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ON RINGS WITH NIL COMMUTATOR IDEAL

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Let R be a ring in which for each x, y in R there exists a positive integer $n = n(x, y)$ such that $(xy)^n - (yx)^n$ is in the center of R. Then R has a nil commutator ideal.

A theorem of Belluce, Herstein and Jain [2] states that, if R is a ring in which for each x, y in R there exists integers $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $(xy)^m = (yx)^n$, then the commutator ideal of R is nil. Our objective is to generalize the above result for the case where $m(x, y) = n(x, y)$. Indeed, we prove that, if R is a ring in which for each x, y in R there exists an integer $n = n(x, y) \geq 1$ such that $(xy)^n - (yx)^n$ is in the center of R, then R has a nil commutator.

In preparation for the proofs of our main theorem, we first consider the following lemmas. Throughout, R will denote a ring, Z will denote the center of R, and J the Jacobson radical of R. We use the standard notation $[x, y] = xy - yx$.

The first two lemmas are known and we omit their proofs.

LEMMA 1. *If* $[x, y]$ *commutes with* x, *then* $[x^k, y] = kx^{k-1}[x, y]$.

LEMMA 2. *Let* d *be a derivation of* R. *If* $x \in R$ *is such that* $d^2(x) = 0$ *then* $d^k(x^k) = k!(d(x))^k$ *for all* $k \geq 1$.

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LEMMA 3. *If R is a ring in which for each x, y in R, there exists an integer $n = n(x, y) \geq 1$ such that $(xy)^n - (yx)^n \in Z$. Then for each $a \in J$, $x \in R$ there exist integers $n = n(x, a) \geq 1$ and $m = m(x, a) \geq 1$ such that $(4m)!\,[a, x^{2n}]^{4m} = 0$.*

Proof. Let $a \in J$, $x \in R$. $(1+a)$ is formally invertible (R need not have an identity element). Using the hypothesis for the elements $(1+a)x$ and $x(1+a)^{-1}$, there exists an integer $n \geq 1$ such that

$$\left((1+a)x^2(1+a)^{-1}\right)^n - x^{2n} \in Z .$$

Thus

$$(1+a)x^{2n} - x^{2n}(1+a) = (1+a)\left((1+a)x^{2n}(1+a)^{-1} - x^{2n}\right) ,$$

$$\left((1+a)x^{2n} - x^{2n}(1+a)\right)(1+a) = (1+a)\left((1+a)x^{2n} - x^{2n}(1+a)\right) .$$

Hence

(1) $$\left(ax^{2n} - x^{2n}a\right)a = a\left(ax^{2n} - x^{2n}a\right) , \quad a \in J , \quad x \in R .$$

Let $d(y) = ay - ya$. d is a derivation of R. Using (1), $d^2\left(x^{2n}\right) = 0$. Applying (1) for x^{4n} instead of x, there exists an integer $m \geq 1$ such that

$$\left(a\left(x^{4n}\right)^{2m} - \left(x^{4n}\right)^{2m}a\right)a = a\left(a\left(x^{4n}\right)^{2m} - \left(x^{4n}\right)^{2m}a\right) .$$

Thus, $d^2\left(x^{8mn}\right) = 0$. Hence, by Lemma 2,

$$0 = d^{4m}\left(\left(x^{2n}\right)^{4m}\right) = (4m)!\left(d\left(x^{2n}\right)\right)^{4m}$$

and so $(4m)!\,[a, x^{2n}]^{4m} = 0$.

Theorem 1 below is proved in [1] and we omit its proof here.

THEOREM 1. *If R is a semisimple ring in which, for each x, y in R there exists an integer $n = n(x, y) \geq 1$ such that $(xy)^n = (yx)^n \in Z$. Then R is commutative.*

THEOREM 2. *Let R be a ring in which, for each x, y in R there exists an integer $n = n(x, y) \geq 1$ such that $(xy)^n - (yx)^n \in Z$. Then the commutator ideal of R is nil. Equivalently, if R has no nonzero* *nil ideals then R is commutative.*

Proof. To prove that the commutator ideal of R is nil it is enough to show that if R has no nonzero nil ideals then it is commutative. So we suppose that R has no nonzero nil ideals. Then R is a subdirect product of prime rings R_α, having no nonzero nil ideals, such that in each R_α there is a nonnilpotent element b_α in which $b_\alpha^{t(I)} \in I$ for every nonzero ideal I of R_α. Clearly, R_α satisfies the condition $(xy)^n - (yx)^n \in Z_\alpha$[center of R_α]. So we may assume that R is a prime ring, having no nonzero nil ideals, in which there is a nonnilpotent element $b \in R$ such that $b^{t(I)} \in I$ for all nonzero ideals I of R. We may assume that $J \neq 0$, otherwise the result follows from Theorem 1. If char $R = p \neq 0$, then, by (1), for any $x \in R$ and $a \in J$, there exists an integer $n = n(a, x) \geq 1$ such that

$$[a, [a, x^{2n}]] = 0 .$$

Hence, by Lemma 1,

$$[a^p, x^{2n}] = pa^{p-1}[a, x^{2n}] = 0 .$$

So for any $x, y \in J$, $[x^p, y^{2n}] = 0$. This implies by [3] that J is commutative, and therefore R is commutative since it is prime and has a nonzero commutative ideal [4].

So we may assume that char $R = 0$, and since R is prime with char $R = 0$, R is torsion-free.

CLAIM 1. Every zero divisor in R is nilpotent.

To prove Claim 1, suppose that $ac = 0$, with $a \neq 0$ and c nonnilpotent. Let $A = \{x \in R : xc^r = 0 \text{ for some } r \geq 1\}$ and $B = \{x \in R : c^s x = 0 \text{ for some } s \geq 1\}$. Then A is a left ideal of R, and B is a right ideal. $A \neq 0$ since $0 \neq a \in A$. If $x \in A$, then $xc^r = 0$ for some $r \geq 1$, and hence $(c^r x)^2 = 0$. By hypothesis, there exists an integer $n \geq 1$ such that $(c^r(x+c^r))^n - ((x+c^r)c^r)^n \in Z$. This implies that $(c^{2r})^{n-1}c^r x \in Z$. So $0 = c^{(2n-1)r}xc^r = c^r c^{(2n-1)r}x$, and

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hence $c^s x = 0$ for a positive integer s. Thus $x \in B$, and $A \subset B$. Similarly, $B \subset A$. So $A = B$ and hence A is an ideal of R. Since A is a nonzero ideal of R, $b^t \in A$ for some $t \geq 1$. Thus $b^t c^r = 0$ for some $r \geq 1$, and since c is not nilpotent, then b^t is a zero divisor. Now we can repeat the above argument to show that the set $C = \{x \in R : (b^t)^u x = 0 \text{ for some } u \geq 1\}$ is an ideal of R. Since $c^r \neq 0 \in C$, $C \neq 0$, and hence $b^k \in C$ for some $k \geq 1$. So $b^{tu} \cdot b^k = 0$. This contradicts the fact that b is nonnilpotent. This proves Claim 1.

CLAIM 2. R has no nonzero nilpotent elements.

To prove Claim 2, suppose that $u^2 = 0$ with $y \neq 0$. Then every element of yR is a zero divisor, and hence by Claim 1 every element of yR is nilpotent. Thus yR is a nil right ideal, and so

(2) $$yR \subset J .$$

If $Z = 0$, then by hypothesis, for every c, d in R there exists an integer $n = n(c, d) \geq 1$ such that $(cd)^n = (dc)^n$, which implies by [2] that R is commutative. So we may assume that $Z \neq 0$, and let $0 \neq z \in Z$. Since R is prime and $0 \neq z \in Z$, then

(3) $$z \text{ is not a zero divisor, } 0 \neq z \in Z .$$

Let $a \in J$. Using (1) with $(y+z)$ instead of x, there exists an integer $n \geq 1$ such that

(4) $$\left(a(y+z)^{2n}-(y+z)^{2n}a\right)a = a\left(a(y+z)^{2n}-(y+z)^{2n}a\right) .$$

Since $y^2 = 0$ and $z \in Z$, (4) implies

$$\left[a\left(2nz^{2n-1}y+z^{2n}\right)-\left(2nz^{2n-1}y+z^{2n}\right)a, a\right] = 0 ,$$

and hence

$$2nz^{2n-1}[[a, y], a] = 0 ,$$

and since R is torsion-free and z^{2n-1} is not a zero divisor

(5) $$[[a, y], a] = 0 \text{ for all } a \in J , \ y^2 = 0 .$$

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Using induction on the index of nilpotence of nilpotent elements v and proceeding as above yields that

(6) $\quad [[a, v], a] = 0$ for all $a \in J$, and all nilpotents v.

Since yR is nil, (2) and (6) imply that

$$[[a, v], a] = 0 \text{ for all } a, v \text{ in } yR .$$

Hence yR is a nil right ideal satisfying a polynomial identity. So by Lemma 2.1.1 of [4], R has a nonzero nilpotent ideal, a contradiction. Hence $yR = 0$, and so $yx = 0$ for all $x \in R$. Thus every element of R is a zero divisor, and hence nilpotent by Claim 1. This is a contradiction since R has no nonzero nil ideals. Thus $y = 0$ and Claim 2 is now proved.

Now we can complete the proof of Theorem 2. By Lemma 3, for each $a \in J$, $x \in R$, there exist integers $n = n(x, a) \geq 1$ and $m = m(x, a) \geq 1$ such that $(4m)!\left[a, x^{2n}\right]^{4m} = 0$. Using Claim 2, and that R is torsion-free, we get $\left[a, x^{2n}\right] = 0$. Thus for every $x, a \in J$ there exists an integer $n = n(x, a) \geq 1$ such that $\left[a, x^{2n}\right] = 0$, and hence J is commutative [3]. R is prime, and has a commutative nonzero ideal J, hence R is commutative [4]. This completes the proof of Theorem 2.

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