REMARKS ON COMPLEMENTATION IN THE LATTICE OF ALL TOPOLOGIES

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0. Our aim is to prove that certain topologies have complements in the lattice of all the topologies on a given set. Lattices of topologies were studied in **(1–8)**. In **(7)** Hartmanis points out that the lattice of all the topologies on a finite set is complemented and poses the question whether this is so if the set is infinite. A positive answer is given here for denumerable sets. This result was announced in **(6)**. The case of higher powers remains unsettled, although quite a few topologies turn out to have complements. As far as the author knows, no one has proved the existence of a topology that has no complement.

1. By a topology τ on a given set X we mean the collection of open sets of this topology, that is, τ is a family of subsets of X containing $\emptyset$ and X as members and closed under finite intersections and arbitrary unions. By a base of τ we mean a subset of τ such that every member of τ is a union of elements belonging to it. If $Y \subseteq X$, then "$\mathrm{Cl}(Y)$" denotes the closure of Y, that is,

$$\mathrm{Cl}(Y) = X - \bigcup\{Z|Z \in \tau \text{ and } Z \cap Y = \emptyset\}.$$

We recall the τ is a T_0-topology if, for all p, q, $\in X$ with $p \neq q$ we have $p \notin \mathrm{Cl}(\{q\})$ or $q \notin \mathrm{Cl}(p)$; it is a T_1-topology if $\mathrm{Cl}(\{p\}) = \{p\}$ for all $p \in X$. The set of all topologies on X together with the inclusion relation, $\subseteq$, forms a lattice that is easily seen to be complete. This lattice has a maximal element that is the discrete topology, $I = \{Y|Y \subseteq X\}$, and a minimal element, $0 = \{\emptyset, X\}$. The least upper bound, $\tau_1 \vee \tau_2$, of two topologies τ_1 and τ_2 is the topology whose base is $\{A \cap B|A \in \tau_1, B \in \tau_2\}$, that is, it consists of all unions of sets of the form $A \cap B$ where $A \in \tau_1$ and $B \in \tau_2$. The greatest lower bound, $\tau_1 \wedge \tau_2$, of τ_1 and τ_2 is $\tau_1 \cap \tau_2$. The lattice is complemented if for every τ_1 there is a τ_2 such that $\tau_1 \vee \tau_2 = I$ and $\tau_1 \wedge \tau_2 = 0$.

If τ is a topology on X and $Y \subseteq X$, then $\tau|Y$ is defined as $\{V \cap Y|V \in \tau\}$. Clearly $(\tau_1 \vee \tau_2)|Y = \tau_1|y \vee \tau_2|y$ and $(\tau_1 \wedge \tau_2)|Y = \tau_1|y \wedge \tau_2|y$.

2. (1) LEMMA. *If the lattice of all topologies on a set X is complemented, so is the lattice of all topologies on Y whenever $|Y| \leqslant |X|$ ($|Y|$ = power of Y).*

Proof. If $|Y| \leqslant |X|$, one can assume without loss of generality that $Y \subseteq X$. Given a topology τ on Y define τ_1 on X to be $\{V|V \cap Y \in \tau\}$. Let τ'_1 be the complement of τ_1, and put $\tau' = \tau'_1|Y$. Then $\tau \wedge \tau' = \tau_1 \wedge \tau'_1|Y = 0|Y$ and $\tau \vee \tau' = \tau_1 \vee \tau'_1|Y = I|Y$ and thus τ' is a complement of τ.

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(2) LEMMA. *If every T_0-topology (T_1-topology) on X has a complement and $|Y| \leqslant |X|$, then every T_0-topology (T_1-topology) on Y has a complement.*

Proof. This is the same as the proof of (1).

(3) LEMMA. *Let τ be a topology on X. Assume that, for every $Y \subseteq X$, if $\tau|Y$ is a T_0-topology, then it has a complement. Under this assumption τ has a complement.*

Proof. Define $x \approx y$ as follows for every $V \in \tau$, $x \in V$ if and only if $y \in V$. Obviously, this is an equivalence relation. Using the axiom of choice, let X_1 be a subset of X whose intersection with every equivalence class is exactly one point. The set X_1 can be described also as a maximal subset Y for which $\tau|Y$ is a T_0-topology. Put $\tau_1 = \tau|X_1$. Since τ_1 is a T_0-topology, it has a complement τ'_1. Let

$$\tau' = \{A|A \cap X_1 \in \tau'_1\}.$$

If $V \in \tau \cap \tau'$, then $V \cap X_1 \in \tau_1 \cap \tau'_1$. Hence $V \cap X_1$ is either X_1 or $\emptyset$. If $V \cap X_1 = X_1$, then $V \supseteq X_1$, and since $V \in \tau$, it contains with every member x all y's such that $y \approx x$; hence $V = X$. If $V \cap X_1 = \emptyset$, then $V = \emptyset$; otherwise, if for some x, $x \in V$, V must contain the point y of X_1 for which $y \approx x$. If $x \in X_1$, then there are $V_1 \in \tau_1$ and $V'_1 \in \tau'_1$ such that $V_1 \cap V'_1 = \{x\}$. For some $V \in \tau$, $V_1 = V \cap X_1$. Since $V'_1 \subseteq X_1$, we get $\{x\} = V \cap V'_1$. By definition $V'_1 \in \tau'$. If $x \notin X_1$, then $\{x\} \in \tau'$, and we have $\{x\} = X \cap \{x\}$. Thus τ' is a complement of τ.

(4) COROLLARY. *If every T_0-topology on X has a complement, so does every topology on X.*

Proof. Let τ be a topology on X. If $Y \subseteq X$ and $\tau|Y$ is a T_0-topology, then the hypothesis of (4) and (2) imply that $\tau|Y$ has a complement. Hence, by (3), τ has a complement.

(5) THEOREM. *Let τ be a topology on X such that, for every $Y \subseteq X$, if $\tau|Y$ is a T_1-topology, then it has a complement. Under this assumption τ has a complement.*

Proof. By (3) it suffices to show that $\tau|Y$ has a complement whenever $Y \subseteq X$ and $\tau|Y$ is a T_0-topology. If $Y' \subseteq Y \subseteq X$, then $\tau|Y' = (\tau|Y)|Y'$. Therefore if τ satisfies the assumption of (5), so does $\tau|Y$ for all $Y \subseteq X$. Hence with no loss of generality, τ can be assumed to be a T_0-topology.

For $x, y \in X$, define $x \ll y$ as: $y \in \mathrm{Cl}(\{x\})$ (that is, $y \in V \Rightarrow x \in V$, for all $V \in \tau$). Obviously this relation is transitive. Since τ is a T_0-topology, $x \ll y$ and $y \ll x$ imply $x = y$. Also $\tau|Y$ is a T_1-topology if and only if neither $x \ll y$ nor $y \ll x$, for every two distinct points x, y of Y. If Y has this property, we say that Y is a T_1-*set*. If, in addition, $Y \subseteq Z$ ($Z \subseteq X$), we say that Y is a T_1-*subset* of Z.

For $x \in X$ and $Y \subseteq X$ define $x \prec Y$ if for some $y \in Y$, $x \ll y$. Similarly, define $Y \prec x$ if for some $y \in Y$, $y \ll x$. If Y is a T_1-subset of X, then we put

$$Y^+ = \{x|x \notin Y \text{ and } Y \prec x\} \text{ and } Y^- = \{x|x \notin Y \text{ and } x \prec Y\}.$$

The set Y^+ is referred to as the *right side of* Y and Y^- as the *left side of* Y.

If $x \in Y^+ \cap Y^-$, there would be y_1, y_2 in Y such that $y_1 \ll x$ and $x \ll y_2$; hence $y_1 \ll y_2$; if $y_1 = y_2$, then $x = y_1$, which is impossible, and if $y_1 \neq y_2$, this contradicts the assumption that Y is a T_1-set. Thus $Y^+ \cap Y^- = \emptyset$.

As is easily seen, if $Y_1 \subseteq Y^+$, then $Y_1^+ \subseteq Y^+$, and if $Y_1 \subseteq Y^-$, then $Y_1^- \subseteq Y^-$.

Let Z be a subset of X. By the axiom of choice there is a maximal T_1-subset of Z (that is, a T_1-subset of Z which is not properly included in any other T_1-subset of Z). Let Y be such a set. If $z \in Z - Y$, then either $Y \prec z$ or $z \prec Y$; otherwise $Y \cup \{z\}$ would be a T_1-set contradicting the maximality of Y. Hence

$$Z = (Z \cap Y^-) \cup Y \cup (Z \cap Y^+).$$

The following construction is carried through by transfinite induction. Let X_0 be a maximal T_1-subset of X. If $X_0^+ \neq \emptyset$, let X_1 be a maximal T_1-subset of X_0^+. In general, if X_λ is defined for all $\lambda < \nu$ and if $\bigcap_{\lambda<\nu} X_\lambda^+ \neq \emptyset$, let X_ν be a maximal T_1-subset of $\bigcap_{\lambda<\nu} X_\lambda^+$. Similarly if $X_0^- \neq \emptyset$, let X_{-1} be a maximal T_1-subset of X_0^-, and in general, if $\bigcap_{\lambda<\nu} X_{-\lambda}^- \neq \emptyset$, let $X_{-\nu}$ be a maximal T_1-subset of $\bigcap_{\lambda<\nu} X_{-\lambda}^-$. In this way we obtain a family of disjoint subsets $\{\ldots, X_{-\lambda'}, \ldots, X_{-1}, X_0, X_1, \ldots, X_\lambda, \ldots\}_{\lambda<\mu,\lambda'<\mu'}$ in which X_ν is a maximal T_1-subset of $\bigcap_{\lambda<\nu} X_\lambda^+$ for all $\nu < \mu$, $X_{-\nu}$ is a maximal T_1-subset of $\bigcap_{\lambda<\nu} X_{-\lambda}^-$ for all $\nu < \mu'$, and $\bigcap_{\lambda<\mu} X_\lambda^+ = \bigcap_{\lambda<\mu'} X_{-\lambda}^- = \emptyset$. For convenience, let λ, ν range over ordinals, and let α, β, γ range over ordinals $<\mu$ as well as over symbols of the form $-\lambda$ where $\lambda < \mu'$, and put $-\lambda < \nu$ and $-\lambda < -\lambda'$ whenever $\lambda > \lambda'$. Finally put $-0 = 0$.

Let $\tau_\alpha = \tau|X_\alpha$. Since every X_α is a T_1-set, every τ_α has a complement. Let τ'_α be a complement of τ_α. Let τ' be the topology generated by the following sets:

(i) $\{x\}$, where $x \notin \bigcup_\alpha X_\alpha$.

(ii) V, where $V \in \tau'_\alpha$ and α is such that for every $W \in \tau$ and every $\beta > \alpha$, $W \supseteq X_\alpha$ implies that $W \cap \bigcup_{\beta \leqslant \gamma} X_\gamma \neq \emptyset$. (If μ is not a limit ordinal and $\alpha = \mu - 1$, this condition is satisfied vacuously.)

(iii) $V \cup \bigcup_{\beta\leqslant\gamma} X_\gamma$, where $V \in \tau'_\alpha$ and α, β are such that $\beta > \alpha$ and, for some $W \in \tau$, $W \supseteq X_\alpha$ and $(\bigcup_{\beta\leqslant\gamma} X_\gamma) \cap W = \emptyset$.

We wish to show that τ' is a complement for τ.

First let $x \in X$. If $x \notin \bigcup_\alpha X_\alpha$, then $\{x\} \in \tau'$.

If $x \in X_\alpha$, then $\{x\} = V \cap V'$, where $V \in \tau_\alpha$ and $V' \in \tau'_\alpha$; thus $V = U \cap X_\alpha$ where $U \in \tau$. If $W \supseteq X_\alpha$ implies $W \cap \bigcup_{\beta<\gamma} X_\gamma \neq \emptyset$ for every $W \in \tau$ and every $\beta > \alpha$, then $V' \in \tau'$ and we have $\{x\} = U \cap V'$. If, for some $W \in \tau$ and some $\beta > \alpha$, $W \supseteq X_\alpha$ and $W \cap \bigcup_{\beta\leqslant\gamma} X_\gamma = \emptyset$, then $V'' = V' \cup \bigcup_{\beta<\gamma} X_\gamma \in \tau'$ and $(U \cap W) \cap V'' = \{x\}$. Hence $\tau \vee \tau'$ is the discrete topology. It remains to show that $\tau \wedge \tau' = \{\emptyset, X\}$.

Every set of τ' is the union of sets of the forms (i), (ii), and (iii). To verify this, note that the intersection of a set of the form (i) with another set of the form (i), (ii), or (iii) is empty, that the intersection of a set of the form (ii) with a set of the form (ii) or (iii) is of the form (ii), and that the intersection of two sets of the form (iii) is the union of sets of the forms (ii) and (iii).

To elaborate the last point, let $W_i = V_i \cup \bigcup_{\gamma \geqslant \beta_i} X_\gamma$, $i = 1, 2$, be two sets of the form (iii), where $V_i \in \tau'_{\alpha_i}$, $i = 1, 2$. Let $\beta_3 = \max\{\beta_1, \beta_2\}$. If $\alpha_1 = \alpha_2$, then $W_1 \cap W_2 = V_1 \cap V_2 \cup \bigcup_{\gamma \geqslant \beta_3} X_\gamma$ and this is of the form (iii). If $\alpha_2 > \alpha_1$, then $W_1 \cap W_2 = W_2$ if $\alpha_2 \geqslant \beta_1$, and $W_1 \cap W_2 = U_{\gamma \geqslant \beta_3} X_\gamma$ if $\alpha_2 \leqslant \beta_1$; this last set is a union of sets of the forms (ii) and (iii).

For the rest of the proof we assume that $V \in \tau \cap \tau'$ and $V \neq \emptyset$. The following will now be proved in the indicated order:

(I) For all α, if $V \cap X_\alpha \neq \emptyset$, then $V \supseteq X_\alpha$.

(II) For some α, $V \cap X_\alpha \neq \emptyset$.

(III) If $V \supseteq X_\alpha$ and $\beta \leqslant \alpha$, then $V \supseteq X_\beta$.

(IV) For all α, $V \supseteq X_\alpha$.

(V) $V = X$.

(I) $V \cap X \in \tau_\alpha$. The definition of τ' implies that $V \cap X_\alpha \in \tau'_\alpha$. Hence $V \cap X_\alpha \neq \emptyset$ implies that $V \cap X_\alpha = X_\alpha$.

(II) Let $x \in V$, $x \notin \bigcup_\alpha X_\alpha$. We have to show that $V \cap \bigcup_\alpha X_\alpha \neq \emptyset$. Since X_0 is a maximal T_1-subset of X, x must be either in the right side or in the left side of X_0. If $x \in X_0{}^+$, then $X \prec x$; and since $V \in \tau$, it follows that $V \cap X_0 \neq \emptyset$.

Assume that $x \in X_0{}^-$. Let λ^* be the least upper bound of $\{\lambda | x \in X_{-\lambda}{}^-\}$. If $x \in X_{-\lambda}{}^-$, then also $x \in X_{-\lambda'}{}^-$, whenever $\lambda' \leqslant \lambda$; consequently

$$x \in \bigcap_{\lambda < \lambda^*} X_{-\lambda}{}^-.$$

The maximality of $X_{-\lambda^*}$ implies that $x \in X_{-\lambda^*}{}^+$ or $x \in X_{-\lambda^*}{}^-$. If $x \in X_{-\lambda^*}{}^+$, then $V \cap X_{-\lambda^*} \neq \emptyset$; and it $x \in X_{-\lambda^*}{}^-$, then $x \in X^{\pm}_{-(\lambda^*+1)}$, and $V \cap X_{-(\lambda^*+1)} \neq \emptyset$.

(III) If $\beta \leqslant \alpha \leqslant 0$, then $X_\beta \subseteq X_\alpha{}^-$; thus $x \prec X_\alpha$ whenever $x \in X_\beta$. Therefore $V \supseteq X_\alpha$ implies that $V \supseteq X_\beta$. If $0 \leqslant \beta \leqslant \alpha$, then $X_\alpha \subseteq X_\beta{}^+$, that is, $X_\beta \prec x$ whenever $x \in X_\alpha$. Hence if $V \supseteq X_\alpha$, then $V \cap X_\beta \neq \emptyset$. Therefore by (II), $V \supseteq X_\beta$. Finally, if $\beta \leqslant 0 \leqslant \alpha$, it follows from the two previous cases that $V \supseteq X_\alpha$ implies $V \supseteq X_0$ and this yields $V \supseteq X_\beta$.

(IV) By (I) and (II), there is an α for which $V \supseteq X_\alpha$. Since V is the union of sets of the forms (i), (ii), and (iii), there is a set U of the form (ii) or (iii) such that $U \cap X_\alpha \neq \emptyset$ and $V \supseteq U$. If U is of the form (ii), $V \in \tau$ and $V \supseteq X_\alpha$ imply that $V \cap \bigcup_{\beta < \gamma} X_\gamma \neq \emptyset$, for all $\beta > \alpha$. If U is of the form (iii), then $U \cap \bigcup_{\beta \leqslant \gamma} X \neq \emptyset$ for all $\beta > \alpha$, and the same holds true replacing U by V. It follows now from (II) and (III) that $V \supseteq X_\beta$ for all β.

(V) If $x \notin \bigcup_\alpha X_\alpha$, then $x \in X_0{}^-$ implies, since $V \supseteq X_0$, that $x \in V$. If $x \in X_0{}^+$, let ν be the smallest ordinal for which $x \notin X_\nu{}^+$. Since $x \in \bigcap_{\lambda < \nu} X_\lambda{}^+$, we must have $x \in X_\nu{}^+$ or $x \in X_\nu{}^-$; hence $x \in X_\nu{}^-$. Again, since $V \supseteq X_\nu$, we have $x \in V$.

(6) Corollary. *If every T_1-topology on X has a complement, so does every topology on X.*

Proof. The same as the proof of (4) from (3).

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(7) COROLLARY. *If X is finite or countable, then every topology on X has a complement.*

Proof. By (6), it suffices to show that every T_1-topology τ on X has a complement. Put $X_1 = \{x|\{x\} \in \tau\}$ and $X_2 = X - X_1$. If $X_2 = \emptyset$, then τ is the discrete topology and the assertion is obviously true. Hence assume that $X_2 \neq \emptyset$, and let $x_1, x_2, \ldots, x_n, \ldots$ be an enumeration of X_2, where the sequence is either infinite or finite. We distinguish two cases: (a) $X_2 \notin \tau$ and (b) $X_2 \in \tau$.

In case (a), let τ' be the topology consisting of $\emptyset$, X, X_2 and all sets of the form $\{x_1, \ldots, x_n\}$, where n runs over all natural numbers if X_2 is infinite, and over some proper initial segment of the natural numbers if X_2 is finite. Given any x_i, since τ is a T_1-topology, there is a $V \in \tau$ such that

$$V \cap \{x_1, \ldots, x_i\} = \{x_i\}.$$

If $x \neq x_i$ for all i, then $x \in X_i$, $\{x\} \in \tau$, and $\{x\} = \{x\} \cap X$. Thus $\tau_1 \vee \tau_2$ is the discrete topology. On the other hand, no set of the form $\{x_1, \ldots, x_n\}$ is in τ, since in that case we would have $\{x_1\} = \{x_1, \ldots, x_n\} \cap V \in \tau$ where V is the member of τ satisfying $x_1 \in V$ and $x_2, \ldots, x_n \notin V$. Hence $\tau \wedge \tau' = \{\emptyset, X\}$.

In case (b), let τ' consist of $\emptyset$, X and all the sets of the form $X_1 \cup \{x_1, \ldots, x_n\}$, $n = 1, 2, \ldots$. Now X_2 must be infinite; for if $X_2 = \{x_1, \ldots, x_m\}$, then $\{x_1\} = X_2 \cap V \in \tau$ for some suitable $V \in \tau$. One proves in a similar way that τ' is a complement of τ.

(8) COROLLARY. *Let $X = \bigcup_\xi X_\xi$, where every X_ξ is countable and the X_ξ's are pairwise disjoint. Let τ be a topology of X such that no $V \in \tau$, $V \neq X$, $V \neq \emptyset$, is the union of X_ξ's. Then τ has a complement.*

Proof. Put $\tau_\xi = \tau|X$ and let τ'_ξ be a complement of τ_ξ. Let τ' consist of all the sets of the form $\bigcup_\xi V_\xi$, where $V_\xi \in \tau_\xi$ for all ξ. It is easily seen that to each $x \in X$ there are $V \in \tau$ and $V' \in \tau'$ such that $\{x\} = V \cap V'$. On the other hand, if $V \in \tau \cap \tau'$, then $V \cap X_\xi \in \tau_\xi \cap \tau'_\xi$. Hence, for all ξ, $V \cap X_\xi$ is either X_ξ or $\emptyset$ which means that V is a union of X_ξ's. Consequently $V = \emptyset$ or $V = X$.

This corollary was communicated to the author by Manuel Berri, who used the result announced in **(6)** for the construction of τ' as it is given here. The same construction was used independently by the author to prove a weaker version in which the sets X_ξ were assumed to be dense in X.

(9) COROLLARY. *If τ is a topology on X that has a countable base whose members are all of the same power, then τ has a complement.*

Proof. One shows that X and τ satisfy the conditions of (8). Let

$$V_1, V_2, \ldots, V_n, \ldots$$

be the sets of the base. If every V_n is countable, then X is countable. Assume that the V_i's are uncountable. Well order each V_i so that the order type is the initial ordinal ν of its cardinality (ν is the same for all V_i's). Let X_0 be the set consisting of the first points of $V_1, V_2, \ldots, V_n, \ldots$, and in general, let X_λ be the set consisting of all the first points of

$$V_1 - \bigcup_{\mu<\lambda} X_\mu, \ldots, V_n - \bigcup_{\mu<\lambda} X_\mu, \ldots.$$

As is easily seen, $\{X_\lambda\}_{\lambda<\nu}$ is a collection of pairwise disjoint, dense, countable sets whose union is X; hence they satisfy the conditions of (8).

3. A more general and interesting problem is that of finding the cases where a relative complement exists. That is, given the topologies τ_1, τ_2, τ_3, all on a set X, so that $\tau_1 \subseteq \tau_2 \subseteq \tau_3$, does there exist a topology τ'_2 satisfying $\tau_2 \wedge \tau'_2 = \tau_1$ and $\tau_2 \vee \tau'_2 = \tau_3$? That one cannot always expect a positive answer even for finite sets is shown by the following examples.

Put $X = \{1,2,3\}$, $\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, X, \{1\}\}$, and

$$\tau_3 = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}.$$

Then $\tau_1 \vee \tau'_2 = \tau_3$ implies that $\{1,2\}, \{1,3\} \in \tau'_2$; but then $\{1\} \in \tau'_2$. Hence $\tau_2 \wedge \tau'_2 \neq \tau_1$.

One can also construct an example where τ_3 is the discrete topology. Again let $X = \{1,2,3\}$; put $\tau_1 = \{\emptyset, X, \{1\}, \{1,2\}\}$, $\tau_2 = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$, and let τ_3 be the discrete topology. If $\tau_2 \wedge \tau'_2 = \tau_1$, then $\{3\} \notin \tau'_2$. Therefore $\tau_2 \vee \tau'_2 = \tau_3$ implies that $\{3\} = \{1, 3\} \cap V$, where $V \in \tau'_2$ and $V \neq \{3\}$. Hence $V = \{2,3\}$. But $\{2,3\} \in \tau'_2$ would yield $\{2\} = \{1,2\} \cap \{2,3\} \in \tau'_2$, which is impossible.

If, for a fixed X, τ_1 consists of all the sets whose complements are finite, then $\{\tau | \tau \supseteq \tau_1\}$ is the set of all the T_1-topologies. Thus, asking whether for every τ_2 such that $\tau_2 \supseteq \tau_1$ there is a τ'_2 such that $\tau_2 \wedge \tau'_2 = \tau_1$, and that $\tau_2 \vee \tau'_2$ is the discrete topology, is the same as asking whether the lattice of all the T_1-topologies on X is complemented. For finite X this is trivial, but no answer is known for an infinite X even if X is denumerable.

References

1. R. W. Bagley, *On the characterization of the lattice of topologies*, J. London Math. Soc., *30* (1955), 247–249.
2. R. W. Bagley and David Ellis, *On the topolattice and permutation group of an infinite set*, Math. Japon., *3* (1954), 63–70.
3. David Ellis, *On the topolattice and permutation group of an infinite set* II, Proc. Cambridge Philos. Soc., *50* (1954), 485–487.
4. V. K. Balachandran, *On the lattice of convergence topologies*, J. Madras Univ. Sect. B, *28* (1958), 129–146.
5. G. Birkhoff, *On the combination of topologies*, Fund. Math., *26* (1936), 156–166.
6. Haim Gaifman, *The lattice of all topologies cn a denumerable set* (Abstract), Amer. Math. Soc. Not., *8* (1961), 356.
7. Juris Hartmanis, *On the lattice of topologies*, Can. J. Math., *10* (1958), 547–553.
8. R. Vaidyanathaswami, *Treatise on set topology* (Madras, 1947).

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