MAXIMAL CONNECTED TOPOLOGIES

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If $(X, \mathscr{T})$ is a set X with topology $\mathscr{T}$ we shall say that $\mathscr{T}$ is connected if $(X, \mathscr{T})$ is a connected topological space. We shall investigate the existence of and the properties of maximal connected topologies.

If $A \subset X$ the interior of A will be denoted Int (A) and the closure of A will be $\mathrm{Cl}(A)$. If it is necessary to distinguish between topologies on the same set we shall use subscripts. For example $\mathrm{Cl}_2(A)$ will denote the closure of set A with respect to topology $\mathscr{T}_2$. If S is a set of subsets of X, $\Phi(S)$ will denote the topology generated by S. If $V \subset X$ we let $(V, \mathscr{T}|V)$ denote the set V with the topology induced by $\mathscr{T}$. If $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on X we shall denote $K(\mathscr{T}_1, \mathscr{T}_2) = \{x \in X | x$ has a neighborhood for $\mathscr{T}_2$ which is not a neighborhood of x for $\mathscr{T}_1\}$.

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DEFINITION 1. A topology $\mathscr{T}$ on a set X will be said to be *finer* than a topology $\mathscr{T}_1$ on X if $\mathscr{T}_1 \subset \mathscr{T}$. If in addition we have $\mathscr{T} \neq \mathscr{T}_1$ we say that $\mathscr{T}$ is *strictly finer* than $\mathscr{T}_1$. A connected topology $\mathscr{T}$ will be said to be *maximal connected* if $\mathscr{T}_1$ strictly finer than $\mathscr{T}$ implies $\mathscr{T}_1$ is not connected.

EXAMPLE 1. Let X be any non-empty set and let $x \in X$. Define $\mathscr{T}$ by letting $V \in \mathscr{T}$ if $V \subset X$ and $V = \emptyset$ or $x \in V$. Then $\mathscr{T}$ is maximal connected.

EXAMPLE 2. Let X be any set, $x \in X$. Let $V \subset X$ belong to $\mathscr{T}$ if and only if $x \notin V$ or $V = X$. Then $\mathscr{T}$ is maximal connected.

We note that each of these examples is T_0. This condition is necessary.

THEOREM 1. *Let $\mathscr{T}_1$ be a maximal connected topology on X. Then $(X, \mathscr{T}_1)$ is T_0.*

PROOF. Suppose $x, y \in X$, $x \in \mathrm{Cl}_1(\{y\})$, $y \in \mathrm{Cl}_1(\{x\})$, $x \neq y$. Let $\mathscr{T}_2 = \Phi(\mathscr{T}_1 \cup \{\{y\}\})$. Then $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$, hence $\mathscr{T}_2$ is not connected. Suppose (A, B) is an open partition of $(X, \mathscr{T}_2)$. Then either $\{x, y\} \subset A$ or $\{x, y\} \subset B$ for the neighborhoods of x are the same for $\mathscr{T}_1$ and $\mathscr{T}_2$. So suppose $\{x, y\} \subset A$. Then $A \in \mathscr{T}_1$ for there exists an open $\mathscr{T}_1$ neighborhood V of x, $V \subset A$. But V is a neighborhood of y contained in A, and every other point of A has a neighborhood contained in A since the neighborhoods of $z \neq y$ are the same for $\mathscr{T}_1$ and $\mathscr{T}_2$. Clearly $B \in \mathscr{T}_1$. Thus (A, B) is an open partition of $(X, \mathscr{T}_1)$, a contradiction.

Neither of the topological spaces in the previous examples are T_1. However, there do exist maximal connected topologies which are T_1, as the following example will show. To the author's knowledge it is an open question whether there are maximal connected T_2 topologies.

EXAMPLE 3. Let X be an infinite set and let $\mathscr{F}$ be the filter of complements of finite sets, $\mathscr{U}$ an ultrafilter finer than $\mathscr{F}$. Let $\mathscr{T}_1 = \Phi(\mathscr{U})$. Then $\mathscr{T}_1$ is maximal connected and T_1.

THEOREM 2. *Let $(X, \mathscr{T}_1)$ be a finite connected topological space. Then there exists a maximal connected topology $\mathscr{T}_2$ on X such that $\mathscr{T}_1 \subset \mathscr{T}_2$.*

The proof is easy and will be left to the reader.

THEOREM 3. *Let $(X, \mathscr{T}_1)$ be a maximal connected topological space and let V be an open connected subset* of $(X, \mathscr{T}_1)$. *Then $(V, \mathscr{T}_1|V)$ is maximal connected.*

PROOF. If $(V, \mathscr{T}_1|V)$ is not maximal connected let $\mathscr{S}_1$ be a topology on V such that $\mathscr{S}_1$ is strictly finer than $\mathscr{T}_1|V$ and $(V, \mathscr{S}_1)$ is connected. Let $W \subset V$ be such that $W \in \mathscr{S}_1 - (\mathscr{T}_1|V)$. Then $\mathscr{S}_2 = \Phi(\mathscr{T}_1|V \cup \{W\})$ is connected and is strictly finer than $\mathscr{T}_1|V$. Let $\mathscr{T}_2 = \Phi(\mathscr{T}_1 \cup \{W\})$. Then $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$ so let (A, B) be an open partition of $(X, \mathscr{T}_2)$. Then either $V \subset A$ or $V \subset B$, for otherwise $(V \cap A, V \cap B)$ is an open partition of $(V, \mathscr{T}_2|V) = (V, \mathscr{S}_2)$, so assume $V \subset A$. Hence $K(\mathscr{T}_1, \mathscr{T}_2) \subset W \subset V \subset A$, which implies $A \in \mathscr{T}_1$ since $V \in \mathscr{T}_1$. Clearly $B \in \mathscr{T}_1$. Hence (A, B) is an open partition of $(X, \mathscr{T}_1)$ which is impossible.

DEFINITION 2. Let a and b be points of a set S and let $H_1, H_2, \cdots H_n$ be a finite collection of subsets of S. This collection is said to be a *simple chain from a to b* if and only if

i) $a \in H_1 - H_2$, $b \in H_n - H_{n-1}$

ii) $H_i \cap H_j \neq \emptyset$, if and only if $|i-j| \leqq 1$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$.

The above definition is a slight modification of that given by [1] in that we require $a \notin H_2$ and $b \notin H_{n-1}$. It is easy to see that with this change the following theorem, also from [1], remains valid.

THEOREM 4. *A space S is connected if and only if given any two points a and b of S and any open covering $\{G_\alpha\}$ of S, there exists a finite subcollection of $\{G_\alpha\}$ which is a simple chain from a to b.*

THEOREM 5. *Let $(X, \mathscr{T}_1)$ be a topological space where X has at least two elements and $\mathscr{T}_1$ is such that every intersection of open sets is open. Let*

I be the set of all isolated points of $(X, \mathscr{T}_1)$ *and let* $J = X - I$. *If* $x \in J$ *let* V_x *be the smallest neighborhood of* x, (i.e. the intersection of all open neighborhoods of x). *Then in order that* $(X, \mathscr{T}_1)$ *be maximal connected it is necessary and sufficient that all of the following three statements be true*:

i) $\bigcup_{x \in J} V_x = X$.

ii) *If* $x \neq x'$, x *and* $x' \in J$, *then* $V_x \cap V_{x'}$ *has at most one point.*

iii) *If* $a, b \in (X, \mathscr{T}_1)$ *then there exists exactly one simple chain from* a *to* b *of open sets* V_x.

Proof. Suppose $(X, \mathscr{T}_1)$ is maximal connected.

i) If $z \in X$ is in no V_x, then $\{z\}$ is open and closed which is impossible.

ii) We shall show first that $V_x - \{x\} \subset I$ for every $x \in J$. Suppose $z \in V_x \cap J$, $z \neq x$. Let $\mathscr{T}_2 = \Phi(\mathscr{T}_1 \cup \{\{z\}\})$, which is strictly finer than $\mathscr{T}_1$. Hence, we can let (A, B) be an open partition of $(X, \mathscr{T}_2)$. Since $K(\mathscr{T}_1, \mathscr{T}_2) = \{z\}$ and $x \notin \{z\}$, we may assume $V_x \subset A$. But then (A, B) is an open partition of $(X, \mathscr{T}_1)$, which is impossible. Thus $z \in I$.

From the above we see that if $x, x' \in J$ then $V_x \cap V_{x'} \subset I$. Suppose $\{y_1, y_2\} \subset V_x \cap V_{x'}$, $y_1 \neq y_2$ and let $V'_{x'} = V_{x'} - \{y_2\}$. Let $\mathscr{T}_3 = \Phi(\mathscr{T}_1 \cup \{V'_{x'}\})$, which is strictly finer than $\mathscr{T}_1$. Let (A, B) be an open partition of $(X, \mathscr{T}_3)$ and suppose $x' \in A$. Then $V'_{x'} \subset A$ and $V_x \not\subset A$ since $K(\mathscr{T}_1, \mathscr{T}_3) = \{x'\}$. Thus we have $y_1 \in A$ and $y_2 \in B$. But then $x \in \mathrm{Cl}_3(A) \cap \mathrm{Cl}_3(B)$ since $\{y_1, y_2\} \subset V_x$. This is a contradiction.

iii) That there exists a simple chain of elements V_x from a to b is a consequence of connectedness (Theorem 4). Hence, suppose there are two such simple chains, $C_1 = \{V_{x_i}\}$, $i = 1, \cdots, n$ and $C_2 = \{C_{y_j}\}$, $j = 1, \cdots, m$. Let $S = \bigcup V_x$, $V_x \in C_1 \cup C_2$. Then S is open and connected and hence maximal connected for $\mathscr{T}_1|S$ by Theorem 3. Let j be the smallest integer such that $V_{x_j} \neq V_{y_j}$. If $j = 1$, we know that $V_{x_j} \cap V_{y_j} = \{a\}$ by ii). If $j \geqq 2$ we have $V_{y_j} \cap V_{y_{j-1}} = V_{y_j} \cap V_{x_{j-1}}$, a single point, say c, again by ii). Let $y = a$ if $j = 1$ and $y = c$ otherwise. Let $V'_{y_j} = V_{y_j} - \{y\}$, and let $\mathscr{T}_2 = \Phi(\mathscr{T}_1 \cup \{V'_{y_j}\})$, which makes $\mathscr{T}_2|S$ strictly finer than $\mathscr{T}_1|S$, so $(S, \mathscr{T}_2|S)$ is not connected. But $S_2 = V'_j \cup (\cup V_{y_i}, i = j+1, \cdots, m)$ is connected for $\mathscr{T}_2|S$, $S_1 = \cup V_x$, $V_x \in C_1$, is connected for $\mathscr{T}_2|S$, and $b \in S_1 \cap S_2$, so $S_1 \cup S_2 = S$ is connected for $\mathscr{T}_2|S$, which is a contradiction.

To show that the conditions are sufficient suppose they hold for $(X, \mathscr{T}_1)$. We show first that $(X, \mathscr{T}_1)$ is connected. Let $a \in (X, \mathscr{T}_1)$. With each $b \in X$ there is a simple chain C_b of sets V_x from a to b, by iii), and clearly $\cup V_x, V_x \in C_b$ is connected. Hence $X = \cup V_x$, $b \in X$, $V_x \in C_b$, is connected.

Suppose now that $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$ and that $(X, \mathscr{T}_2)$ is

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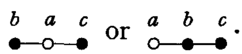
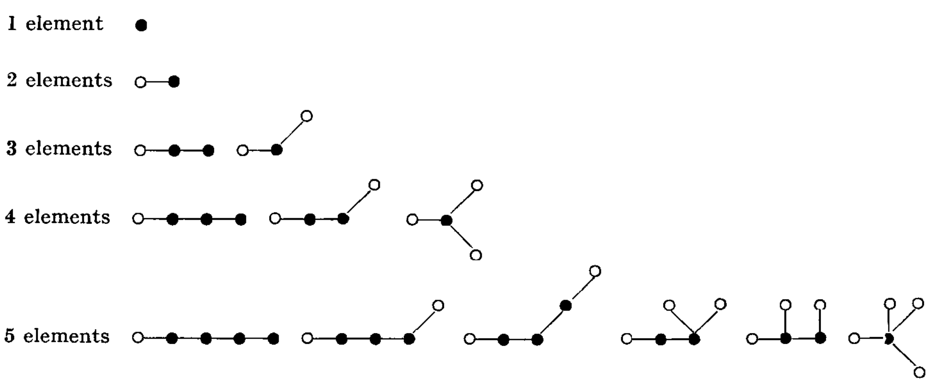
connected. Let $V \in \mathscr{T}_2 - \mathscr{T}_1$. We may assume $x \in V \subset V_x$ for some $x \in J$. Then $V_x - V$ consists entirely of isolated points, by ii), and is non-empty. Let M be the open covering of $(X, \mathscr{T}_2)$ consisting of all V_y such that $y \neq x$, $y \in J$, all $\{z\}$ such that $z \in V_x - V$, and V. Let $w \in V_x - V$. Let $C' = \{M_1, \cdots, M_n\}$ be a simple chain of elements of M from x to w. Note that since V is the only element of C which contains x we have $M_1 = V$. Hence, let $\{v\} = V \cap M_2$ and let $C' = C - \{V\}$. Then C' is a simple chain from v to w of elements of M. Furthermore, $V \notin C'$ and $V_x \notin C'$, so C' consists entirely of sets V_y, $y \neq x$, and hence is a simple chain from v to w of elements V_y. But $C'' = \{V_x\}$ is also such a chain which contradicts iii). Thus $(X, \mathscr{T}_1)$ is maximal connected. This completes the proof.

The preceding theorem give us a means of quickly determining all the maximal connected topologies on small finite sets. We represent the members of I by solid dots and the members of J by open dots. We represent V_x by a line segment on which we place the dots representing x and the isolated points which are in V_x. The order of the dots on the line segment is immaterial. Thus if $X = \{a, b, c\}$ and $\mathscr{T} = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ our sketch is



We can thus easily see and sketch all the non-homeomorphic maximal connected topologies on small finite sets by disregarding the naming of elements. We sketch below all the non-empty maximal connected topologies with less than 6 elements.



It should be an interesting counting problem to discover the number of maximal connected topologies on a set with n elements. To the author's knowledge this question is as yet unanswered.

Theorem 5 gives us a good source of examples for answering general questions about maximal connected topological spaces.

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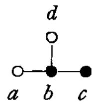
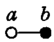
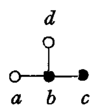
EXAMPLE 4. The quotient of a maximal connected topological space by an equivalence relation is not necessarily maximal connected. For, let $(X, \mathscr{T})$ be



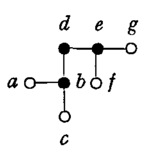
and let $R = \{\{a, b\}, \{c, d\}\}$. Note that $\{a, b\}$ is neither open nor closed. Then the quotient topology on X/R is the trivial topology which is not T_0 and hence not maximal connected by Theorem 1.

EXAMPLE 5. The product of maximal connected spaces is not necessarily maximal. For  is maximal connected but the product topology $\mathscr{T}_1 = \{\phi, \{(b, b)\}, \{(b, a), (b, b)\}, \{(a, b), (b, b)\}, X \times X\}$ is not maximal since $\Phi(\{\{(x, y)\} | (x, y) \in X \times X, (x, y) \neq (a, a)\})$ is connected and strictly finer than $\mathscr{T}_1$.

EXAMPLE 6. A door space $(X, \mathscr{T})$ is a topological space having the property that if $A \subset X$ then either $A \in \mathscr{T}$ or $X - A \in \mathscr{T}$. Examples 1, 2, 3 and Theorem 6 suggest the possibility that every maximal connected space is a door space. This is not the case for



is not a door space since $\{a, b\}$ is neither open nor closed. A semi-door space $(X, \mathscr{T})$ is a space having the property that for $A \subset X$ there exists $B \in \mathscr{T}$ such that either $B \subset A \subset \text{Cl}(B)$ or $B \subset X - A \subset \text{Cl}(B)$. The space



is maximal connected but not semi-door. For $A = \{b, c, d, f\}$ does not satisfy the condition.

THEOREM 6. *In order that $(X, \mathscr{T}_1)$ be maximal connected it is necessary that whenever $A \subset X$ and A is connected and $X - A$ is connected, $A \in \mathscr{T}_1$ or $X - A \in \mathscr{T}_1$.*

PROOF. If either A or $X - A$ is empty the proposition is trivial so suppose $A \neq \emptyset$, $X - A \neq \emptyset$. Suppose neither A nor $X - A$ is open. Let $\mathscr{T}_2 = \Phi(\mathscr{T}_1 \cup \{A\})$. Then $\mathscr{T}_2$ is strictly finer than $\mathscr{T}_1$ hence not connected, so there exist $U, V \in \mathscr{T}_2$ such that (U, V) is an open partition of $(X, \mathscr{T}_2)$. Suppose $U \cap (X-A)$ and $V \cap (X-A)$ are non-empty. Then $(U \cap (X-A), V \cap (X-A))$ is an open partition of $X-A$ for $\mathscr{T}_2$. But

$$((X-A), \mathscr{T}_1|(X-A)) = ((X-A), \mathscr{T}_2|(X-A))$$

so $(U \cap (X-A), V \cap (X-A))$ is an open partition of $(X-A, \mathscr{T}_1|(X-A))$ which is a contradiction. Thus either $U \subset A$ or $V \subset A$, so assume $U \subset A$. If $U = A$ we have $V = X-A \in \mathscr{T}_2$, and hence $V \in \mathscr{T}_1$, which is impossible. Hence, $V \cap A \neq \emptyset$, and therefore $(U, V \cap A)$ is an open partition of $(A, \mathscr{T}_2|A)$. But if $x \in A$ the neighborhoods of x for $\mathscr{T}_1|A$ are the same as those for $\mathscr{T}_2|A$, hence $(U, V \cap A)$ is an open partition of $(A, \mathscr{T}_1|A)$ which is a contradiction, hence the result.

NOTE: The conditions of the above theorem are not sufficient for they hold for the reals R with the order topology $\mathscr{T}$, but $\Phi(\mathscr{T} \cup \{Q\})$, where Q denotes the rationals, is strictly finer than $\mathscr{T}$ and is connected.

Reference

[1] Dick Wick Hall and Guilford L. Spencer, II, *Elementary Topology* (John Wiley and Sons, New York 1955).

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