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SOME NEW GENERALISATIONS OF INEQUALITIES OF HARDY AND LEVIN–COCHRAN–LEE

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In this paper finite versions of Hardy's discrete, Hardy's integral and the Levin–Cochran–Lee inequalities will be considered and some new generalisations of these inequalities will be derived. Moreover, it will be shown that the constant factors involved in the right-hand sides of the integral results obtained are the best possible.

1. INTRODUCTION

During the 1920's, G. H. Hardy (see [6], or [8]) proved two highly important classical inequalities, the so-called Hardy's discrete and integral inequalities. They are stated in the following two theorems.

THEOREM A. *If $p \in \mathbb{R}$, $p > 1$, and $\mathbf{a} = (a_1, a_2, \dots)$ is a sequence of non-negative real numbers, then the inequality*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

holds, unless $a_n = 0$ for all $n \in \mathbb{N}$. The constant $\left(p/(p-1)\right)^p$ is the best possible.

THEOREM B. *Let $p, k \in \mathbb{R}$, $p > 1$ and $k \neq 1$. Suppose f is a non-negative measurable function such that $x^{1-k/p} f \in L^p(0,\infty)$, and the function F is defined on $(0,\infty)$ by*

$$F(x) = \begin{cases} \displaystyle\int_0^x f(t)\,dt, & k > 1, \\ \displaystyle\int_x^\infty f(t)\,dt, & k < 1. \end{cases}$$

Then

$$\int_0^\infty x^{-k} F^p(x)\,dx < \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx, \tag{2}$$

unless $f \equiv 0$. The constant $\left(p/|k-1|\right)^p$ is the best possible.

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We also need to consider a pair of well-known exponential integral inequalities, discovered much later, in the eighties. They are given in

THEOREM C. *Let $\alpha, \gamma \in \mathbf{R}$, $\alpha \neq 0$, and f be a positive measurable function on $(0,\infty)$ such that $\int_0^\infty x^{\gamma-1} f(x)\,dx < \infty$. Then the inequalities*

$$\int_0^\infty x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right]dx \leqslant e^{\gamma/\alpha}\int_0^\infty x^{\gamma-1} f(x)\,dx, \tag{3}$$

for $\alpha > 0$, and

$$\int_0^\infty x^{\gamma-1} \exp\left[-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right]dx \leqslant e^{\gamma/\alpha}\int_0^\infty x^{\gamma-1} f(x)\,dx, \tag{4}$$

for $\alpha < 0$, hold. The constant $e^{\gamma/\alpha}$ is the best possible for both inequalities.

Inequality (3) is due to Cochran and Lee, [3], while inequality (4) represents its companion result, proved by Love in [7] (see also [8] for both results). Because of the reasons explained in [4] and [5], these results will be called the Levin–Cochran–Lee inequalities.

A natural problem closely related to this topic is to investigate inequalities of the same type as relations (1)–(4), but with one difference: the series on both sides of (1) are replaced by their partial sums and the outer integrals in (2)–(4), instead of being over $(0,\infty)$, are taken over some of its subsets.

In our paper [4] (see also [5]), it was shown that if both series in Hardy's discrete inequality are restricted to a finite number of terms, n, then the best possible constant for (1) can be replaced with a smaller constant dependent on n. That result is

THEOREM D. *If p and a are as in Theorem A and $n \in \mathbf{N}$, $n > 1$, is arbitrary, then*

$$\sum_{k=1}^n \left(\frac{1}{k}\sum_{l=1}^k a_l\right)^p < n^{1-p}\left(\sum_{k=1}^n k^{-1/p}\right)^p \sum_{k=1}^n a_k^p, \tag{5}$$

unless $a_1 = \cdots = a_n = 0$.

On the other hand, two papers, [1] and [2] appeared recently, that deal with some new generalisations of Hardy's integral inequality (2). Bicheng, Zhuohua and Debnath proved the following:

THEOREM E. *Let real numbers p and b be such that $p > 1$ and $b > 0$, and let f be a non-negative measurable function.*

(i) *If $0 < \int_0^b f^p(x)\,dx < \infty$, then*

$$\int_0^b x^{-p}\left(\int_0^x f(t)\,dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{(p-1)/p}\right] f^p(x)\,dx. \tag{6}$$

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The constant $(p/(p-1))^p$ *is the best possible.*

(ii) *If* $0 < \int_b^\infty x^p f^p(x)\,dx < \infty$, *then*

$$\int_b^\infty \left(\int_x^\infty f(t)\,dt\right)^p dx < p^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{1/p}\right] x^p f^p(x)\,dx. \tag{7}$$

The constant p^p *is the best possible.*

Note that Theorem E considered only two particular cases of the parameter k from inequality (2): $k = p$ in (i), and $k = 0$ in (ii).

The main objective of this paper is to obtain generalisations of all the previously mentioned discrete and integral inequalities. First, Theorem D will be improved by providing a smaller upper bound for the left-hand side of inequality (5). That result will prove that the best possible constant factor for the infinite case is not the best possible for the finite series. Furthermore, Theorem E will be generalised to cover all the admissible choices of the parameter k from Hardy's integral inequality (2), that is, the analysis of the finite versions of inequality (2) will be completed. Moreover, the same procedure will be applied to the Levin–Cochran–Lee inequalities (3) and (4) to derive their versions for intervals $(0, b)$ and (b, ∞). Finally, it will be shown that the constants involved in the right-hand sides of all these integral inequalities are the best possible.

The method that will be used in the proofs is mainly based on mixed–means inequalities related to discrete and integral power means, obtained in [4] and [9] (see also [5]). The desired relations will be proved by a careful analysis of the proofs of Theorem A, Theorem B and Theorem C given in [4].

2. Generalisation of Hardy's discrete inequality

In this section we give an improvement of Theorem D.

Theorem 1. *Let* $p \in \mathbf{R}$, $p > 1$, *and let* $\mathbf{a} = (a_1, a_2, \dots)$ *be a sequence of non-negative real numbers. Then the inequality*

$$\sum_{k=1}^n \left(\frac{1}{k}\sum_{l=1}^k a_l\right)^p \leqslant n^{1-p}\left(\sum_{l=1}^n l^{-1/p}\right)^p \sum_{k=1}^n \left(1 - \frac{\sum_{l=1}^{k-1} l^{-1/p}}{\sum_{l=1}^n l^{-1/p}}\right) a_k^p \tag{8}$$

holds for all $n \in \mathbf{N}$. *Equality holds if and only if* $n > 1$ *and* $a_1 = \cdots = a_n = 0$, *or* $n = 1$.

Proof: Let $n \in \mathbf{N}$ be fixed. The discrete non-weighted mixed $(1, p)$–means inequality, obtained in [9], is

$$\left[\frac{1}{n}\sum_{k=1}^n \left(\frac{1}{k}\sum_{l=1}^k a_l\right)^p\right]^{1/p} \leqslant \frac{1}{n}\sum_{k=1}^n \left(\frac{1}{k}\sum_{l=1}^k a_l^p\right)^{1/p}, \tag{9}$$

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with equality only if $a_1 = \cdots = a_n$. Denote $S_n = \sum_{l=1}^{n} l^{-1/p}$. Starting from (9), a simple calculation yields the following sequence of relations:

$$\begin{aligned}
\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{l=1}^{k}a_l\right)^p &\leqslant n^{1-p}\left[\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{l=1}^{k}a_l^p\right)^{1/p}\right]^p \\
&= n^{1-p}S_n^p\left[\sum_{k=1}^{n}\frac{k^{-1/p}}{S_n}\left(\sum_{l=1}^{k}a_l^p\right)^{1/p}\right]^p \leqslant n^{1-p}S_n^p\sum_{k=1}^{n}\frac{k^{-1/p}}{S_n}\sum_{l=1}^{k}a_l^p. \qquad (10)
\end{aligned}$$

The inequality in the second line of (10) is obtained by applying Jensen's inequality to the convex function $x \mapsto x^p$. The last term in this inequality is equal to

$$\begin{aligned}
n^{1-p}S_n^{p-1}\sum_{k=1}^{n}k^{-1/p}\sum_{l=1}^{k}a_l^p &= n^{1-p}S_n^{p-1}\sum_{k=1}^{n}\left(\sum_{l=k}^{n}l^{-1/p}\right)a_k^p \\
&= n^{1-p}\left(\sum_{l=1}^{n}l^{-1/p}\right)^p\sum_{k=1}^{n}\left(1-\frac{\sum_{l=1}^{k-1}l^{-1/p}}{\sum_{l=1}^{n}l^{-1/p}}\right)a_k^p,
\end{aligned}$$

so (8) is proved. To complete the proof, we discuss the conditions for achieving equality in (8). The case $n = 1$ is trivial. Observe that for $n > 1$ equality will occur if and only if there is equality in (9) and in Jensen's inequality, that is, only if

$$a_1 = \cdots = a_n \quad \text{and} \quad a_1^p = a_1^p + a_2^p = \cdots = a_1^p + a_2^p + \cdots + a_n^p,$$

that is, $a_1 = \cdots = a_n = 0$. $\square$

REMARK 1. Note that $\sum_{l=1}^{n} l^{-1/p} < \int_0^n x^{-1/p}\,dx = (p/(p-1))n^{(p-1)/p}$, since the left-hand side of this inequality is obviously the lower Darboux sum for $\int_0^n x^{-1/p}dx$. Therefore, inequality (8) is sharper than the relation

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{l=1}^{k}a_l\right)^p \leqslant \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\left(1-\frac{\sum_{l=1}^{k-1}l^{-1/p}}{\sum_{l=1}^{n}l^{-1/p}}\right)a_k^p.$$

Unfortunately, we do not know whether the constant $n^{1-p}\left(\sum_{l=1}^{n} l^{-1/p}\right)^p$ is the best possible constant factor α_n involved in the right-hand side of (8) or not. However, the contribution of Theorem 1 is in providing an explicit upper bound for α_n, dependent on n and less than $(p/(p-1))^p$.

3. Generalisation of Hardy's integral inequality

Our next step is to obtain an integral version of Theorem 1. The results that generalise the inequalities (6) and (7) are given in the following theorem.

THEOREM 2. *Let p, k and b be real numbers such that $p > 1$, $k \neq 1$ and $b > 0$, and let f be a non-negative measurable function.*

(i) *If $k > 1$ and $0 < \int_0^b x^{p-k} f^p(x)\,dx < \infty$, then*

$$\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx < \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)\,dx. \tag{11}$$

(ii) *If $k < 1$ and $0 < \int_b^\infty x^{p-k} f^p(x)\,dx < \infty$, then*

$$\int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx < \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] x^{p-k} f^p(x)\,dx. \tag{12}$$

The constant $\left(p/|k-1|\right)^p$ is the best possible for both inequalities.

PROOF: The proof is based on a careful analysis of some results obtained in [4] (see also [5]). Consider the case $k > 1$ first. In [4] we proved the following inequality for non-negative measurable functions:

$$\begin{aligned}(13)\quad &\left\{\frac{1}{(b-a)^\gamma}\int_a^b (x-a)^{\gamma-1}\left[\frac{1}{(x-a)^\alpha}\int_a^x (t-a)^{\alpha-1} f^r(t)\,dt\right]^{s/r} dx\right\}^{1/s} \\ &\qquad \leqslant \left\{\frac{1}{(b-a)^\alpha}\int_a^b (x-a)^{\alpha-1}\left[\frac{1}{(x-a)^\gamma}\int_a^x (t-a)^{\gamma-1} f^s(t)\,dt\right]^{r/s} dx\right\}^{1/r},\end{aligned}$$

where the parameters $a, b, \alpha, \gamma, r, s \in \mathbf{R}$ are such that $a < b$ and $r < s$, $r, s \neq 0$. If $r = 1$, $s = p > 1$, $\alpha = 1$, $\gamma = p - k + 1$, $a = 0$ and $b > 0$ is arbitrary, (13) can be written in the form

$$\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx \leqslant b^{1-k}\left\{\int_0^b x^{((k-1)/p)-1}\left[\int_0^x t^{p-k} f^p(t)\,dt\right]^{1/p} dx\right\}^p. \tag{14}$$

Since $\int_0^b x^{((k-1)/p)-1}\,dx = \left(p/(k-1)\right) b^{(k-1)/p}$, the right-hand side of (14) is equal to

$$\begin{aligned}&\left(\frac{p}{k-1}\right)^p \left\{\frac{1}{\int_0^b x^{((k-1)/p)-1}\,dx}\int_0^b x^{((k-1)/p)-1}\left[\int_0^x t^{p-k} f^p(t)\,dt\right]^{1/p} dx\right\}^p \\ &\qquad < \left(\frac{p}{k-1}\right)^p \frac{1}{\int_0^b x^{((k-1)/p)-1}\,dx}\int_0^b x^{((k-1)/p)-1}\int_0^x t^{p-k} f^p(t)\,dt\,dx \\ &\qquad = \left(\frac{p}{k-1}\right)^{p-1} b^{(1-k)/p}\int_0^b x^{((k-1)/p)-1}\int_0^x t^{p-k} f^p(t)\,dt\,dx \\ &\qquad = \left(\frac{p}{k-1}\right)^{p-1} b^{(1-k)/p}\int_0^b t^{p-k} f^p(t)\int_t^b x^{((k-1)/p)-1}\,dx\,dt \\ (15)\quad &\qquad = \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{t}{b}\right)^{(k-1)/p}\right] t^{p-k} f^p(t)\,dt,\end{aligned}$$

so (11) is proved. The inequality in the second line of (15) is obtained by using Jensen's inequality for the convex function $x \mapsto x^p$, applied to the term $\left[\int_0^x t^{p-k} f^p(t)\,dt\right]^{1/p}$, while the equality in the fourth line of (15) is a consequence of Fubini's theorem. Note that the inequality sign in (15) is strict, owing to the conditions on f from the statement of Theorem 2.

We prove that the constant $B = \left(p/(k-1)\right)^p$ is the best possible for inequality (11). If it is not true, then there exists a smaller constant C, $0 < C < B$, such that

$$\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx < C\int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)\,dx. \tag{16}$$

Since $0 < \left(p/(k-1+y)\right)^p \nearrow B$ as $y \searrow 0$, there exists a small positive number ε such that $0 < C < \left(p/(k-1+\varepsilon)\right)^p < B$. Let the function $f_\varepsilon : (0,b] \to \mathbf{R}$ be defined by $f_\varepsilon(x) = x^{((k-1+\varepsilon)/p)-1}$. Then we have

$$\int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f_\varepsilon^p(x)\,dx \leqslant \int_0^b x^{p-k} f_\varepsilon^p(x)\,dx = \int_0^b x^{\varepsilon-1}dx = \frac{b^\varepsilon}{\varepsilon}$$

and therefore,

$$\begin{aligned}\int_0^b x^{-k}\left(\int_0^x f_\varepsilon(t)\,dt\right)^p dx &= \left(\frac{p}{k-1+\varepsilon}\right)^p \int_0^b x^{\varepsilon-1}dx \\ &= \left(\frac{p}{k-1+\varepsilon}\right)^p \cdot \frac{b^\varepsilon}{\varepsilon} > C\cdot\frac{b^\varepsilon}{\varepsilon} \geqslant C\int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f_\varepsilon^p(x)\,dx.\end{aligned}$$

This contradicts (16), so B is the best possible constant for (11).

The proof of (12) is similar. It is based on the relation

$$\begin{aligned}&\left\{\frac{1}{b^\gamma}\int_b^\infty x^{\gamma-1}\left[\frac{1}{x^\alpha}\int_x^\infty t^{\alpha-1} f^r(t)\,dt\right]^{s/r} dx\right\}^{1/s} \\ &\qquad\leqslant \left\{\frac{1}{b^\alpha}\int_b^\infty x^{\alpha-1}\left[\frac{1}{x^\gamma}\int_x^\infty t^{\gamma-1} f^s(t)\,dt\right]^{r/s} dx\right\}^{1/r},\end{aligned} \tag{17}$$

also obtained in [4], that holds for non-negative measurable function f and $\alpha, \gamma, b, r, s \in \mathbf{R}$ such that $b > 0$ and $r < s$, $r, s \neq 0$. Inequality (12) follows by starting from (17), rewritten with $r = 1$, $s = p > 1$, $\alpha = 1$, $\gamma = p - k + 1$ and arbitrary $b > 0$ as parameters, then applying Jensen's inequality, and finally, by using Fubini's theorem. Now, it is only left to prove that $D = \left(p/(1-k)\right)^p$ is the best possible constant for (12). If it is not true, then the constant factor D on the right-hand side of this relation can be replaced with a smaller constant $0 < K < D$. Considering $\varepsilon > 0$ such that $K < \left(p/(1-k+\varepsilon)\right)^p < D$, and the function $g_\varepsilon : [b,\infty) \to \mathbf{R}$ defined by $g_\varepsilon(x) = x^{((k-1-\varepsilon)/p)-1}$, as in the previous case we obtain

$$\int_b^\infty x^{-k}\left(\int_x^\infty g_\varepsilon(t)\,dt\right)^p dx > K\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] x^{p-k} g_\varepsilon^p(x)\,dx.$$

This is a contradiction, and hence D is the best possible constant for (12). That completes the proof. $\square$

4. Some improvements of the Levin–Cochran–Lee inequalities

To conclude this paper, we establish generalisations of inequalities (3) and (4) from Theorem C of the same type as the improvements of (1) and (2) obtained in the previous two sections. We also discuss the best possible constants for these inequalities. The results are given in the next theorem.

Theorem 3. *Let $\alpha, \gamma, b \in \mathbf{R}$ be such that $\alpha \neq 0$ and $b > 0$, and let f be a positive measurable function.*

(i) *If $\alpha > 0$ and $0 < \int_0^b x^{\gamma-1} f(x)\,dx < \infty$, then*

$$\int_0^b x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right]dx < e^{\gamma/\alpha}\int_0^b \left[1-\left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)\,dx. \tag{18}$$

(ii) *If $\alpha < 0$ and $0 < \int_b^\infty x^{\gamma-1} f(x)\,dx < \infty$, then*

$$\int_b^\infty x^{\gamma-1} \exp\left[-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right]dx < e^{\gamma/\alpha}\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{-\alpha}\right] x^{\gamma-1} f(x)\,dx. \tag{19}$$

The constant $e^{\gamma/\alpha}$ is the best possible for both inequalities.

Proof: First, let $\alpha > 0$. To begin the proof, we consider the following inequality from [4] (see also [5]):

$$\left\{\frac{1}{(b-a)^\gamma}\int_a^b (x-a)^{\gamma-1}\left[\exp\left(\frac{\alpha}{(x-a)^\alpha}\int_a^x (t-a)^{\alpha-1}\log f(t)\,dt\right)\right]^s dx\right\}^{1/s}$$
$$\leqslant \exp\left\{\frac{\alpha}{(b-a)^\alpha}\int_a^b (x-a)^{\alpha-1}\log\left[\frac{1}{(x-a)^\gamma}\int_a^x (t-a)^{\gamma-1} f^s(t)\,dt\right]^{1/s} dx\right\},$$

where $a, b, s \in \mathbf{R}$ are such that $a < b$ and $s > 0$. If this is rewritten with $s = 1$ and $a = 0$, we have

$$\int_0^b x^{\gamma-1}\exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right]dx \tag{20}$$
$$\leqslant b^\gamma \exp\left\{\frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\log\left[\frac{1}{x^\gamma}\int_0^x t^{\gamma-1} f(t)\,dt\right]dx\right\}.$$

Since $\int_0^b x^{\alpha-1}\log x\,dx = (b^\alpha/\alpha)\big(\log b - (1/\alpha)\big)$, the second line of (20) is equal to

$$b^\gamma \exp\left\{\frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\log\frac{1}{x^\gamma}\,dx + \frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\log\left[\int_0^x t^{\gamma-1} f(t)\,dt\right]dx\right\}$$
$$= e^{\gamma/\alpha}\exp\left\{\frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\log\left[\int_0^x t^{\gamma-1} f(t)\,dt\right]dx\right\}$$
$$< e^{\gamma/\alpha}\cdot\frac{\alpha}{b^\alpha}\cdot\int_0^b x^{\alpha-1}\int_0^x t^{\gamma-1} f(t)\,dt\,dx. \tag{21}$$

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The inequality in (21) was obtained by Jensen's inequality for the convex function $x \mapsto e^x$, applied to the term $\log\left[\int_0^x t^{\gamma-1} f(t)\,dt\right]$. Note that under the conditions of the theorem the inequality really is strict. Further, using Fubini's theorem the last line in (21) is equal to

$$e^{\gamma/\alpha}\cdot\frac{\alpha}{b^\alpha}\cdot\int_0^b t^{\gamma-1}f(t)\int_t^b x^{\alpha-1}dx\,dt = e^{\gamma/\alpha}\int_0^b\left[1-\left(\frac{t}{b}\right)^\alpha\right]t^{\gamma-1}f(t)\,dt,$$

so inequality (18) is proved. Consider the case $\alpha < 0$ now. The companion result (19) can be derived by the same technique, but this time we start from the relation

$$(22)\quad \left\{\frac{1}{b^\gamma}\int_b^\infty x^{\gamma-1}\left[\exp\left(-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right)\right]^s dx\right\}^{1/s} \leqslant \exp\left\{-\frac{\alpha}{b^\alpha}\int_b^\infty x^{\alpha-1}\log\left[\frac{1}{x^\gamma}\int_x^\infty t^{\gamma-1}f^s(t)\,dt\right]^{1/s}dx\right\},$$

where s is a positive real parameter. This is obtained in [4] (se also [5]). Inequality (19) is a consequence of (22) rewritten for $s = 1$, Jensen's inequality and Fubini's theorem.

We prove that $e^{\gamma/\alpha}$ is the best possible constant for (18). For any $\varepsilon > 0$ and the function $f_\varepsilon : (0, b] \to \mathbf{R}$ defined by $f_\varepsilon(x) = \alpha e^{-\gamma/\alpha}x^{\alpha\varepsilon-\gamma}$, the left-hand side of (18) is equal to:

$$\begin{aligned} L_\varepsilon &= \int_0^b x^{\gamma-1}\exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f_\varepsilon(t)\,dt\right]dx \\ &= \int_0^b x^{\gamma-1}\exp\left[\frac{\alpha}{x^\alpha}\log\left(\alpha e^{-\gamma/\alpha}\right)\int_0^x t^{\alpha-1}dt + \frac{\alpha}{x^\alpha}(\alpha\varepsilon-\gamma)\int_0^x t^{\alpha-1}\log t\,dt\right]dx \\ &= \alpha e^{-\varepsilon}\int_0^b x^{\alpha\varepsilon-1}dx = \frac{b^{\alpha\varepsilon}}{\varepsilon}\cdot e^{-\varepsilon}, \end{aligned}$$

while on the right-hand side of (18) we have

$$\begin{aligned} R_\varepsilon &= e^{\gamma/\alpha}\int_0^b\left[1-\left(\frac{x}{b}\right)^\alpha\right]x^{\gamma-1}f_\varepsilon(x)\,dx \leqslant e^{\gamma/\alpha}\int_0^b x^{\gamma-1}f_\varepsilon(x)\,dx \\ &= \alpha\int_0^b x^{\alpha\varepsilon-\gamma}dx = \frac{b^{\alpha\varepsilon}}{\varepsilon}. \end{aligned}$$

The desired result follows immediately, since

$$1 \leqslant \frac{R_\varepsilon}{L_\varepsilon} \leqslant e^\varepsilon \to 1, \quad \text{as } \varepsilon \to 0.$$

The proof that $e^{\gamma/\alpha}$ is the best possible constant for (19) is similar, if the function $g_\varepsilon : [b,\infty) \to \mathbf{R}$, $g_\varepsilon(x) = -\alpha e^{-\gamma/\alpha}x^{\alpha\varepsilon-\gamma}$, is considered. $\square$

References

[1] Y. Bicheng, Z. Zhuohua and L. Debnath, 'On new generalizations of Hardy's integral inequality', *J. Math. Anal. Appl.* **217** (1998), 321–327.

[2] Y. Bicheng and L. Debnath, 'Generalizations of Hardy's integral inequalities', *Internat. J. Math. Math. Sci.* **22** (1999), 535–542.

[3] J.A. Cochran and C.-S. Lee, 'Inequalities related to Hardy's and Heinig's', *Math. Proc. Cambridge Philos. Soc.* **96** (1984), 1–7.

[4] A. Čižmešija and J. Pečarić, 'Mixed means and Hardy's inequality', *Math. Inequal. Appl.* **1** (1998), 491–506.

[5] A. Čižmešija and J. Pečarić, 'Classical Hardy's and Carleman's inequalities and mixed means', in *Survey on Classical Inequalities*, (T. M. Rassias, Editor) (Kluwer Academic Publishers, Dordrecht, Boston, London, 2000), pp. 27–65.

[6] G. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, (second edition) (Cambridge University Press, Cambridge, 1967).

[7] E.R. Love, 'Inequalities related to those of Hardy and of Cochran and Lee', *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 395–408.

[8] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Inequalities involving functions and their integrals and derivatives* (Kluwer Academic Publishers, Dordrecht, Boston, London, 1991).

[9] B. Mond and J. Pečarić, 'A mixed means inequality', *Austral. Math. Soc. Gazette* **23** (1996), 67–70.

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