BULL. AUSTRAL. MATH. SOC. 54C35, 54A25
VOL. 10 (1974), 403-408.

Ascoli theorems and the pseudocharacter of mapping spaces

J.A. Guthrie

The pseudocharacter of the space Y^X of continuous functions from X to Y is studied. For certain topologies on Y^X sufficient conditions, which are shown in some cases also to be necessary, for Y^X to have a specified pseudocharacter are given. An Ascoli theorem is a theorem which characterizes the compact subsets of Y^X. Several Ascoli theorems are obtained, including ones which utilize results on pseudocharacter.

1. Preliminaries

While other cardinal invariants such as cardinality and density have been studied for mapping spaces, see for example Comfort and Hager [5] and Vidossich [14] respectively, the pseudocharacter seems to have received much less attention. In Section 2 we obtain results which relate the pseudocharacter of the mapping space to the pseudocharacter of the range space and the density of the domain.

We adopt the following notation and terminology. All spaces are assumed to be Hausdorff. The cardinality of the set A is denoted by $|A|$. We define the *pseudocharacter* $\psi(p, X)$ of the point p in the topological space X by

$$\psi(p, X) = \inf\{|\mathcal{U}| \mid \mathcal{U} \text{ is a collection of open subsets of } X \text{ and } \cap\mathcal{U} = \{p\}\} .$$

We call $\psi(X) = \sup\{\psi(p, X) \mid p \in X\}$ the pseudocharacter of the space X.

Received 5 February 1974.

In case $\psi(p, X) \leq \aleph_0$ we say p is a G_δ in X. Aull [3] has called X an E_0-space provided $\psi(X) \leq \aleph_0$. Other authors such as Anderson [1] call such a space a G_δ-space. The *density* of X, denoted $\delta(X)$, is defined by $\delta(X) = \inf\{|S| \mid \mathrm{Cl}(S) = X\}$.

We represent by Y^X the set of all maps (= continuous functions) from X into Y. The only topologies we shall consider for Y^X will be the ones Arens and Dugundji [2] call set-open. That is, we consider topologies which have a subbase which can be described as follows. Let $\mathcal{A}$ be a collection of subsets of X and let $(A, U) = \{f \in Y^X \mid f(A) \subset U\}$. Then $\{(A, U) \mid A \in \mathcal{A}$ and U is open in $Y\}$ forms a subbase for the set-open topology determined by $\mathcal{A}$.

We use special notation in the following cases. If $\mathcal{A}$ is the set of all singletons in X, denote this *topology of pointwise convergence* by $C_p(X, Y)$. If $\mathcal{A}$ is the set of all convergent sequences in X (a convergent sequence includes its limit point), we denote this *cs-open topology* by $C_{cs}(X, Y)$. If $\mathcal{A}$ is the set of all compact subsets of X, denote this *compact-open topology* by $C_k(X, Y)$. The symbol $C^\alpha(X, Y)$, where α is an infinite cardinal, signifies that the topology under consideration is a set-open topology determined by a collection $\mathcal{A}$ such that $\{x\} \in \mathcal{A}$ for every $x \in X$, and for each $A \in \mathcal{A}$, $\delta(A) \leq \alpha$.

2. Pseudocharacter of mapping spaces

Note the obvious fact that if σ and τ are topologies for X and $\sigma \subset \tau$, then $\psi(X, \tau) \leq \psi(X, \sigma)$. In particular, the pseudocharacter of $C_p(X, Y)$ gives an upper bound for the pseudocharacter of any finer topology. Note also that $\psi(Y^X) \geq \psi(Y)$, for it is clear that if $A \subset X$, $\psi(A) \leq \psi(X)$, and the constant functions form a subspace of Y^X homeomorphic to Y.

THEOREM 1. $\psi(Y) \leq \psi(C_p(X, Y)) \leq \psi(Y)\delta(X)$.

Proof. Let D be dense in X with $|D| = \delta(X)$, and let

$f \in C_p(X, Y)$. For each $d \in D$ let $\mathcal{U}(f(d))$ be a collection of open sets such that $|\mathcal{U}(f(d))| \le \psi(Y)$ and $\cap\mathcal{U}(f(d)) = \{f(d)\}$. Let $\mathcal{V} = \{(d, U) \mid U \in \mathcal{U}(f(d))\}$. Then $|\mathcal{V}| \le \psi(Y)\delta(X)$ and $\cap\mathcal{V} = \{f\}$.

THEOREM 2. *Let X be completely regular and let Y contain a non-degenerate path. Then the following are true:*

(1) *if* $\psi(C^{\alpha}(X, Y)) \le \beta$ *, then* $\delta(X) \le \alpha\beta$ *;*

(2) $\psi(C^{\alpha}(X, Y)) \le \alpha\psi(Y)$ *if and only if* $\delta(X) \le \alpha\psi(Y)$.

Proof. To show (1), let $g : [0, 1] \to Y$ be a nondegenerate path such that $g(0) \ne g(1)$. Let f be the constant function which is zero on X . Then $gf \in C^{\alpha}(X, Y)$, so there exists a collection $\mathcal{V}$ of subbasic open sets of the form (A, U) such that $|\mathcal{V}| \le \psi(Y)$ and $\cap\mathcal{V} = \{gf\}$. For each A , $\delta(A) \le \alpha$, so we may choose a set D such that $|D| \le \alpha\psi(Y)$, and such that D is dense in the union of all A such that $(A, U) \in \mathcal{V}$ for some U . Now if D is not dense in X , there is a nonempty open set W such that $W \cap D = \emptyset$. Pick $x \in W$, and by complete regularity choose a function $h : X \to [0, 1]$ such that $h(x) = 1$ and $h(X\backslash W) = 0$. Now gh and gf agree on D and hence on each A . Thus $gh \in \cap\mathcal{V}$ and by construction $gh(x) \ne gf(x)$. This contradiction means D must be dense in X .

For (2) simply combine (1) with Theorem 1 and the observation that $\psi(C_p(X, Y)) \ge \psi(C^{\alpha}(X, Y))$ for all α .

As was pointed out in the review by Katětov [9], not all the implications claimed in the corollary on page 759 of [1] are true. The following corollary generalizes the portions which are correct.

COROLLARY 3. *Let X be completely regular, and let R be the real numbers. Then the following are equivalent for an infinite cardinal* α *:*

(1) $\delta(X) \le \alpha$ *;*

(2) $\psi(C_p(X, R)) \le \alpha$ *;*

(3) *there exists* $f \in R^X$ *such that* $\psi(f, C_p(X, R)) \le \alpha$ *;*

(4) $\psi(C^{\alpha}(X, R)) \le \alpha$ *;*

(5) *there exists* $f \in R^X$ *such that* $\psi(f, C^{\alpha}(X, R)) \leq \alpha$.

Note that if each compact subset of X is separable, which is the case if X is, for example, metrizable or stratifiable, then we may add to the list of equivalences in the corollary

(6) $\psi(C_k(X, R)) \leq \alpha$;

(7) *there exists an* $f \in R^X$ *such that* $\psi(f, C_k(X, R)) \leq \alpha$.

Much work has been done, see for example [7], [8], [11], [13], on obtaining theorems of the following type: Let X have property P and let Y have property Q , then the space of maps from X to Y has property T . In each such theorem in the references just cited the mapping space was of the type $C^{\aleph_0}(X, Y)$, Q was a property of the reals, and the property T was one which implies that each point is a G_{δ} . In each of these theorems the condition P implied that X was separable. In [11] Michael showed that in each of his theorems the separability of X was necessary. The results of this section show that this requirement cannot be relaxed in any of these theorems.

3. Ascoli theorems

The problem under consideration in this section is that of determining the compact subsets of a mapping space. We call theorems which answer this question Ascoli theorems. Kelley [10] gives several Ascoli theorems for $C_k(X, Y)$. See also Bagley and Yang [4]. Noble [12] gives other Ascoli theorems, generalizations, and an extensive bibliography.

Recall that a space X is *sequential* when a subset of X is closed if and only if its intersection with every convergent sequence is closed. Let $F(x)$ denote $\{f(x) \mid f \in F\}$. Definitions of all other terms used below may be found in Kelley [10].

When Theorems 4 and 5 are read omitting the parenthetical portions they are analogues for the cs-open topology of Theorem 18, page 234, and Theorem 21, page 236, of Kelley [10] dealing with the compact-open topology. The proofs are straightforward adaptations of Kelley's techniques and are omitted. When parenthetical portions are included, the

https://doi.org/10.1017/S0004972700041083 Published online by Cambridge University Press

significance of the theorems is that under the stated hypotheses on X and Y, conditions which are weaker than the corresponding parts of Kelley's Theorems become sufficient.

THEOREM 4. *Let X be a sequential space (which is separable) and Y a uniform space (which has each point a G_δ), then $\mathcal{F} \subset Y^X$ is compact in the cs-open topology (and the compact-open topology) if and only if*

(a) $\mathcal{F}$ *is closed,*

(b) $\mathrm{Cl}(\mathcal{F}(x))$ *is compact for every* $x \in X$,

(c) $\mathcal{F}$ *is equicontinuous on every convergent sequence in* X.

THEOREM 5. *Let X be a sequential space (which is separable) and Y a regular space (which has each point a G_δ), then $\mathcal{F} \subset Y^X$ is compact in the cs-open topology (and the compact-open topology) if and orly if*

(a) $\mathcal{F}$ *is closed,*

(b) $\mathrm{Cl}(\mathcal{F}(x))$ *is compact for every* $x \in X$,

(c) $\mathcal{F}$ *is evenly continuous on every convergent sequence in* X.

Proof of Theorems 4 and 5. By Theorem 4 of Guthrie [8], if X is a separable space in which each compact set is sequentially compact, and Y has each point a G_δ, then $C_k(X, Y)$ and $C_{cs}(X, Y)$ have the same compact subsets. Since by Franklin [6] every compact subset of a sequential space is sequentially compact, the theorem applies. Thus *(a)*, *(b)*, and *(c)* are also sufficient in the compact-open topology.

COROLLARY 6. *Let X be a separable sequential space and Y a metric space with metric d. Then $\mathcal{F} \subset Y^X$ is compact in the compact-open topology if and only if*

(a) $\mathcal{F}$ *is closed,*

(b) $\mathrm{Cl}(\mathcal{F}(x))$ *is compact for every* $x \in X$,

(c) for every convergent sequence $\{z_i\}$ converging to z_0, and for every $\varepsilon > 0$, there exists an N such that for every $f \in \mathcal{F}$, $d(f(z_i), f(z_0)) < \varepsilon$ whenever $i \geq N$.

https://doi.org/10.1017/S0004972700041083 Published online by Cambridge University Press

References

[1] Frank W. Anderson, "A lattice characterization of completely regular G_δ-spaces", *Proc. Amer. Math. Soc.* 6 (1955), 757-765.

[2] Richard Arens and James Dugundji, "Topologies for function spaces", *Pacific J. Math.* 1 (1951), 5-31.

[3] C.E. Aull, "A certain class of topological spaces", *Prace Mat.* 11 (1967), 49-53.

[4] R.W. Bagley and J.S. Yang, "On k-spaces and function spaces", *Proc. Amer. Math. Soc.* 17 (1966), 703-705.

[5] W.W. Comfort and Anthony W. Hager, "Estimates for the number of real-valued continuous functions", *Trans. Amer. Math. Soc.* 150 (1970), 619-631.

[6] S.P. Franklin, "Spaces in which sequences suffice", *Fund. Math.* 57 (1965), 107-115.

[7] J.A. Guthrie, "A characterization of $\aleph_0$-spaces", *General Topology and Appl.* 1 (1971), 105-110.

[8] J.A. Guthrie, "Mapping spaces and cs-networks", *Pacific J. Math.* 47 (1973), 465-471.

[9] M. Katětov, Review of [1], *Math. Reviews* 17 (1956), 287.

[10] John L. Kelley, *General topology* (Van Nostrand, Toronto, New York, London, 1955).

[11] E. Michael, "$\aleph_0$-spaces", *J. Math. Mech.* 15 (1966), 983-1002.

[12] N. Noble, "Ascoli theorems and the exponential map", *Trans. Amer. Math. Soc.* 143 (1969), 393-411.

[13] Paul O'Meara, "On paracompactness in function spaces with the compact-open topology", *Proc. Amer. Math. Soc.* 29 (1971), 183-189.

[14] Giovanni Vidossich, "A remark on the density character of function spaces", *Proc. Amer. Math. Soc.* 22 (1969), 618-619.

Department of Mathematics, University of Pittsburgh,
Pennsylvania, USA.