PGNF-**functions and** PGNF-**norms made as smooth as possible.** In the previous section we did not consider the degree of smoothness outside of the special point when constructing a PGNF-function or a PGNF-norm. Now we will deal with the following problem. If we have a norm on the space with some degree of smoothness, can we "preserve" this in the construction of an equivalent PGNF-norm? Below we present two such renorming procedures.

THEOREM 3. *Let X be a separable Banach space. Then X admits an equivalent norm which is not Fréchet differentiable at some nonzero point and whose dual norm is strictly convex.*

PROOF. We will think of X as $Y \times \mathbb{R}$ endowed with the norm $\|(y, t)\| = \|y\| + |t|$, $(y, t) \in Y \times \mathbb{R}$. Then $(Y \times \mathbb{R})^*$ can be thought of as $Y^* \times \mathbb{R}$ with the norm $\|(\zeta, r\| = \max(\|\zeta\|, |r|)$, $(\zeta, r) \in Y^* \times \mathbb{R}$. Let $\{y_n\}$ be a dense sequence in the unit ball of Y and define $T\colon l_2 \longrightarrow Y$ by $T(\{\lambda_n\}) = \sum n^{-2}\lambda_n y_n$, $\{\lambda_n\} \in l_2$. Then T is a linear compact operator. Define a norm $|||\cdot|||$ on $Y^* \times \mathbb{R}$ by

$$|||(\zeta, r)|||^2 = \max(\|\zeta\|^2, r^2) + \|T^*\zeta\|_{l_2}^2 + r^2, \quad (\zeta, r) \in Y^* \times \mathbb{R}.$$

Clearly $|||\cdot|||$ is an equivalent norm on $Y^* \times \mathbb{R}$. Moreover it is weakly* lower semicontinuous, so that it is a dual norm. We will use the symbol $|||\cdot|||$ also for the corresponding norm on $Y \times \mathbb{R}$.

Let $\{\zeta_n\}$ be a JN-sequence in Y^*. Without loss of generality we may assume that $\|\zeta_n\| \longrightarrow 1$. Thus we have $(\zeta_n, 1) \longrightarrow (0,1)$ weakly*, and $|||(0,1)||| = \sqrt{2}$. Now $T^*\zeta_n \longrightarrow 0$ weakly* and, as T^* is norm-compact (because T is), $T^*\zeta_n \longrightarrow 0$ in l_2-norm. Hence

$$|||(\zeta_n, 1)|||^2 = \max(\|\zeta_n\|^2, 1) + \|T^*\zeta_n\|_{l_2}^2 + 1 \longrightarrow 2 \quad \text{as } n \longrightarrow \infty.$$

Therefore putting $\{\xi_n\} = \{(\zeta_n, 1)/|||(\zeta_n, 1)|||\}$, and $\xi = (0, \sqrt{2}/2)$ we get $|||\xi_n||| = |||\xi||| = 1$ and $\xi_n \longrightarrow \xi$ weakly*. On the other hand we shall show that ξ is a norm attaining functional. Indeed, taking $x = (0, \sqrt{2})$ in $Y \times \mathbb{R}$, we have $\langle \xi, x\rangle = \langle (0, \sqrt{2}/2), (0, \sqrt{2})\rangle = 1$ while

$$\begin{aligned} |||x||| &= \sup\{\langle (\zeta, r), (0, \sqrt{2})\rangle : (\zeta, r) \in Y^* \times \mathbb{R}, |||(\zeta, r)||| \le 1\} \\ &= \sup\{\sqrt{2}r : (\zeta, r) \in Y^* \times \mathbb{R}, \max(\|\zeta\|^2, r^2) + \|T^*\zeta\|_{l_2}^2 + r^2 \le 1\} \\ &\le \sup\{\sqrt{2}r : r \in \mathbb{R}, 2r^2 \le 1\} = 1. \end{aligned}$$

Let us summarize what we have shown so far about $\{\xi_n\}$: $|||\xi_n||| = |||\xi||| = 1 = \langle \xi, x\rangle = |||x|||$, $\xi_n \longrightarrow \xi$ weakly* and a simple computation yields that $|||\xi_n - \xi||| \longrightarrow \sqrt{2}/2$. Whence, by Šmulyan's test, the norm $|||\cdot|||$ on $Y \times \mathbb{R}$ is not Fréchet differentiable at x.

The last thing that has to be checked is the strict convexity of the dual norm $|||\cdot|||$. So let $(\zeta_i, r_i) \in Y^* \times \mathbb{R}$, $i = 1, 2$, be such that $|||(\zeta_1, r_1)||| = |||(\zeta_2, r_2)||| = 1/2|||(\zeta_1 + \zeta_2, r_1 + r_2)|||$. From convexity and the definition of $|||\cdot|||$ we immediately get $r_1 = r_2$ and $\|T^*\zeta_1\|||_{l_2} = \|T^*\zeta_2\|||_{l_2} = 1/2\|T^*(\zeta_1 + \zeta_2)\|_{l_2}$. Now, as $\|\cdot\|_{l_2}$ is strictly convex, we have $T^*\zeta_1 = T^*\zeta_2$, and because T has dense range, $\zeta_1 = \zeta_2$. The strict convexity of the dual norm $|||\cdot|||$ is thus verified. Note that in consequence $|||\cdot|||$ on X is Gateaux differentiable at all non-zero points. ∎

Of course the above theorem and the corollary below are interesting, only if X is Asplund. Indeed, otherwise, according to Ekeland and Lebourg [Ph, Corollary 4.6], any equivalent norm on X is not Fréchet differentiable at some nonzero point. Let us mention here [DGZ, Theorem III.1.9] characterizing separable Banach spaces admitting a norm which is everywhere (except origin) Gateaux differentiable but nowhere Fréchet differentiable. See also [GMS].

COROLLARY 1. *Assume $X = Y \times Z$ where Y is separable and infinite dimensional and Z^* has a dual strictly convex norm $\|\cdot\|$; in particular let X be a* WCG *space. Then the conclusion of Theorem 3 holds.*

PROOF. According to Theorem 3, there exists a PGNF-norm $|||\cdot|||$ on Y such that its dual norm is strictly convex. Define a norm $|\cdot|$ on X by

$$|(y,z)|^2 = |||y|||^2 + \|z\|^2, \quad (y,z) \in Y \times Z.$$

Then

$$|(y^*, z^*)|^2 = |||y^*|||^2 + \|z^*\|^2, \quad (y^*, z^*) \in Y^* \times Z^* (\cong X^*)$$

and so the strict convexity of the dual norms $|||\ |||$ and $||\ ||$ is inherited by the dual norm $|\ |$ Further $|(y,0)| = |||y|||$ for all $y \in Y$ Hence $|\ |$ is not Fréchet differentiable at $(y,0)$ if $|||\ |||$ is not Fréchet differentiable at y

Now let X be a WCG space Then, there is a linear bounded projection $P\ X \longrightarrow X$ with separable infinite dimensional range Thus X is isomorphic with $PX \times (I-P)X$, where $(I-P)X$ is WCG, so it has an equivalent norm whose dual norm is strictly convex [D1, p 148] ∎

Next we improve the distance function d_K so as to obtain, first a smooth PGNF-function and then, actually a smooth PGNF-norm

THEOREM 4 *Let X be a* WCG *Asplund space Then X admits a convex function, which is Fréchet differentiable at each* $x \in X \setminus \{0\}$ *and which is Gateaux but not Fréchet differentiable at 0 Moreover X admits an equivalent norm which is Gateaux differentiable at each* $x \in X \setminus \{0\}$ *and which is Fréchet differentiable exactly at each* $x \in X \setminus \mathbb{R}x_0$, *where* $x_0 \in X$ *is a fixed nonzero point*

PROOF Let $||\ ||$ be a Fréchet smooth norm on X [DGZ, Corollary VII 1 13] Since X is WCG, it has a separable complemented subspace [DGZ, Section VI 2] Then, following the text immediately below the proof of Proposition 2, we can construct a convex symmetric weakly compact set $K \subset X$ with $\operatorname{int} K = \emptyset$ and $\overline{\cup nK} = X$ (If X is nonreflexive, then the Krein-Šmulyan theorem provides us with such a K immediately) Consider a set $C \subset \mathbb{R}^2$ defined by

$$C = \Big([0,1] \times \Big[0, \frac{1}{2}\Big]\Big) \cup \Big(\Big[0, \frac{1}{2}\Big] \times \Big[\frac{1}{2}, 1\Big]\Big) \\ \bigcup \Big\{(t,s) \in \Big[\frac{1}{2}, 1\Big] \times \Big[\frac{1}{2}, 1\Big] \quad \Big(t - \frac{1}{2}\Big)^2 + \Big(s - \frac{1}{2}\Big)^2 \le \frac{1}{4}\Big\}$$

and let φ be Minkowski's functional associated with C (we take $\inf \emptyset = +\infty$) Since

$$\Big[0, \frac{1}{2}\Big] \times \Big[0, \frac{1}{2}\Big] \subset C \subset [0,1] \times [0,1],$$

we have for $t \ge 0$, $s \ge 0$

$$\max(t,s) \le \varphi(t,s) \le 2\max(t,s)$$

Clearly φ is convex and differentiable at any $(t,s) \in \mathbb{R}^2$ with $t > 0$, $s > 0$ We also remark that

$$\varphi(t,s) = \varphi(s,t) = t \quad \text{if } 0 \le s < \frac{1}{2}t$$

We define $f\ X \longrightarrow \mathbb{R}$ by

$$f(x) = \varphi\big(||x||^2, d_K(x)\big), \quad x \in X,$$

where $d_K(x)$ is the distance of x from K We can easily see that f is convex, continuous, nonnegative and $f(0) = 0$

Let us check the promised differentiability properties of f Take any ξ in $\partial f(0)$ Then for any $k \in K$ and any $t \in (0,1)$ we have from the definition of f and by a property of φ,

$$\langle \xi, tk\rangle \leq f(tk) - f(0) = f(tk) = \|tk\|^2,$$

so $\langle \xi, k\rangle \leq 0$ for any $k \in K$ As $\cup nK$ is dense in X, we can conclude that $\xi = 0$ This proves f is Gateaux differentiable at 0, with derivative equal to 0 Consider now $x \in X \setminus \{0\}$ Assume first that $x \in K$ Then $d_K(x) = 0 < \frac{1}{2}\|x\|^2$ Hence there is a neighbourhood N of x such that $d_K(x') < \frac{1}{2}\|x'\|^2$ for all $x' \in N$ Then by the same property of φ, we have $f(x') = \|x'\|^2$ for all $x' \in N$ Therefore f is Frechet differentiable at x Assume now that $x \in X \setminus K$ Then $\|x\|^2 > 0$ and $d_K(x) > 0$ Now, it is well known that d_K inherits the Frechet smoothness of $\| \ \|$ at points outside K whenever K is weakly compact Thus, the chain rule implies that f is Fréchet differentiable at x Finally, let us show that f is not Fréchet differentiable at 0 By another property of φ we have $f(x) \geq d_K(x)$ for all $x \in X$ Moreover $f(0) = d_K(0)$ and, according to Proposition 2, d_K is not Fréchet differentiable at 0 since $\operatorname{int} K = \emptyset$ A fortiori, the same holds true for f

We may now, with some work, build a norm with the properties announced in the statement of the theorem Write $X = Y \times \mathbb{R}$, then Y is also WCG and Asplund By the preceding argument, there is a convex symmetric continuous function f $Y \longrightarrow [0,+\infty)$, which is everywhere Fréchet differentiable except at 0 where it is only Gateaux differentiable, with $f(0) = 0$ and $f'(0) = 0$ We may also observe that $f(y) \longrightarrow +\infty$ as $\|y\| \longrightarrow +\infty$

First of all we will cultivate f Let ψ $[0,1] \longrightarrow [0,1]$ be defined by $\psi(t) = 1 - \sqrt{1-t^2}$, $t \in [0,1]$ Clearly ψ is convex, increasing, differentiable at each $t \in (0,1)$ and $\psi'_-(1) = +\infty$ Define

$$g(y) = \varphi\Big(\psi\big(f(y)\big), \frac{1}{8}f(y)\Big), \quad \text{if } y \in Y \text{ and } f(y) \leq 1,$$

where φ is our maximum-like function used above Clearly $g(0) = 0 \leq g(y)$ whenever $f(y) \leq 1$ and g is symmetric since f is Also g is convex since f, ψ, φ are and ψ, φ are increasing Further g is Gateaux differentiable at 0 with $g'(0) = 0$ as $g(y) \leq 2\max\Big(\psi\big(f(y)\big), \frac{1}{8}f(y)\Big) \leq 2f(y)$ By the chain rule we get that g is Fréchet differentiable at each $0 \neq y \in Y$ satisfying $f(y) < 1$ Finally we will show g is not Fréchet differentiable at 0 For this find $0 < \delta < 1$ such that $\psi(t) \leq t/16$ whenever $0 \leq t < \delta$, this is possible since $\psi'_+(0) = 0$ Find $\Delta > 0$ such that $f(y) < \delta$ whenever $\|y\| < \Delta$ Then $\|y\| < \Delta$ implies $g(y) = \frac{1}{8}f(y)$ by a property of φ and so g cannot be Fréchet differentiable at 0 as f is not

Define the set

$$D = \{(y,t) \in Y \times \mathbb{R} \quad g(y) - 1 \leq t \leq -g(y) + 1, f(y) \leq 1\}$$

This is a convex, symmetric, closed and bounded set with nonempty interior Let $|||\ |||$ be Minkowski's functional for D We will show that $|||\ |||$ is the norm we are looking for We can easily check that the boundary ∂D of D is

$$\{(y,t) \in Y \times \mathbb{R} \quad f(y) \leq 1 \text{ and either } g(y) - 1 = t \text{ or } -g(y) + 1 = t\}$$

Fix $(y_0, t_0) \in \partial D$, with $g(y_0) < 1$; then necessarily $f(y_0) < 1$. Also $|||(y_0, t_0)||| = 1$ and t_0 is equal to $g(y_0) - 1$, say. Take (ζ, r) in $\partial ||| \cdot |||(y_0, t_0)$. Thus for all $h \in Y$, $||h|| \leq 1$, and all sufficiently small $\tau > 0$ we have

$$1 = \langle \zeta, y_0 \rangle + r\big(g(y_0) - 1\big) \geq \langle \zeta, y_0 + \tau h \rangle + r\big(g(y_0 + \tau h) - 1\big).$$

Hence for these h and τ

$$\langle \zeta, h \rangle \leq (-r) \frac{g(y_0 + \tau h) - g(y_0)}{\tau}.$$

Since g is differentiable at y_0,

$$\langle \zeta, h \rangle \leq (-r) \langle g'(y_0), h \rangle \quad \text{for all } h \in Y,$$

that is, $\zeta = -rg'(y_0)$. Using this, we have

$$1 = \langle -rg'(y_0), y_0 \rangle + r\big(g(y_0) - 1\big)$$

and hence (ζ, r) is uniquely determined. Therefore $\partial ||| \cdot |||(y_0, t_0)$ consists of one point only and so $||| \cdot |||$ is Gateaux differentiable at (y_0, t_0). If $t_0 = -g(y_0) + 1$ we proceed analogously or we can use the symmetry of D and g. In particular we have that $||| \cdot |||$ is Gateaux differentiable at $(0, -1)$ and $(0, 1)$ and with $||| \cdot |||'(0, -1) = -||| \cdot |||'(0, 1) = (0, -1)$.

Next we will show that $||| \cdot |||$ is not Fréchet differentiable at $(0, -1)$. Since g is not Fréchet differentiable at 0, there are $y_n \in Y$, $y_n \longrightarrow 0$, such that $\inf_n ||g'(y_n)|| > 0$. By the preceding paragraph we know that $||| \cdot |||'\big(y_n, g(y_n) - 1\big) = \big(-r_n g'(y_n), r_n\big)$ for large n and appropriate $r_n \in \mathbb{R}$. Since $||| \cdot |||'(0, 1) = (0, -1)$, we have from weak* convergence that $r_n \longrightarrow -1$. Thus

$$\begin{aligned}\liminf_n \Big|\Big|\Big|\big(-r_n g'(y_n), r_n\big) - (0, -1)\Big|\Big|\Big| &= \liminf_n \Big|\Big|\Big|\big(g'(y_n), -1\big) - (0, -1)\Big|\Big|\Big| \\ &\geq c \liminf_n ||g'(y_n)|| > 0,\end{aligned}$$

where the constant c comes from the equivalence of the norms involved. This shows that $||| \cdot |||$ is not Fréchet differentiable at the points of the line $0 \times \mathbb{R}$.

Fix $0 \neq y_0 \in Y$ with $g(y_0) < 1$. We will show $||| \cdot |||$ is Fréchet differentiable at (y_0, t_0), where $t_0 = g(y_0) - 1$. So consider $(y_n, t_n) \longrightarrow (y_0, t_0)$. We are to show that $||| \cdot |||'(y_n, t_n) \longrightarrow ||| \cdot |||'(y_0, t_0)$ in norm. By homogeneity we may assume $|||(y_n, t_n)||| = 1$ and hence $t_n = g(y_n) - 1$ for large n, say, for simplicity, for all n. Thus we already know that $||| \cdot |||'(y_n, t_n) = \big(-r_n g'(y_n), r_n\big)$ with appropriate r_n, for $n = 0, 1, 2, \ldots$. Now because $\big(-r_n g'(y_n), r_n\big) \longrightarrow \big(-r_0 g'(y_0), r_0\big)$ weakly* we have in particular $r_n \longrightarrow r_0$. Also, as $y_n \longrightarrow y_0$ and g is Fréchet differentiable at y_0, we get $g'(y_n) \longrightarrow g'(y_0)$ in norm. Therefore $||| \cdot |||'(y_n, t_n) \longrightarrow ||| \cdot |||'(y_0, t_0)$ in norm, which proves the Fréchet differentiability of $||| \cdot |||$ at (y_0, t_0).

It remains to investigate the differentiability of $||| \cdot |||$ at points $(y, 0)$, with $g(y) = 1$. We claim that $g(y) < 1$ whenever $f(y) < 1$. In fact, if $f(y) \leq 1/2$, then

$$g(y) = \varphi\Big(\psi\big(f(y)\big), \frac{1}{8} f(y)\Big) \leq 2 \max\Big(\psi\big(f(y)\big), \frac{1}{8} f(y)\Big) < 2f(y) \leq 1.$$

Further, if $1/2 < f(y) < 1$, then

$$\frac{1}{2}\psi\big(f(y)\big) = \frac{1}{2}(1 - \sqrt{1-f(y)^2} \geq \frac{1}{8}f(y);$$

so $g(y) = \psi\big(f(y)\big) < 1$. Put

$$B = \{y \in Y : f(y) \leq 1\}.$$

According to the claim, we have $B\times\{0\} = D\cap(Y\times\{0\})$ and so Minkowski's functional $|\cdot|$ for B will be nothing else than $|||(\cdot,0)|||$.

We will show that $|\cdot|$ is Fréchet differentiable on Y. Fix any $y \in Y$, with $|y| = 1$, and any $\xi \in \partial|\cdot|(y)$; then $\langle \xi, y\rangle = 1 = |\xi|$. Let $h \in Y$ be such that $\langle \xi, h\rangle = 0$ and let $t \in \mathbb{R}$ be given. Then $|y+th| \geq \langle \xi, y+th\rangle = 1$. Fix an arbitrary $\epsilon > 0$. Then $|(1+\epsilon)(y+th)| > 1$. Hence $(1+\epsilon)(y+th) \notin B$, that is, $f\big((1+\epsilon)(y+th)\big) > 1$. Letting $\epsilon > 0$ go to 0 we get $f(y+th) \geq 1\big(= f(y)\big)$. This holds for any $t \in \mathbb{R}$. Hence $\langle f'(y), h\rangle = 0$. Recall that we have obtained this for any $h \in Y$ satisfying $\langle \xi, h\rangle = 0$. It follows that $\xi = \lambda f'(y)$ for an appropriate $\lambda \in \mathbb{R}$. But $1 = \langle \xi, y\rangle = \lambda\langle f'(y), y\rangle$; so $\xi = \langle f'(y), y\rangle^{-1} f'(y)$. Thus ξ is uniquely determined and so $|\cdot|$ is Gateaux differentiable at y with

$$|\cdot|'(y) = \langle f'(y), y\rangle^{-1} f'(y).$$

From this it follows that $|\cdot|'$ is norm continuous (because f is convex and Fréchet differentiable), so we have shown Fréchet differentiability of $|\cdot|$ at every nonzero point of Y. This will help us in proving that $|||\cdot|||$ is Fréchet differentiable at points $(y,0)$.

First we will show $|||\cdot|||$ is Gateaux differentiable at $(y_0,0)$ with $f(y_0) = 1$, that is with $g(y_0) = 1$. Take any $(\zeta, r) \in \partial|||\cdot|||(y_0,0)$. Then $\langle \zeta, y_0\rangle = \langle (\zeta,r),(y_0,0)\rangle = \sup\langle(\zeta,r), D\rangle$.

Hence for $\tau > 0$ small enough we have

$$\langle \zeta, y_0 - \tau y_0\rangle + r\big(g(y_0-\tau y_0) - 1\big) \leq \langle \zeta, y_0\rangle,$$

or

$$r\frac{g(y_0-\tau y_0) - g(y_0)}{\tau} \leq \langle \zeta, y_0\rangle.$$

Let us remark that for $\tau > 0$ sufficiently small we have from convexity

$$\frac{1}{\tau}\big(g(y_0-\tau y_0) - g(y_0)\big) = \frac{1}{\tau}\Big(\psi\big(f(y_0-\tau y_0)\big) - 1\Big)$$
$$= -\frac{1}{\tau}\sqrt{f(y_0)^2 - f(y_0-\tau y_0)^2}$$

$$\ldots f(y_0 - \tau y_0) \ldots \longrightarrow -\infty \quad \text{as } \tau \downarrow 0.$$

Hence ... about $Y\times\{0\}$, we obtain ... $(\cdot,0)||| = |\cdot|$, we have $\zeta \in \partial$ ... erentiability of $|||\cdot|||$ at $(y_0, 0$ ...

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It remaıns to check that $|||\ \ |||$ ıs Frechet dıfferentıable at $(y_0, 0)$ wıth $g(y_0) = 1$ Consıder $(\zeta_n, r_n) \in Y^* \times \mathbb{R}$, wıth $|||(\zeta_n, r_n)||| = 1$, and such that $\langle(\zeta_n, r_n), (y_0, 0)\rangle \longrightarrow 1$ $\big(= \langle(|\ \ |'(y_0), 0), (y_0, 0)\rangle\big)$ Agaın, usıng the weak* convergence we get $r_n \longrightarrow 0$ Further we remark that

$$\sup\langle\zeta_n, B\rangle = \sup\langle(\zeta_n\ r_n), B \times \{0\}\rangle \leq \sup\langle(\zeta_n, r_n), D\rangle = 1$$

so $|\zeta_n| \leq 1$ Also $\langle\zeta_n, y_0\rangle = \langle(\zeta_n, r_n), (y_0, 0) \longrightarrow 1 = \langle|\ \ |'(y_0), y_0\rangle$ Hence $|\zeta_n - |\ \ |'(y_0)| \longrightarrow 0$ by the Frechet dıfferentıabılıty of $|\ \ |$ Therefore

$$\lim_n \Big|\Big|\Big|(\zeta_n, r_n) - \big(|\ \ |'(y_0), 0\big)\Big|\Big|\Big| = \lim |\zeta_n - |\ \ |'(y_0)| = 0$$

Thıs proves $|||\ \ |||$ ıs Frechet dıfferentıable at $(y_0, 0)$ ■

It should be noted that for $X = l_2$ a renormıng much lıke the one ın Theorem 4 ıs gıven by V Klee [K]

Appendix.

Ørno's paper ıs not avaılable ın a prınted form We have thus chosen, followıng a referee's suggestıon, to ınclude Ørno's proof ın ıts entırety

THEOREM 5 *A Banach space X ıs sequentıally reflexıve ıf and only ıf ℓ_1 ıs not ıso morphıc to a subspace of X*

PROOF Assume first that ℓ_1 ıs not ısomorphıc to a subspace of X and let $\{x_n^*\}_{n\ 1}^\infty$ be a weak* null sequence ın X^* for whıch the sequence $\{\langle x_n^*, x_n\rangle\}_{n\ 1}^\infty$ converges to zero for every weakly null sequence $\{x_n\}_{n\ 1}^\infty$ ın X It ıs easy to see that ıt ıs enough to check that such a sequence $\{x_n^*\}_{n\ 1}^\infty$ must converge ın norm to zero (See Lemma 2 1 ın [B]) If not, by passıng to a subsequence we can a sequence $\{x_n\}_{n\ 1}^\infty$ ın the unıt ball of X wıth $\{\langle x_n^*, x_n\rangle\}_{n\ 1}^\infty$ bounded away from zero By passıng to a further subsequence, we can as sume by Rosenthal's theorem [D2, Chapter XI] on Banach spaces whıch do not contaın ısomorphs of ℓ_1 that $\{x_n\}_{n\ 1}^\infty$ ıs weakly Cauchy Sınce $\{x_n^*\}_{n\ 1}^\infty$ converges weak* to zero by passıng to further subsequences and replacıng $\{x_n\}_{n\ 1}^\infty$ wıth a subsequence of dıffer ences $\frac{x_{n}\ x_{2n\ 1}}{2}$, we can assume moreover that $\{x_n\}_{n\ 1}^\infty$ ıs weakly null Thıs contradıctıon completes the proof of the first dırectıon

To go the other way, suppose that Y ıs a subspace of X whıch ıs ısomorphıc to ℓ_1 and let $\{e_n\}_{n\ 1}^\infty$ be the ımage of the unıt vector basıs under some ısomorphısm from ℓ_1 onto Y Define a bounded lınear operator from Y ınto $L_\infty[0, 1]$ by mappıng e_n to the n th Rademacher functıon r_n By the ınjectıve property of $L_\infty[0, 1]$, thıs operator extends to a bounded lınear operator T from X ınto $L_\infty[0, 1]$ Let r_n^* be the n th Rademacher functıon ın $L_1[0, 1]$ consıdered as a subspace of $L_\infty[0, 1]^*$ Thus the sequence $\{r_n^*\}_{n\ 1}^\infty$ beıng equıvalent to an orthonormal sequence ın a Hılbert space, converges weakly to zero Sınce $L_\infty[0, 1]$ has the Dunford Pettıs property (*cf* [D2, p 113]), $\{r_n^*\}_{n\ 1}^\infty$ converges ın the Mackey topology to zero, *a fortıorı* $\{T^* r_n^*\}_{n\ 1}^\infty$ converges $\tau(X^*\ X)$ to zero But $\langle T^* r_n^*, e_n\rangle = \langle r_n^*, r_n\rangle = 1$, so $\{T^* r_n^*\}_{n\ 1}^\infty$ does not converge to zero ın norm ■

Theorem 5 extends and simplifies work in [B], where it was shown that Asplund spaces are sequentially reflexive and the full result was conjectured. The paper [B] also contains a variety of applications related to the present work.

ADDED IN PROOF. The authors have recently discovered that Theorem 5 may also be deduced from [E].

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