Canad. Math. Bull. Vol. **15** (2), 1972

ON MINIMALLY THIN SETS IN A STOLZ DOMAIN

BY
H. L. JACKSON

Let D denote the open right half plane and

$$K = \{z \in D : |\text{Arg}\, z| \le \theta_0 < \pi/2\}$$

a Stolz domain in D with vertex at the origin. If h is a minimal harmonic function on D with pole at the origin then $E \subset D$ is minimally thin at the origin iff $R_h^E \not\equiv h$ where R_h^E is the reduced function of h on E in the sense of Brelot. We now define

$$I_n = \{z \in D : s^{n+1} \le |z| < s^n\}$$

where s shall be fixed to be $1/e$. For the set $E \cap I_n$ we shall let c_n denote the outer ordinary capacity (see [1, pp. 320–321]), λ_n the outer logarithmic capacity, and σ_n the outer Green capacity with respect to D. If $E \subset K$, Mme. Lelong [3, p. 131] was able to prove that E is minimally thin at the origin iff $\sum_{n=1}^{\infty} \sigma_n < +\infty$. Since one cannot easily relate the classical measure theoretic properties of a plane set with its Green capacity, it would appear desirable to find some other criteria for minimal thinness. If $E \subset \{z : |z| < \frac{1}{2}\}$ it is known (see [1, p. 320]) that

$$c_n = \frac{1}{\log(1/\lambda_n)} \quad \text{if } \lambda_n > 0$$

and that

$$c_n = 0 \qquad \text{iff } \lambda_n = 0.$$

Furthermore the logarithmic capacity can be directly related to some of the classical measure theoretic properties of the set in question (see [4, pp. 84–85]). In an earlier paper (see [2, Theorem 1]), the author was able to prove that $E \subset K$ is minimally thin at 0 if $\overline{\lim}_{n\to\infty} (nc_n) < 1$ and $\sum_{n=1}^{\infty} c_n < +\infty$. It turns out that these conditions, taken together, are sufficient for minimal thinness but not necessary. On the other hand the condition $\sum_{n=1}^{\infty} c_n < +\infty$, by itself is necessary for minimal thinness but not sufficient.

The main purpose of this paper is to give necessary and sufficient conditions for minimal thinness at 0 for a set $E \subset K$ in terms of ordinary and logarithmic capacity. We shall verify that these new conditions provide an improvement of Theorem 1 in [2]. We shall now state and prove a version of our main theorem.

Received by the editors January 27, 1971.

https://doi.org/10.4153/CMB-1972-040-8 Published online by Cambridge University Press

THEOREM 1. *If $E \subset K$, then E is minimally thin at* 0 *iff the following conditions are satisfied*:

(i) *There exists n_0 such that $nc_n < 1$ for all $n \geq n_0$.*

(ii) $\sum_{n=n_0}^{\infty} \frac{c_n}{1-nc_n} < +\infty.$

Proof. We shall employ the following inequalities that were established in [2].

(I) (see inequality (vi) in the proof of Theorem 1 [2]):

$$\frac{\sigma_n}{1+(A+n)\sigma_n} \leq c_n$$

where $A = 1+\log(1/(2\cos\theta_0))$. We note that $A > 0$ for every $\theta_0 \in [0, \pi/2)$.

(II) (see inequality (vi′) in the proof of Theorem 5 [2]):

$$\frac{\sigma_n}{1+(n-\log 2)\sigma_n} \geq c_n.$$

In future we shall ignore all terms where $c_n = 0$ or equivalently $\sigma_n = 0$. We first combine inequalities I and II to obtain

$$\text{(i)} \qquad \frac{\sigma_n}{1+(A+n)\sigma_n} \leq c_n \leq \frac{\sigma_n}{1+(n-\log 2)\sigma_n}$$

or

$$\text{(ii)} \qquad \frac{1}{\sigma_n}+A+n \geq \frac{1}{c_n} \geq \frac{1}{\sigma_n}+n-\log 2.$$

If we subtract n from both inequalities in (ii) we obtain

$$\text{(iii)} \qquad \frac{1}{\sigma_n}+A \geq \frac{1}{c_n}-n \geq \frac{1}{\sigma_n}-\log 2$$

or

$$\text{(iv)} \qquad \frac{1+A\sigma_n}{\sigma_n} \geq \frac{1-nc_n}{c_n} \geq \frac{1-\log 2\sigma_n}{\sigma_n}.$$

We now suppose that E is minimally thin at 0. Then $\sum_{n=1}^{\infty} \sigma_n < +\infty$ which implies that $\lim_{n\to\infty} \sigma_n = 0$ and which in turn implies that $\lim_{n\to\infty} (1/\sigma_n - \log 2) = +\infty$. By the right side of (iii) it follows that $(1/c_n) - n = (1-nc_n)/c_n > 0$ for all n sufficiently large, and therefore $1-nc_n > 0$ for all n sufficiently large. We choose n_0 such that $1/\sigma_n - \log 2 > 0$ if $n \geq n_0$, and note that $1-nc_n > 0$ also when $n \geq n_0$. If $n \geq n_0$ all terms in (iv) are greater than zero so that (iv) is equivalent to

$$\text{(v)} \qquad \frac{\sigma_n}{1+A\sigma_n} \leq \frac{c_n}{1-nc_n} \leq \frac{\sigma_n}{1-\log 2\sigma_n} \quad \text{when } n \geq n_0.$$

Since $\lim_{n\to\infty} \sigma_n = 0$, therefore $(1-\log 2\sigma_n) > \frac{1}{2}$ or $1/(1-\log 2\sigma_n) < 2$, for all n sufficiently large, and it follows that $\sum_{n=1}^{\infty} \sigma_n < +\infty$ implies

$$\sum_{n=n_0}^{\infty} \frac{\sigma_n}{1-\log 2\sigma_n} < +\infty$$

which in turn implies

$$\sum_{n=n_0}^{\infty} \frac{c_n}{1-nc_n} < +\infty.$$

The necessity part of our theorem follows. For the sufficiency we note from (v) that the given conditions of our theorem imply that $\sum_{n=1}^{\infty} [\sigma_n/(1+A\sigma_n)] < +\infty$. The function $f(x) = x/(1+Ax)$ is continuous and monotone strictly increasing on $(-1/A, +\infty)$, and possesses its only zero at $x=0$. The convergence of the series $\sum_{n=1}^{\infty} [\sigma_n/(1+A\sigma_n)]$ implies $\lim_{n\to\infty} (\sigma_n/(1+A\sigma_n)) = 0$, which in turn implies $\lim_{n\to\infty} \sigma_n = 0$. Hence $1+A\sigma_n < 2$ or $1/(1+A\sigma_n) > \frac{1}{2}$ for all n sufficiently large, and it follows that $\sum_{n=1}^{\infty} (\sigma_n/(1+A\sigma_n)) < +\infty$ implies $\sum_{n=1}^{\infty} \sigma_n < +\infty$, which in turn implies minimal thinness of E at 0. This proves the sufficiency and the theorem.

REMARK 1. If $\overline{\lim}_{n\to\infty} nc_n < 1$, then

$$\sum_{n=n_0}^{\infty} \frac{c_n}{1-nc_n} < +\infty \quad \text{iff} \quad \sum_{n=1}^{\infty} c_n < +\infty,$$

so that the sufficiency part of our new theorem is at least as strong as Theorem 1 in [2]. We shall demonstrate by example that it is stronger.

LEMMA. *It is possible for $E \subset K$ to be minimally thin at* 0 *and satisfy the condition that* $\overline{\lim}_{n\to\infty} nc_n = 1$.

Proof. Let us define $E \subset K$ to be a sequence of intervals so that

$$\left.\begin{aligned} c_n &= \frac{1}{n^2} && \text{if } n \neq m^4 \\ &= \frac{1}{n} - \frac{1}{n\sqrt{n}} && \text{if } n = m^4 \end{aligned}\right\}$$

where m runs through the natural numbers. It follows that

$$\left.\begin{aligned} 1-nc_n &= 1-\frac{1}{n} && \text{if } n \neq m^4 \\ &= \frac{1}{\sqrt{n}} && \text{if } n = m^4 \end{aligned}\right\},$$

https://doi.org/10.4153/CMB-1972-040-8 Published online by Cambridge University Press

and that

$$\left.\begin{aligned}\frac{c_n}{1-nc_n} &= \frac{1}{n(n-1)} \quad \text{if } n \neq m^4\\ &= \frac{1}{\sqrt{n}}-\frac{1}{n} \quad \text{if } n = m^4\end{aligned}\right\}.$$

Hence

$$\sum_{n=1}^{\infty}\left(\frac{c_n}{1-nc_n}\right) \le \sum_{n=2}^{\infty}\frac{1}{n(n-1)}+\sum_{m=1}^{\infty}\left(\frac{1}{m^2}-\frac{1}{m^4}\right).$$

Since the series $\sum_{n=1}^{\infty}(c_n/(1-nc_n))$ is a positive series and all series on the right side of the above inequality converge, it follows that

$$\sum_{n=1}^{\infty}\left(\frac{c_n}{1-nc_n}\right) < +\infty$$

and hence E is minimally thin at 0. Nevertheless

$$\left.\begin{aligned}nc_n &= 1-\frac{1}{\sqrt{n}} \quad \text{if } n = m^4\\ &= \frac{1}{n} \qquad\quad \text{if } n \neq m^4\end{aligned}\right\}$$

and it is evident that

$$\overline{\lim_{n\to\infty}}\, nc_n = \lim_{m\to\infty}\left(1-\frac{1}{m^2}\right) = 1.$$

The lemma follows. We shall now rephrase Theorem 1 in terms of logarithmic capacity.

THEOREM 1′. *$E\subset K$ is minimally thin at 0 iff the following conditions are satisfied*:

(i) There exists n_0 such that $(\lambda_n e^n)<1$ for all $n\ge n_0$.

(ii) $\displaystyle\sum_{n=n_0}^{\infty}\frac{1}{\log\left(\frac{1}{\lambda_n e^n}\right)} < +\infty.$

Proof. As before we restrict ourselves to $\{z:|z|<\frac{1}{2}\}$. Then

$$\begin{aligned}c_n &= \frac{1}{\log(1/\lambda_n)} \quad \text{where } 0<\lambda_n<1\\ &= 0 \qquad\qquad\quad \text{if } \lambda_n = 0.\end{aligned}$$

It follows that $nc_n<1$ iff $\log 1/\lambda_n>n$, or $1/\lambda_n>e^n$. The condition of Theorem 1 that

https://doi.org/10.4153/CMB-1972-040-8 Published online by Cambridge University Press

there exists n_0 such that $nc_n < 1$ if $n \geq n_0$ is therefore equivalent to the condition that $\lambda_n e^n < 1$ if $n \geq n_0$. If $c_n \neq 0$, then

$$\frac{c_n}{1-nc_n} = \frac{1}{(1/c_n)-n} = \frac{1}{\log(1/\lambda_n)-n} = \frac{1}{\log(1/\lambda_n e^n)}$$

so that

$$\sum_{n=n_0}^{\infty} \frac{c_n}{1-nc_n} < \infty \quad \text{iff} \quad \sum_{n=n_0}^{\infty} \frac{1}{\log(1/\lambda_n e^n)} < +\infty.$$

It follows that Theorem 1′ is a rephrasing of Theorem 1 in terms of logarithmic capacity.

Remark 2. If $E \subset K$ it would be of interest to develop an integral criterion for minimal thinness in terms of ordinary or logarithmic capacity comparable to the one developed by Brelot (see [1, pp. 334–336]) for ordinary thinness.

Remark 3. In [2, Theorem 5] the author was able to prove that the condition of minimal thinness for $E \subset K$ strictly implies that it is an r-set of finite logarithmic length. Let us examine a set E such that each $E \cap I_n$ is a disk of radius r_n. Then $\lambda_n = r_n$, and E is an r-set of finite logarithmic length iff $\sum_{n=1}^{\infty} \lambda_n e^n < +\infty$. In the particular case where $(\lambda_n e^n)$ is a monotone decreasing sequence the minimal thinness of $E \subset K$ implies that

$$\frac{1}{\log(1/\lambda_n e^n)} < \frac{1}{n \log n}$$

for all n sufficiently large, so that

$$\frac{1}{\lambda_n e^n} > n^n \quad \text{or} \quad \lambda_n e^n < \frac{1}{n^n}$$

for all n sufficiently large. One can easily provide more stringent inequalities but the one above provides evidence that the condition of finite logarithmic length is not a good approximation for minimal thinness.

References

1. M. Brelot, *Points irréguliers et transformations continues en théorie du potentiel*, J. Math. Pures App. **19** (1940), 319–337.

2. H. L. Jackson, *Some results on thin sets in a half-plane*, Ann. Inst. Fourier (Grenoble) (2) **20** (1970), 201–218.

3. J. Lelong, *Étude au voisinage de la frontière des fonctions surharmoniques positives dans un demi-espace*, Ann. de L'École Normale Sup. **66** (1949), 125–159.

4. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.

McMaster University,
Hamilton, Ontario