BULL. AUSTRAL. MATH. SOC. 43A15, 43A20, 46H05, 47B48, 47B07
VOL. 76 (2007) [49–54]

COMPLETELY CONTINUOUS ELEMENTS OF BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

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Let G be a locally compact group and $L_0^\infty(G)$ be the Banach space of all essentially bounded measurable functions on G vansihing an infinity. Here, we study some families of right completely continuous elements in the Banach algebra $L_0^\infty(G)^*$ equipped with an Arens type product. As the main result, we show that $L_0^\infty(G)^*$ has a certain right completely continuous element if and only if G is compact.

1. INTRODUCTION

Let G denote a locally compact group with a fixed left Haar measure λ. The group algebra $L^1(G)$ is defined as in [6] equipped with the convolution product $*$ and the norm $\|\cdot\|_1$. Also, let $L^\infty(G)$ denote the usual Lebesgue space as defined in [6] equipped with the essential supremum norm $\|\cdot\|_\infty$. Then $L^\infty(G)$ is the dual of $L^1(G)$ for the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)d\lambda(x).$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ that vanish at infinity; that is, for each $\varepsilon > 0$, there is a compact subset K of G for which $\|f\chi_{G\backslash K}\|_\infty < \varepsilon$, where $\chi_{G\backslash K}$ denotes characteristic function of $G\backslash K$ on G. For every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we denote by ng the function in $L^\infty(G)$ defined by

$$\langle ng, \phi\rangle = \left\langle n, \frac{1}{\Delta}\tilde{\phi} * g\right\rangle$$

for all $\phi \in L^1(G)$, where $\tilde{\phi}(x) = \phi(x^{-1})$ for all $x \in G$ and Δ denotes the modular function of G. The space $L_0^\infty(G)$ is left introverted in $L^\infty(G)$; that is, for each $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we have $ng \in L_0^\infty(G)$. This lets us endow $L_0^\infty(G)^*$ with the first *Arens product* "." defined by

$$\langle m \cdot n, g\rangle = \langle m, ng\rangle$$

Received 18th October, 2006
The authors wish to thank the referee for his invaluable remarks. They also would like to acknowledge the support of the Center of Excellence for Mathematics at the Isfahan University of Technology.

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https://doi.org/10.1017/S0004972700039460 Published online by Cambridge University Press

for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$. Then $L_0^\infty(G)^*$ with this product is a Banach algebra. This Banach algebra was introduced and studied by Lau and Pym [**9**]; see also Isik, Pym and Ülger [**8**] for the compact group case. The functional $m \cdot \mu \in L_0^\infty(G)^*$ is defined in a similar way for all $m \in L_0^\infty(G)^*$ and $\mu \in M(G)$. The measure algebra of G as defined in [**6**] endowed with the convolution product $*$ and the total variation norm.

Let $\mathfrak{A}$ be a Banach algebra; a bounded operator $T : \mathfrak{A} \to \mathfrak{A}$ is called a *right multiplier* if $T(ab) = aT(b)$ for all $a, b \in \mathfrak{A}$. For any $a \in \mathfrak{A}$, the right multiplier $b \mapsto ba$ on $\mathfrak{A}$ is denoted by ρ_a; also, a is said to be a *right completely continuous element of* $\mathfrak{A}$ if ρ_a is a compact operator on $\mathfrak{A}$. Compact right multipliers on the second dual algebras $L^1(G)^{**}$ and $M(G)^{**}$ have been studied by Ghahramani and Lau in [**3**] and [**4**]; see also Ghahramani and Lau [**5**] and Losert [**11**]. In [**4**], among other things, they have proved that G is amenable if and only if there is a non-zero compact left multiplier on $L^1(G)^{**}$ or $M(G)^{**}$

In this work, we study compact right multipliers on $L_0^\infty(G)^*$. We prove that G is compact if and only if there is a non-zero compact right multiplier on $L_0^\infty(G)^*$. We also study some families of right completely continuous elements of $L_0^\infty(G)^*$.

2. The results

For each $\phi \in L^1(G)$, let ϕ also denote the functional in $L_0^\infty(G)^*$ defined by

$$\langle \phi, g \rangle := \int_G \phi(x) g(x)\, d\lambda(x) \ \big(g \in L_0^\infty(G)\big).$$

Note that this duality defines a linear isometric embedding of $L^1(G)$ into $L_0^\infty(G)^*$. Also, observe that $\phi \cdot \psi = \phi * \psi$ for all $\phi, \psi \in L^1(G)$. It is well known that $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$; see [**9**]. Furthermore, an easy application of Goldstein's Theorem shows that $L^1(G)$ is weak* dense in $L_0^\infty(G)^*$. For any n in $L_0^\infty(G)^*$, the map $m \mapsto m \cdot n$ is weak*-weak* continuous on $L_0^\infty(G)^*$. For an element m in $L_0^\infty(G)^*$, the map $n \mapsto m \cdot n$ is in general not weak*-weak* continuous on $L_0^\infty(G)^*$ unless m is in $L^1(G)$; see Lau and Ülger [**10**] for details.

We begin with the following result which is needed in the sequel. First, let us remark that any right multiplier T on $L_0^\infty(G)^*$ is of the form ρ_m for some $m \in L_0^\infty(G)^*$; indeed, $T = \rho_{T(u)}$ for all $u \in \Lambda_0(G)$, the set of all mixed identities u with norm one in $L_0^\infty(G)^*$; that is, $\phi \cdot u = u \cdot \phi = \phi$ for all $\phi \in L^1(G)$.

Proposition 2.1. *Let G be a locally compact group and $n \in L_0^\infty(G)^*$. Then $\rho_n : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is compact if and only if $\rho_n|_{L^1(G)} : L^1(G) \to L^1(G)$ is compact.*

Proof: Let m be an element in the unit ball of $L_0^\infty(G)^*$. Then there exists a net (ϕ_α) in $L^1(G)$ with $\|\phi_\alpha\|_1 \leqslant 1$ and $\phi_1 \to m$ in the weak* topology of $L_0^\infty(G)^*$. Thus $\phi_\alpha \cdot n \to m \cdot n$ in the weak* topology of $L_0^\infty(G)^*$. So, if $\rho_n|_{L^1(G)} : L^1(G) \to L^1(G)$ is

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compact, then there exists a subnet (ϕ_β) of (ϕ_α) such that $\phi_\beta \cdot n$ converges to an element of $L^1(G)$ in the norm topology. We therefore have $\|\phi_\beta \cdot n - m \cdot n\| \to 0$. This shows that

$$\{m \cdot n : m \in L_0^\infty(G)^*, \|m\| \leqslant 1\} \subseteq \{\phi \cdot n : \phi \in L^1(G), \|\phi\|_1 \leqslant 1\}^{-\|\cdot\|_1}.$$

It follows that n is a right completely continuous element of $L_0^\infty(G)^*$. The converse is trivial. ∎

In the following, the set of all positive functionals in the C^*-algebras $L_0^\infty(G)^*$ is denoted by $P_0(G)$. Furthermore, for $I \subseteq L_0^\infty(G)^*$, the right annihilator of I is denoted by $\operatorname{ran}(I)$ and is defined by $\{r \in I : I \cdot r = \{0\}\}$. Let us remark that $\operatorname{ran}(L_0^\infty(G)^*)$ is the weak* closed ideal

$$\ker(\mathcal{P}) = \{n - u \cdot n : n \in L_0^\infty(G)^*\}$$

in $L_0^\infty(G)^*$ for all $u \in \Lambda_0(G)$; see Isik, Pym and Ülger [**8**, p. 139].

Theorem 2.2. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is compact.*

(b) *$L_0^\infty(G)^*$ has a non-zero right completely continuous in $P_0(G)$.*

(c) *$L_0^\infty(G)^*$ has a right completely continuous element in $L_0^\infty(G)^* \setminus \operatorname{ran}(L_0^\infty(G)^*)$.*

Proof: If G is compact, then the constant function one 1_G is a non-zero right completely continuous element of $L^1(G)$. So, by Proposition 2.1, $1_G \in P_0(G)$ is a non-zero right completely continuous element of $L_0^\infty(G)^*$. That is (a) implies (b). That (b) implies (c) is clear.

To complete the proof, suppose that $L_0^\infty(G)^*$ has a right completely continuous element in $L_0^\infty(G)^* \setminus \operatorname{ran}(L_0^\infty(G)^*)$. The the right multiplier $\rho_n : L^1(G) \to L^1(G)$ is compact. On the other hand, $L^1(G) \cdot n$ is weak* dense in $L_0^\infty(G)^* \cdot n$ by the continuity properties of the first Arens product. This together with $L_0^\infty(G)^* \cdot n \neq \{0\}$ imply that $L^1(G) \cdot n \neq \{0\}$. That is $\rho_n : L^1(G) \to L^1(G)$ is also non-zero. Now, we only need to recall from Sakai [**12**, Theorem 1] that G is compact if there is a non-zero right compact multiplier on $L^1(G)$. ∎

Corollary 2.3. *Let G be a locally compact group. Then G is compact if and only if there is a non-zero compact right multiplier on $L_0^\infty(G)^*$.*

Proof: This follows immediately from Theorem 2.2 together with the fact that $n \in L_0^\infty(G)^* \setminus \operatorname{ran}(L_0^\infty(G)^*)$ if and only if ρ_n is non-zero. ∎

Corollary 2.4. *Let I be a left ideal in $L_0^\infty(G)^*$ such that $\operatorname{ran}(I) = \{0\}$. If G is not compact, then there is no non-zero compact right multiplier on I.*

Proof: Suppose that $T : I \to I$ is a compact right multiplier. Fix $\iota_1, \iota_2 \in I$. Then $T(\iota_1 \cdot \iota_2)$ is a right completely continuous element of $L_0^\infty(G)^*$; indeed, for each $k \in L_0^\infty(G)^*$

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with $\|k\| \leqslant 1$ we have $\iota_2 \cdot k \in I$, hence

$$\begin{aligned} k \cdot T(\iota_1 \cdot \iota_2) &= k \cdot \iota_1 \cdot T(\iota_2) \\ &= T(k \cdot \iota_1 \cdot \iota_2) \\ &\in \big\{T(\iota) : \iota \in I, \|\iota\| \leqslant \|\iota_1\| \, \|\iota_2\|\big\}. \end{aligned}$$

Since G is not compact, it follows from Theorem 2.2 that

$$T(\iota_1 \cdot \iota_2) \in \operatorname{ran}\big(L_0^\infty(G)^*\big).$$

This together with $T(\iota_1 \cdot \iota_2) \in I$ yield that $T(\iota_1 \cdot \iota_2) \in \operatorname{ran}(I)$, and hence $T(\iota_1 \cdot \iota_2) = 0$ by assumption. Thus $I.T(\iota_2) = \{0\}$, and hence $T(\iota_1) \in \operatorname{ran}(I)$. That is, $T(\iota_1) = 0$. ∎

Let us remark that Corollary 2.4 is, in particular, applicable to $L^1(G)$. So, it is a more general statement of Sakai [**12**, Theorem 1].

Theorem 2.5. *Let G be a locally compact group and $n \in L_0^\infty(G)^* \setminus \operatorname{ran}\big(L_0^\infty(G)^*\big)$. Then n is a right completely continuous element of $L_0^\infty(G)^*$ if and only if G is compact and n has the form $n = \phi + r$ for some $\phi \in L^1(G)$ and $r \in \operatorname{ran}\big(L_0^\infty(G)^*\big)$.*

Proof: Suppose that n is a right completely continuous element of $L_0^\infty(G)^*$. Since $L^1(G)$ is an ideal in $L_0^\infty(G)^*$, it follows that $\rho_n|_{L^1(G)}$ is a compact right multiplier on $L^1(G)$. Thus there exists $\phi \in L^1(G)$ with $\rho_n = \rho_\phi$ on $L^1(G)$; see Akemann [**1**]. Now, let $u \in \Lambda_0(G)$, and choose a bounded approximate identity (e_γ) for $L^1(G)$ such that $e_\gamma \to u$ in the weak* topology of $L_0^\infty(G)^*$; see [**2**]. So $e_\gamma \cdot n = e_\gamma \cdot \phi$ for all γ, and thus

$$u \cdot n = u \cdot \phi = \phi$$

by the weak* continuity properties of the Arens product. Therefore

$$\begin{aligned} m \cdot (n - \phi) &= m \cdot n - m \cdot \phi \\ &= m \cdot n - m \cdot (u \cdot n) \\ &= 0 \end{aligned}$$

for all $m \in L_0^\infty(G)^*$. That is $r := n - \phi \in \operatorname{ran}\big(L_0^\infty(G)^*\big)$. Moreover, G is compact by Theorem 2.2.

For the converse, recall from Akemann [**1**, Theorem 4] that if G is compact, then ϕ is a right completely continuous element of $L^1(G)$, and of course a right completely continuous element of $L_0^\infty(G)^*$ by Proposition 2.1. The proof will be complete if we note that $\rho_{\phi+r} = \rho_\phi$ for all $r \in \operatorname{ran}\big(L_0^\infty(G)^*\big)$. ∎

Let $\mathcal{P} : L_0^\infty(G)^* \to M(G)$ be the map that associates to any bounded functional on $L_0^\infty(G)$ its restricton to $C_0(G)$, the Banach space of all continuous functions on G vanishing at infinity; note that $\mathcal{P}$ is an algebra homomorphism; in fact, for each $m, n \in$

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$L_0^\infty(G)^*$, there exist two nets (ϕ_α) and (ψ_β) in $L^1(G)$ with $\phi_\alpha \to m$ and $\psi_\beta \to n$ in the weak* topology of $L_0^\infty(G)^*$, and so

$$m \cdot n = \text{ weak}^* - \lim_\alpha \text{ weak}^* - \lim_\beta \phi_\alpha * \psi_\beta.$$

Corollary 2.6. *Let G be a locally compact group, and n be a right completely continuous element of $L_0^\infty(G)^*$. Then the following statements hold.*

(i) $\mathcal{P}(n) \in L^1(G)$,

(ii) $n - \mathcal{P}(n) \in \operatorname{ran}(L_0^\infty(G)^*)$,

(iii) $u \cdot n = \mathcal{P}(n)$ *for all* $u \in \Lambda_0(G)$,

(iv) $\mathcal{P}(n)$ *is a right completely continuous element of* $L_0^\infty(G)^*$.

(v) ρ_n *is a linear combination of compact right multipliers* ρ_{ϕ_i} *for some positive functions* $\phi_i \in L^1(G)(i = 1, 2, 3, 4)$.

Proof: The first three statements are immediate consequences of Theorem 2.5. The statment (iv) follows from that $\rho_n = \rho_{\mathcal{P}(n)}$. For (v), note that $\mathcal{P}(n)$ is a linear combination of ϕ_i for some positive functions $\phi_i \in L^1(G)$ $(i = 1, 2, 3, 4)$. Now, if ρ_n is non-zero, then G is compact by Theorem 2.2 and so ρ_{ϕ_i} is a compact right multiplier on $L^1(G)$; see Akemann [**1**, Theorem 4]. Now, apply Proposition 2.1. ∎

In the following, let $\Delta_0(G)$ denote the set of all non-zero multiplicative linear functionals on Banach algebra $L_0^\infty(G)$.

Corollary 2.7. *Let G be a locally compact group. Then the following assertiona are equivalent.*

(a) *G is finite.*

(b) *Any $m \in \Delta_0(G)$ is a right completely continuous element of $L_0^\infty(G)^*$.*

(c) *$L_0^\infty(G)^*$ has a right completely continuous element in $\Delta_0(G)$.*

Proof: The implications (a)⇒(b)⇒(c) are trivial. To complete the proof, suppose that $L_0^\infty(G)^*$ has a right completely continuous element n in $\Delta_0(G)$. Then $\mathcal{P}(n)$ is a non-zero multiplicative linear functional on the Banach algebra $C_0(G)$; indeed, $n \in P_0(G)$ and hence $\|\mathcal{P}(n)\| = \|n\| \neq 0$ by [**9**, Lemma 2.5]. So, there is an element $x \in G$ such that $\mathcal{P}(n)$ is a non-zero scalar multiple of the Dirac measure δ_x at x; see for example [**7**, Exercise 20.52]. This together with Corollary 2.6 yield that δ_x is a right completely continuous element of $L_0^\infty(G)^*$. Therefore, the closed unit ball of $L_0^\infty(G)^*$ is norm compact in $L_0^\infty(G)^*$. Thus, $L_0^\infty(G)^*$ is finite dimensional; or equivalently, G is finite. ∎

In our last result, $P(G)$ denotes the set $P_0(G) \cap L^1(G)$ of all positive functions in $L^1(G)$.

Corollary 2.8. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) *G is compact.*

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(b) *Any* $\phi \in L^1(G)$ *is a right completely continuous element of* $L_0^\infty(G)^*$.

(c) *Any* $\phi \in P(G)$ *is a right completely continuous element of* $L_0^\infty(G)^*$.

(d) $L_0^\infty(G)^*$ *has a non-zero right completely continuous element in* $P(G)$.

(e) $L_0^\infty(G)^*$ *has a non-zero right completely continuous element in* $L^1(G)$.

PROOF: Suppose that G is compact. Then any $\phi \in L^1(G)$ is a completely continuous element of $L^1(G)$; see Akemann [**1**, Theorem 4]. This together with Proposition 2.1 imply that ϕ is a completely continuous element of $L_0^\infty(G)^*$. That is, (a) implies (b). Also, the implications (b)⇒(c)⇒(d)⇒(e) are trivial. Finally, (e) implies (a) by Theorem 2.2. ∎

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